Solving spinor Maxwell equations in cylindrical parabolic coordinates, and spinor space structure

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Abstract. Maxwell equations in any Riemannian space-time can be presented in spinor form on the base of the tetrad method, when the Maxwell field is described by a local 2-nd rank symmetrical spinor. This general covariant equation is specified in terms of cylindrical parabolic coordinates and of a corresponding diagonal tetrad. After separating the variables, we derive the system of four 1-st order differential equations with partial derivatives for three functions which depend on two parabolic coordinates. The mathematical task reduces to one 2-nd order equation with partial derivative for a main function, which determines all the remaining functions. The solutions are constructed in terms of the confluent hypergeometric functions. We study the properties of four types of constructed solutions - they must be continuous and single-valued in the context of vector or spinor space models. It is shown that in a space with vector structure only two variants provide correct solutions; in a spinor space, all four variants are appropriate. It is shown that the diagonalization of the helicity operator for a 2-rank symmetric spinor leads to the system of equations which coincides with the one which emerges from the Maxwell equations, when identifying the eigenvalue with the frequency of electromagnetic solutions, $\sigma = +\omega$. The eigenvalues $\sigma = 0$ and $\sigma = -\omega$ with respective to the eigenstates of the helicity operator are shown to be irrelevant.

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1 The spinor form of Maxwell equations

To introduce spinor notations, let us start with the ordinary Dirac equation [5]

$$(1.1)\qquad (i\gamma^a \partial_a - m)\Psi = 0, \gamma^a = \begin{vmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{vmatrix}, \Psi = \begin{vmatrix} \xi^\alpha \\ \eta_{\dot{\alpha}} \end{vmatrix}, \{\alpha, \dot{\alpha}\} = 1, 2\,;$$

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where $\sigma^a = (I, \sigma^j),\ \bar{\sigma}^a = (I, -\sigma^j)$. In 2-spinor form we have two equations

$$i\sigma^a \partial_a \xi = m\eta\,, \quad i\bar{\sigma}^a \partial_a \eta = m\xi\,. \tag{1.2}$$

It is convenient to attach spinor indices to Pauli matrices, $\sigma^a = (\sigma^a)_{\dot{\beta}\alpha}\,,\ \bar{\sigma}^a = (\bar{\sigma}^a)^{\beta\dot{\alpha}}$. Then eqs. (1.2) read

$$i(\sigma^a \partial_a)_{\dot{\beta}\alpha}\, \xi^\alpha = m\eta_{\dot{\beta}}, \quad i(\bar{\sigma}^a \partial_a)^{\beta\dot{\alpha}}\, \eta_{\dot{\alpha}} = m\xi^\beta\,. \tag{1.3}$$

The electromagnetic tensor is equivalent to a pair of symmetrical 2-rank spinors: $F_{mn} \longleftrightarrow \{\xi^{\alpha\beta}, \eta_{\dot{\alpha}\dot{\beta}}\}$; correspondingly, eight Maxwell equations are obtained as follows

$$(\sigma^a \partial_a)_{\dot{\rho}\alpha}\, \xi^{\alpha\beta} = (\sigma^b)_{\dot{\rho}\alpha} \omega^{\alpha\beta}\, J_b\,, \quad (\bar{\sigma}^a \partial_a)^{\rho\dot{\alpha}}\, \eta_{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}^b)^{\rho\dot{\alpha}} \omega_{\dot{\alpha}\dot{\beta}}\, J_b\,, \tag{1.4}$$

where the second equation is conjugate to the first one. In (1.4) we use spinor metrical matrices [5],

$$(\epsilon_{\alpha\beta}) = i\sigma^2\,, \quad (\epsilon^{\alpha\beta}) = -i\sigma^2\,; \quad (\epsilon_{\dot{\alpha}\dot{\beta}}) = i\sigma^2\,, \quad (\epsilon^{\dot{\alpha}\dot{\beta}}) = -i\sigma^2\,. \tag{1.5}$$

To prove the equivalence of the spinor form (1.4) to the ordinary Maxwell equations in vector notations, we apply notations without spinor indices. To this end, we take into account the identities [5]

$$\begin{gathered} (\xi^{\alpha\beta}) = \Sigma^{mn} F_{mn} \sigma^2\,,\ (\eta_{\dot{\alpha}\dot{\beta}}) = -\bar{\Sigma}^{mn} F_{mn} \sigma^2\,, \\ \Sigma^{mn} = \frac{1}{4}(\bar{\sigma}^m \sigma^n - \bar{\sigma}^n \sigma^m)\,,\ \bar{\Sigma}^{mn} = \frac{1}{4}(\sigma^m \bar{\sigma}^n - \sigma^n \bar{\sigma}^m)\,. \end{gathered} \tag{1.6}$$

Then, (1.4) may be re-written as

$$\sigma^a \partial_a\, \Sigma^{mn} F_{mn} = -\sigma^b J_b\,, \qquad \bar{\sigma}^a \partial_a\, \bar{\Sigma}^{mn} F_{mn} = -\bar{\sigma}^b J_b\,. \tag{1.7}$$

We further take into account the identities

$$\begin{gathered} \Sigma^{mn} F_{mn} = \sigma^1 (F_{01} - iF_{23}) + \sigma^2 (F_{02} - iF_{31}) + \sigma^3 (F_{03} - iF_{12})\,, \\ \bar{\Sigma}^{mn} F_{mn} = \sigma^1 (-F_{01} - iF_{23}) + \sigma^2 (-F_{02} - iF_{31}) + \sigma^3 (-F_{03} - iF_{12})\,. \end{gathered}$$

Using the notations

$$F_{01} = -E^1\,, F_{02} = -E^2\,, F_{03} = -E^3\,, F_{23} = B^1\,, F_{31} = B^2\,, F_{12} = B^3 \tag{1.8}$$

they read

$$\begin{gathered} \Sigma^{mn} F_{mn} = -\sigma^1 (E^1 + iB^1) - \sigma^1 (E^2 + iB^2) - \sigma^1 (E^3 + iB^3) = -\sigma^j a_j, \\ \bar{\Sigma}^{mn} F_{mn} = \sigma^1 (E^1 - iB^1) + \sigma^1 (E^2 - iB^2) + \sigma^1 (E^3 - iB^3) = +\sigma^j b_j, \end{gathered} \tag{1.9}$$

and

$$(\xi^{\alpha\beta}) = \left| \begin{array}{cc} -i(a_1 - ia_2) & ia_3 \\ ia_3 & +i(a_1 + ia_2) \end{array} \right|, (\eta_{\dot{\alpha}\dot{\beta}}) = \left| \begin{array}{cc} -i(b_1 - ib_2) & ib_3 \\ ib_3 & i(b_1 + ib_2) \end{array} \right|.$$

Taking into account (1.9), the equations (1.7) may be presented in the form

$$(\partial_0 + \sigma^l \partial_l)\,(\sigma^k a_k) = J_0 + \sigma^j J_j\,, \quad (\partial_0 - \sigma^l \partial_l)\,(\sigma^k b_k) = -J_0 + \sigma^j J_j\,,$$

whence we derive

$$\sigma^n \partial_0 a_n + (\delta_{lk} + i\omega_{nlk}\sigma^n)\partial_l a_k = J_0 + \sigma^n J_n,$$

$$\sigma^n \partial_0 b_n - (\delta_{lk} + i\omega_{nlk}\sigma^n)\partial_l b_k = -J_0 + \sigma^n J_n\,.$$

Therefore, we have four equations

$$\partial_l a_l = J_0\,, \quad \partial_0 a_n + i\omega_{nlk}\partial_l a_k = J_n\,, \quad \partial_l b_l = J_0\,, \quad \partial_0 b_n - i\omega_{nlk}\partial_l b_k = J_n\,,$$

or, differently,

$$\begin{array}{ll} (1) \quad \partial_l(E^l + iB^l) = J_0\,, & (2) \quad \partial_0(E^l + iB^l) + i\omega_{nlk}\partial_l(E^k + iB^k) = J_n\,, \\ (1') \quad \partial_l(E^l - iB^l) = J_0\,, & (2') \quad \partial_0(E^l - iB^l) - i\omega_{nlk}\partial_l(E^k - iB^l) = J_n\,. \end{array}$$

Summing and subtracting the equations, we obtain

$$1+1', \quad \partial_l E^l = J_0\,, \quad 1-1', \quad \partial_l B^l = 0\,,$$
$$2+2', \quad \partial_0 E^n - \omega_{nlk}\partial_l B^k = J_k\,, \quad 2-2', \quad \partial_0 B^n + \omega_{nlk}\partial_l E^k = 0\,;$$

they may be identified with Maxwell equations in vector form

$$\text{div}\,\mathbf{E} = J^0, \quad \text{div}\,\mathbf{B} = 0\,, \quad \text{rot}\,\mathbf{B} = \partial_0\mathbf{E} + \mathbf{J}\,, \quad \text{rot}\,\mathbf{E} = -\partial_0\mathbf{B}\,, \tag{1.10}$$

where

$$\mathbf{E} = (E^n)\,,\ \mathbf{B} = (B^n)\,, \quad J^0 = J_0\,,\ \mathbf{J} = (J^n) = (-J_n).$$

2 Cylindrical parabolic coordinates

Let us construct the solutions of the spinor Maxwell equations in cylindric parabolic coordinates:

$$\begin{gathered} x_1 = \frac{u^2 - v^2}{2}\,, \quad x_2 = uv\,, \quad x_3 = z\,; \\ v = +\sqrt{-x_1 + \sqrt{x_1^2 + x_2^2}}\,, \quad u = \pm\sqrt{+x_1 + \sqrt{x_1^2 + x_2^2}}\,. \end{gathered} \tag{2.1}$$

The metric of the Minkowski space takes the form (let $x^\alpha = (t, u, v, z)$)

$$g_{\alpha\beta} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -(u^2+v^2) & 0 & 0 \\ 0 & 0 & -(u^2+v^2) & 0 \\ 0 & 0 & 0 & -1 \end{array} \right|. \tag{2.2}$$

Below we shall use the diagonal tetrad[1]:

$$e_{(k)}^{\ \alpha} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{u^2+v^2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{u^2+v^2} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| ,$$

(2.3)

$$e_{(k)\alpha} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -\sqrt{u^2+v^2} & 0 & 0 \\ 0 & 0 & -\sqrt{u^2+v^2} & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| .$$

In order to find the Ricci rotation coefficients, we introduce auxiliary quantities (see in [1]): $\lambda_{abc} = \gamma_{abc} - \gamma_{acb}$. For λ_{abc} we easily derive the following representation

$$\lambda_{abc} = \gamma_{abc} - \gamma_{acb} = (e_{(a)\alpha;\beta} - e_{(a)\beta;\alpha})\, e^{\alpha}_{(c)} e^{\beta}_{(b)}$$

$$= (\partial_\beta e_{(a)\alpha} - \Gamma^{\rho}_{\alpha\beta} e_{(a)\rho} - \partial_\alpha e_{(a)\beta} + \Gamma^{\rho}_{\beta\alpha} e_{(a)\rho})\, e^{\alpha}_{(c)} e^{\beta}_{(b)} ;$$

that is

$$\lambda_{abc} = [\partial_\beta e_{(a)\alpha} - \partial_\alpha e_{(a)\beta}]\, e^{\alpha}_{(c)} e^{\beta}_{(b)} ; \qquad (2.4)$$

according to (2.4), λ_{abc} are calculated with the use of ordinary derivatives. Besides, we have the identity

$$\frac{1}{2}(\lambda_{abc} + \lambda_{bca} - \lambda_{cab}) \equiv \frac{1}{2}(\gamma_{abc} - \gamma_{acb} + \gamma_{bca} - \gamma_{bac} - \gamma_{cab} + \gamma_{cba}) \equiv \gamma_{abc} .$$

We need explicit expressions for λ_{abc}. First of all, we have two relations,

$$a = 0, \quad \lambda_{0bc} = 0, \qquad\qquad a = 3, \quad \lambda_{0bc} = 0 .$$

Further, taking in mind the diagonal structure of the tetrad, we derive the formula

$$\lambda_{1bc} = e^{\beta}_{(b)} e^{1}_{(c)} \partial_\beta e_{(1)1} - e^{1}_{(b)} e^{\beta}_{(c)} \partial_\beta e_{(1)1}$$
$$= e^{1}_{(b)} e^{1}_{(c)} \partial_1 e_{(1)1} + e^{2}_{(b)} e^{1}_{(c)} \partial_2 e_{(1)1} - e^{1}_{(b)} e^{1}_{(c)} \partial_1 e_{(1)1} - e^{1}_{(b)} e^{2}_{(c)} \partial_2 e_{(1)1} ,$$

that is,

$$\lambda_{1[bc]} = [e^{2}_{(b)} e^{1}_{(c)} - e^{1}_{(b)} e^{2}_{(c)}]\, \partial_2 e_{(1)1} = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{1}_{(1)} e^{2}_{(2)} & 0 \\ 0 & e^{1}_{(1)} e^{2}_{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| \partial_2 e_{(1)1} ;$$

and similarly

$$\lambda_{2bc} = e^{\beta}_{(b)} e^{2}_{(c)} \partial_\beta e_{(2)2} - e^{2}_{(b)} e^{\beta}_{(c)} \partial_\beta e_{(2)2}$$
$$= e^{1}_{(b)} e^{2}_{(c)} \partial_1 e_{(2)2} + e^{2}_{(b)} e^{2}_{(c)} \partial_2 e_{(2)2} - e^{2}_{(b)} e^{1}_{(c)} \partial_1 e_{(2)2} - e^{2}_{(b)} e^{2}_{(c)} \partial_2 e_{(2)2} ,$$

[1] We recall that $g_{\alpha\beta}(x) e_{(k)}^{\ \alpha} e_{(l)}^{\ \beta} = \eta_{kl}$.

so that,

$$\lambda_{2[bc]} = [e^1_{(b)} e^2_{(c)} - e^2_{(b)} e^1_{(c)}]\, \partial_1 e_{(2)2} = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & e^1_{(1)} e^2_{(2)} & 0 \\ 0 & -e^1_{(1)} e^2_{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| \partial_1 e_{(2)2} \, .$$

Allowing for the relations

$$e^1_{(1)} = e^2_{(2)} = 1/\sqrt{u^2+v^2}, \; e_{(1)1} = e_{(2)2} = -\sqrt{u^2+v^2} \, ,$$

we obtain the needed formulas for the coefficients $\lambda_{1[bc]}$ and $\lambda_{2[bc]}$:

$$\lambda_{1[bc]} = \frac{1}{(u^2+v^2)^{3/2}} \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & +v & 0 \\ 0 & -v & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| , \lambda_{2[bc]} = \frac{1}{(u^2+v^2)^{3/2}} \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -u & \\ 0 & +u & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| .$$

Thus, the only nonvanishing coefficients are

$$\lambda_{1[12]} = +\frac{v}{(u^2+v^2)^{3/2}} \, , \quad \lambda_{1[21]} = -\frac{v}{(u^2+v^2)^{3/2}} \, ,$$
$$\lambda_{2[12]} = -\frac{u}{(u^2+v^2)^{3/2}} \, , \quad \lambda_{1[21]} = +\frac{u}{(u^2+v^2)^{3/2}} \, . \tag{2.5}$$

Now, when using the formula $\gamma_{[ab]c} = \frac{1}{2}(-\lambda_{c[ab]} + \lambda_{a[bc]} - \lambda_{b[ac]})$, it is convenient to split it into four cases

$$\gamma_{[ab]0} = \frac{1}{2}(-\lambda_{0[ab]} + \lambda_{a[b0]} - \lambda_{b[a0]}) \, , \quad \gamma_{[ab]3} = \frac{1}{2}(-\lambda_{3[ab]} + \lambda_{a[b3]} - \lambda_{b[a3]}) \, ,$$
$$\gamma_{[ab]1} = \frac{1}{2}(-\lambda_{1[ab]} + \lambda_{a[b1]} - \lambda_{b[a1]}) \, , \quad \gamma_{[ab]2} = \frac{1}{2}(-\lambda_{2[ab]} + \lambda_{a[b2]} - \lambda_{b[a2]}) \, .$$

We find the nonvanishing Ricci coefficients

$$\gamma_{[12]1} = -\gamma_{[21]1} = -\lambda_{1[12]} = -\frac{v}{(u^2+v^2)^{3/2}} \, ,$$
$$\gamma_{[12]2} = -\gamma_{[21]2} = -\lambda_{2[12]} = +\frac{u}{(u^2+v^2)^{3/2}} \, .$$

Now let us turn to Maxwell equations, when using the tetrad formalism:

$$\left[\sigma^c e^\alpha_{(c)}(x) \partial_\alpha + \sigma^c (\frac{1}{2}\Sigma^{ab} \otimes I + I \otimes \frac{1}{2}\Sigma^{ab}) \gamma_{[ab]c}(x) \right] \xi(x) = 0 \, , \tag{2.6}$$

where

$$\Sigma^{0j} = \frac{1}{2}\sigma^j, \; \Sigma^{12} = -\frac{i}{2}\sigma^3, \; \Sigma^{23} = -\frac{i}{2}\sigma^1, \; \Sigma^{31} = -\frac{i}{2}\sigma^2 \, .$$

Equation (2.6) takes the form

$$\Big[\sigma^0 \partial_t + \sigma^3 \partial_z + \sigma^1 e^1_{(1)}(x) \partial_u + \sigma^2 e^2_{(2)}(x) \partial_v$$
$$+ \sigma^1 (\Sigma^{12} \otimes I + I \otimes \Sigma^{12}) \gamma_{[12]1} + \sigma^2 (\Sigma^{12} \otimes I + I \otimes \Sigma^{12}) \gamma_{[12]2} \Big] \, \xi(x) = 0 \; ,$$

or differently,

$$\begin{gathered}\big\{\sigma^0\partial_t+\sigma^3\partial_z+\frac{1}{\sqrt{u^2+v^2}}\left(\sigma^1\partial_u+\sigma^2\partial_v\right)+\frac{i/2}{(u^2+v^2)^{3/2}}\\ \times\left[v\sigma^1(\sigma^3\otimes I+I\otimes\sigma^3)-u\sigma^2(\sigma^3\otimes I+I\otimes\sigma^3)\right]\big\}\xi(x)=0.\end{gathered}\tag{2.7}$$

We use the following substitution for the electromagnetic spinor $\xi(x)$:

$$\xi(x)=e^{-i\omega t}e^{ikz}\left|\begin{array}{cc} f(u,v) & h(u,v)\\ h(u,v) & g(u,v)\end{array}\right|,\tag{2.8}$$

where f, g, h stand for some functions over the variables u, v. Taking into account the expressions of the Pauli matrices

$$\sigma^0=\left|\begin{array}{cc}1&0\\0&1\end{array}\right|,\sigma^1=\left|\begin{array}{cc}0&1\\1&0\end{array}\right|,\sigma^2=\left|\begin{array}{cc}0&-i\\i&0\end{array}\right|,\sigma^3=\left|\begin{array}{cc}1&0\\0&-1\end{array}\right|,$$

from (2.7) we derive

$$-i\omega\left|\begin{array}{cc}f&h\\h&g\end{array}\right|+ik\left|\begin{array}{cc}f&h\\-h&-g\end{array}\right|+\frac{1}{\sqrt{u^2+v^2}}\left|\begin{array}{cc}(\partial_u-i\partial_v)h&(\partial_u-i\partial_v)g\\(\partial_u+i\partial_v)f&(\partial_u+i\partial_v)h\end{array}\right|$$

$$+\frac{i}{(u^2+v^2)^{3/2}}\left|\begin{array}{cc}0&-(v+iu)g\\(v-iu)f&0\end{array}\right|=\left|\begin{array}{cc}0&0\\0&0\end{array}\right|.$$

There follow four equations

$$\begin{aligned}(11)\quad & -i\omega f+ikf+\frac{1}{\sqrt{u^2+v^2}}(\partial_u-i\partial_v)h=0\,,\\ (22)\quad & -i\omega g-ikg+\frac{1}{\sqrt{u^2+v^2}}(\partial_u+i\partial_v)h=0\,,\\ (12)\quad & -i\omega h+ikh+\frac{1}{\sqrt{u^2+v^2}}(\partial_u-i\partial_v)g-\frac{i(v+iu)}{(u^2+v^2)^{3/2}}g=0\,,\\ (21)\quad & -i\omega h-ikh+\frac{1}{\sqrt{u^2+v^2}}(\partial_u+i\partial_v)f+\frac{i(v-iu)}{(u^2+v^2)^{3/2}}f=0\,.\end{aligned}\tag{2.9}$$

They may be re-written as

$$\begin{gathered}-i\omega f+ikf+\frac{1}{\sqrt{u^2+v^2}}\partial_u h-\frac{i}{\sqrt{u^2+v^2}}\partial_v h=0\,,\\ -i\omega g-ikg+\frac{1}{\sqrt{u^2+v^2}}\partial_u h+\frac{i}{\sqrt{u^2+v^2}}\partial_v h=0\,,\\ -i\omega h+ikh+\frac{1}{\sqrt{u^2+v^2}}\left(\partial_u+\frac{u}{u^2+v^2}\right)g-\frac{i}{\sqrt{u^2+v^2}}\left(\partial_v+\frac{v}{u^2+v^2}\right)g=0\,,\\ -i\omega h-ikh+\frac{1}{\sqrt{u^2+v^2}}\left(\partial_u+\frac{u}{u^2+v^2}\right)f+\frac{i}{\sqrt{u^2+v^2}}\left(\partial_v+\frac{v}{u^2+v^2}\right)f=0\,.\end{gathered}$$

Taking into account the following two identities for the function g:

$$\frac{1}{\sqrt{u^2+v^2}}\left(\partial_u + \frac{u}{u^2+v^2}\right) g = \frac{1}{u^2+v^2}\partial_u \sqrt{u^2+v^2}g, \quad \sqrt{u^2+v^2}g \equiv \bar{g},$$
$$\frac{1}{\sqrt{u^2+v^2}}\left(\partial_v + \frac{v}{u^2+v^2}\right) g = \frac{1}{u^2+v^2}\partial_v \sqrt{u^2+v^2}g, \quad \sqrt{u^2+v^2}g \equiv \bar{g},$$

and the similar ones for f:

$$\frac{1}{\sqrt{u^2+v^2}}\left(\partial_u + \frac{u}{u^2+v^2}\right) f = \frac{1}{u^2+v^2}\partial_u \sqrt{u^2+v^2}f, \quad \sqrt{u^2+v^2}f \equiv \bar{f},$$
$$\frac{1}{\sqrt{u^2+v^2}}\left(\partial_v + \frac{v}{u^2+v^2}\right) f = \frac{1}{u^2+v^2}\partial_v \sqrt{u^2+v^2}f, \quad \sqrt{u^2+v^2}f \equiv \bar{f},$$

and introducing new functions

$$\sqrt{u^2+v^2}f \equiv \bar{f}, \quad \sqrt{u^2+v^2}g \equiv \bar{g}, \tag{2.10}$$

we present the above system as follows

$$\begin{gathered} -i\omega\bar{f} + ik\bar{f} + \partial_u h - i\partial_v h = 0\,, \quad -i\omega\bar{g} - ik\bar{g} + \partial_u h + i\partial_v h = 0\,, \\ -i\omega h + ikh + \frac{1}{u^2+v^2}\partial_u\bar{g} - \frac{i}{u^2+v^2}\partial_v\bar{g} = 0\,, \\ -i\omega h - ikh + \frac{1}{u^2+v^2}\partial_u\bar{f} + \frac{i}{u^2+v^2}\partial_v\bar{f} = 0\,. \end{gathered} \tag{2.11}$$

Let us sum and subtract the equations within each pair. Then, with the notations

$$\bar{f}+\bar{g} = F\,, \quad \bar{f}-\bar{g} = G\,, \qquad F+G = 2\bar{f}\,, \quad F-G = 2\bar{g} \tag{2.12}$$

we obtain

$$\begin{gathered} -i\omega F + ikG + 2\partial_u h = 0\,, \; -i\omega G + ikF - 2i\partial_v h = 0\,, \\ -2i\omega h + \frac{1}{u^2+v^2}\partial_u F + \frac{i}{u^2+v^2}\partial_v G = 0\,, \\ 2ikh - \frac{1}{u^2+v^2}\partial_u G - \frac{i}{u^2+v^2}\partial_v F = 0\,. \end{gathered} \tag{2.13}$$

From the first two equations we find

$$F = \frac{2}{\omega^2-k^2}[-i\omega\partial_u h - k\partial_v h]\,, \quad G = \frac{2}{\omega^2-k^2}[-ik\partial_u h - \omega\partial_v h]\,, \tag{2.14}$$

and substituting these into the remaining two equations, we get

$$-2i\omega h + \frac{1}{u^2+v^2}\frac{2}{\omega^2-k^2}\left\{-i\omega\partial_u^2 - k\partial_{uv}^2 + k\partial_{uv}^2 - i\omega\partial_v^2\right\} h = 0\,,$$
$$2ikh - \frac{1}{u^2+v^2}\frac{2}{\omega^2-k^2}\left\{-ik\partial_u^2 - \omega\partial_{uv}^2 + \omega\partial_{uv}^2 - ik\partial_v^2\right\} h = 0\,.$$

Hence, after simple regrouping of the terms, there follow the two coinciding equations

$$
\begin{aligned}
-2i\omega h + \frac{1}{u^2+v^2}\frac{2}{\omega^2-k^2}\left\{-i\omega(\partial_u^2+\partial_v^2) - k\partial^2_{uv} + k\partial^2_{uv}\right\} h = 0\,,\\
2ikh - \frac{1}{u^2+v^2}\frac{2}{\omega^2-k^2}\left\{-ik(\partial_u^2+\partial_v^2) - \omega\partial^2_{uv} + \omega\partial^2_{uv}\right\} h = 0\,.
\end{aligned}
$$

Thus, we arrive at the equation for $h(u,v)$:

$$
h + \frac{1}{u^2+v^2}\frac{1}{\omega^2-k^2}(\partial_u^2+\partial_v^2)h = 0\,;
$$

which can be presented as (let $\lambda^2 = \omega^2 - k^2 > 0$)

$$
\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \lambda^2 u^2 + \lambda^2 v^2\right) h = 0\,. \tag{2.15}
$$

In this equation, the variables may be separated by the substitution $h(u,v) = U(u)\,V(v)$:

$$
\left(\frac{1}{U}\frac{d^2}{dv^2}U + \lambda^2 u^2\right) + \left(\frac{1}{V}\partial_v^2 V + \lambda^2 v^2\right) = 0\,.
$$

We find the two separate equations

$$
\begin{aligned}
&\frac{1}{U}\partial_u^2 U + \lambda^2 u^2 = -A \;\Longrightarrow\; \left(\frac{d^2}{du^2} + \lambda^2 u^2 + A\right) U = 0\,,\\
&\frac{1}{V}\partial_v^2 V + \lambda^2 v^2 = +A \;\Longrightarrow\; \left(\frac{d^2}{dv^2} + \lambda^2 v^2 - A\right) V = 0\,.
\end{aligned} \tag{2.16}
$$

Let us transform eqs. (2.16) using the new variables:

$$
\begin{aligned}
X = u^2,\; \left(\frac{d^2}{dX^2} + \frac{1}{2X}\frac{d}{dX} + \frac{\lambda^2}{4} + \frac{A}{4X}\right) U = 0\,;\\
Y = v^2,\; \left(\frac{d^2}{dY^2} + \frac{1}{2Y}\frac{d}{dY} + \frac{\lambda^2}{4} - \frac{A}{4Y}\right) V = 0\,.
\end{aligned}
$$

Their solutions are searched in the form

$$
U(X) = X^{\alpha} e^{\beta X} f(X), \qquad V(Y) = Y^{\rho} e^{\sigma Y} g(Y)\,. \tag{2.17}
$$

We further obtain

$$
\left[\frac{d^2}{dX^2} + (\frac{2\alpha}{X} + \frac{1}{2X} + 2\beta)\frac{d}{dX} + \frac{2\alpha\beta}{X} + \frac{\beta}{2X} + \frac{A}{4X} + \frac{\alpha(\alpha-1)}{X^2} + \frac{\alpha/2}{X^2} + \beta^2 + \frac{\lambda^2}{4}\right] f(X) = 0\,,
$$

$$
\left[\frac{d^2}{dY^2} + (\frac{2\rho}{Y} + \frac{1}{2Y} + 2\sigma)\frac{d}{dY} + \frac{2\rho\sigma}{Y} + \frac{\sigma}{2Y} - \frac{A}{4Y} + \frac{\rho(\rho-1)}{Y^2} + \frac{\rho/2}{Y^2} + \sigma^2 + \frac{\lambda^2}{4}\right] g(Y) = 0\,.
$$

Imposing the restrictions $\alpha = 0, +\frac{1}{2},\; 2\beta = \pm i\lambda\,;\; \rho = 0, +\frac{1}{2},\; 2\sigma = \pm i\lambda$, we get

$$
\begin{aligned}
\left[X\frac{d^2}{dX^2} + (2\alpha + \frac{1}{2} + 2\beta X)\frac{d}{dX} + 2\alpha\beta + \frac{\beta}{2} + \frac{A}{4}\right] f = 0\,,\\
\left[Y\frac{d^2}{dY^2} + (2\rho + \frac{1}{2} + 2\sigma Y)\frac{d}{dY} + 2\rho\sigma + \frac{\sigma}{2} - \frac{A}{4}\right] g = 0\,.
\end{aligned}
$$

Let us fix the two parameters $2\beta = -i\lambda\,,\ 2\sigma = -i\lambda$, this yields

$$\Big[X\frac{d^2}{dX^2} + (2\alpha + 1/2 - i\lambda X)\frac{d}{dX} - i\lambda\alpha - i\lambda/4 + A/4\Big]\, f(X) = 0\,,$$
$$\Big[Y\frac{d^2}{dY^2} + (2\rho + 1/2 - i\lambda Y)\frac{d}{dY} - i\lambda\rho - i\lambda/4 - A/4\Big]\, g(Y) = 0\,.$$

After transforming these equations to the new variables

$$i\lambda X = x = i\lambda u^2, \qquad i\lambda Y = y = i\lambda v^2, \quad A/4i\lambda = \Lambda\,, \tag{2.18}$$

we obtain

$$\begin{gathered}\Big[x\frac{d^2}{dx^2} + (2\alpha + 1/2 - x)\frac{d}{dx} - \alpha - 1/4 + \Lambda\Big]\, f(x) = 0\,,\\ \Big[y\frac{d^2}{dy^2} + (2\rho + 1/2 - y)\frac{d}{dy} - \rho - 1/4 - \Lambda\Big]\, g(y) = 0\,.\end{gathered} \tag{2.19}$$

For definiteness, let us take the values $\alpha = 0, \rho = 0$:

$$\begin{gathered}\Big[x\frac{d^2}{dx^2} + (1/2 - x)\frac{d}{dx} - (1/4 - \Lambda)\Big]\, f(x) = 0\,,\\ U(x) = e^{x/2} f(x), \quad x = i\lambda u^2;\\ \Big[y\frac{d^2}{dy^2} + (1/2 - y)\frac{d}{dy} - (1/4 + \Lambda)\Big]\, g(y) = 0\,,\\ V(y) = e^{y/2} g(y), \quad y = i\lambda v^2.\end{gathered} \tag{2.20}$$

They both can be identified with the confluent hypergeometric equation

$$zF'' + (c - z)F' - aF = 0, \quad c = 1/2,\ a = 1/4 \pm \Lambda\ . \tag{2.21}$$

Two linearly independent solutions may be used:

$$F_1 = \Phi(a, c; z), \quad F_2 = z^{1-c}\Phi(a - c + 1, 2 - c; z)\,. \tag{2.22}$$

Therefore, for the two equations in (2.20), we have respectively the pairs of solutions:

$$\begin{gathered}U_1 = e^{x/2}\Phi(\frac{1}{4} - \Lambda, \frac{1}{2};\, x), \quad U_2 = e^{x/2}\sqrt{x}\ \Phi(\frac{3}{4} - \Lambda, \frac{3}{2} - c; x)\,;\\ V_1 = e^{y/2}\Phi(\frac{1}{4} + \Lambda, \frac{1}{2};\, y), \quad V_2 = e^{y/2}\sqrt{y}\ \Phi(\frac{3}{4} + \Lambda, \frac{3}{2}; y)\,;\end{gathered} \tag{2.23}$$

we recall that $\sqrt{x} = \sqrt{i\lambda}\, u$ and $\sqrt{y} = \sqrt{i\lambda}\, v$.

3 Continuity and spinor space structure

Let us find the explicit form for the constructed solutions relative to the initial Cartesian tetrad. It is known that the 2-spinor ψ_0 in Cartesian tetrad is related to the

2-spinor ψ in cylindric parabolic tetrad, by the following local gauge transformation

$$\psi = s\psi_0, \quad s = \frac{1}{(u^2+v^2)^{1/4}} \left| \begin{array}{cc} \sqrt{u+iv} & 0 \\ 0 & \sqrt{u-iv} \end{array} \right|, \qquad s^{-1} = s^{+} = \frac{1}{(u^2+v^2)^{1/4}} \left| \begin{array}{cc} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{array} \right|. \tag{3.1}$$

Therefore, the 2-rank spinors relate to each other by the transformation

$$\xi = S\xi_0 = (s \otimes s)\xi_0 = s\xi_0 s\,, \; \xi_0 = S^{-1}\xi = (s^{-1} \otimes s^{-1})\xi = s^{-1}\xi s^{-1}\,; \tag{3.2}$$

so for the electromagnetic spinor in Cartesian basis

$$\xi_0 = \frac{1}{(u^2+v^2)^{1/2}} \left| \begin{array}{cc} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{array} \right| \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| \left| \begin{array}{cc} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{array} \right|$$

we get

$$\xi_0 = \left| \begin{array}{cc} \frac{(u-iv)}{\sqrt{u^2+v^2}} f & h \\ h & \frac{(u+iv)}{\sqrt{u^2+v^2}} g \end{array} \right|. \tag{3.3}$$

Evidently, the primary one is the function h, while the two other ones f and g are determined by h. Taking into account the expressions for f and g,

$$f = \frac{1}{2\sqrt{u^2+v^2}}(F+G), \qquad g = \frac{1}{2\sqrt{u^2+v^2}}(F-G)\,,$$

and allowing for the formulas

$$F = \frac{2}{\omega^2-k^2}[-i\omega\partial_u h - k\partial_v h]\,, \quad G = \frac{2}{\omega^2-k^2}[-ik\partial_u h - \omega\partial_v h]\,, \tag{3.4}$$

we derive

$$f = \frac{1}{\omega-k}\frac{1}{\sqrt{u^2+v^2}}(-i\partial_u - \partial_v)\,h, \qquad g = \frac{1}{\omega+k}\frac{1}{\sqrt{u^2+v^2}}(-i\partial_u + \partial_v)\,h. \tag{3.5}$$

Further, we should take in mind the product formula $h(u,v) = U(u)\,V(v)$ and the two possibilities for each multiplier (see (2.23)):

$$U_1 = e^{x/2}\Phi(1/4-\Lambda, 1/2;\, x), \quad U_2 = e^{x/2}\sqrt{x}\;\Phi(3/4-\Lambda, 3/2; x),$$

$$V_1 = e^{y/2}\Phi(1/4+\Lambda, 1/2;\, y), \quad V_2 = e^{y/2}\sqrt{y}\;\Phi(3/4+\Lambda, 3/2; y),$$

where

$$x = i\lambda u^2, \quad y = i\lambda v^2 \implies \frac{\partial}{\partial u} = 2i\lambda u\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial v} = 2i\lambda v\frac{\partial}{\partial y}\,. \tag{3.6}$$

Let us change (3.5) to its other form

$$
(3.7)\qquad
\begin{aligned}
f &= \frac{2i\lambda}{\omega - k}\frac{1}{\sqrt{u^2+v^2}}(-iu\frac{\partial}{\partial x} - v\frac{\partial}{\partial y})\, h = A(-iu\frac{\partial}{\partial x} - v\frac{\partial}{\partial y})\, h\,,\\
g &= \frac{2i\lambda}{\omega + k}\frac{1}{\sqrt{u^2+v^2}}(-iu\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\, h = B(-iu\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\, h\,.
\end{aligned}
$$

Besides, we should take in mind the rule for differentiating the confluent hypergeometric function

$$\frac{d}{dx}\Phi(a,c;x) = \frac{a}{c}\Phi(a+1,c+1;x).$$

We are to follow all the four possibilities for the function $h(u,b)$:

$$
(3.8)\qquad
\begin{aligned}
h_{11} &= U_1(x)V_1(y)\,, \quad h_{12} = U_1(x)V_2(y)\,,\\
h_{21} &= U_2(x)V_1(y)\,, \quad h_{22} = U_1(x)V_2(y)\,.
\end{aligned}
$$

Consider the variant h_{11}:

$$f_{11} = A(-iu\frac{\partial}{\partial x} - v\frac{\partial}{\partial y})\, U_1(x)V_1(y)$$

$$= A\Big\{ -iue^{x/2}\Big[\frac{1}{2}\Phi(1/4-\Lambda,1/2;\,x) + \frac{1/4-\Lambda}{1/2}\Phi(5/4-\Lambda,3/2;\,x)\Big]\,V_1(y)$$

$$-vU_1(x)e^{y/2}\Big[\frac{1}{2}\Phi(1/4+\Lambda,1/2;\,y) + \frac{1/4+\Lambda}{1/2}\Phi(5/4+\Lambda,3/2;\,y)\Big]\Big\}\,,$$

$$g_{11} = B(-iu\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\, U_1(x)V_1(y)$$

$$= B\Big\{ -iue^{x/2}\Big[\frac{1}{2}\Phi(1/4-\Lambda,1/2;\,x) + \frac{1/4-\Lambda}{1/2}\Phi(5/4-\Lambda,3/2;\,x)\Big]\,V_1(y)$$

$$+vU_1(x)e^{y/2}\Big[\frac{1}{2}\Phi(1/4+\Lambda,1/2;\,y) + \frac{1/4+\Lambda}{1/2}\Phi(5/4+\Lambda,3/2;\,y)\Big]\Big\}\,.$$

Consider the variant h_{12}:

$$f_{12} = A(-iu\frac{\partial}{\partial x} - v\frac{\partial}{\partial y})\, U_1(x)V_2(y)$$

$$= A\Big\{ -iue^{x/2}\Big[\frac{1}{2}\Phi(1/4-\Lambda,1/2;\,x) + \frac{1/4-\Lambda}{1/2}\Phi(5/4-\Lambda,3/2;\,x)\Big]\,V_2(y)$$

$$-vU_1(x)e^{y/2}\sqrt{y}\Big[\frac{1}{2}\Phi(3/4+\Lambda,3/2;y) + \frac{1}{2y}\Phi(3/4+\Lambda,3/2;y)$$

$$+\frac{3/4+\Lambda}{3/2}\Phi(7/4+\Lambda,5/2;y)\Big]\Big\}\,,$$

$$g_{12} = A(-iu\frac{\partial}{\partial x} + v\;\frac{\partial}{\partial y})\, U_1(x)V_2(y)$$

$$= B\left\{-iue^{x/2}\left[\frac{1}{2}\Phi(1/4-\Lambda,1/2;\,x)+\frac{1/4-\Lambda}{1/2}\Phi(5/4-\Lambda,3/2;\,x)\right]V_2(y)\right.$$

$$+vU_1(x)e^{y/2}\sqrt{y}\left[\frac{1}{2}\Phi(3/4+\Lambda,3/2;y)+\frac{1}{2y}\Phi(3/4+\Lambda,3/2;y)\right.$$

$$\left.\left.+\frac{3/4+\Lambda}{3/2}\Phi(7/4+\Lambda,5/2;y)\right]\right\}.$$

Consider the variant h_{21}:

$$f_{21} = A(-iu\frac{\partial}{\partial x}-v\frac{\partial}{\partial y})\,U_2(x)V_1(y)$$

$$= A\left\{-iue^{x/2}\sqrt{x}\left[\frac{1}{2}\Phi(3/4-\Lambda,3/2;x)+\frac{1}{2x}\Phi(3/4-\Lambda,3/2;x)\right.\right.$$

$$\left.+\frac{3/4-\Lambda}{3/2}\Phi(7/4+\Lambda,5/2;x)\right]V_1(y)$$

$$\left.-vU_2(x)e^{y/2}\left[\frac{1}{2}\Phi(1/4+\Lambda,1/2;\,y)+\frac{1/4+\Lambda}{1/2}\Phi(5/4+\Lambda,3/2;\,y)\right]\right\},$$

$$g_{21} = B(-iu\;\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})\,U_2(x)V_1(y)$$

$$= A\left\{-iue^{x/2}\sqrt{x}\left[\frac{1}{2}\Phi(3/4-\Lambda,3/2;x)+\frac{1}{2x}\Phi(3/4-\Lambda,3/2;x)\right.\right.$$

$$\left.+\frac{3/4-\Lambda}{3/2}\Phi(7/4+\Lambda,5/2;x)\right]V_1(y)$$

$$\left.+vU_2(x)e^{y/2}\left[\frac{1}{2}\Phi(1/4+\Lambda,1/2;\,y)+\frac{1/4+\Lambda}{1/2}\Phi(5/4+\Lambda,3/2;\,y)\right]\right\}.$$

Consider the variant h_{22}:

$$f_{22} = A(-iu\frac{\partial}{\partial x}-v\frac{\partial}{\partial y})\,U_2(x)V_2(y)$$

$$= A\left\{-iue^{x/2}\sqrt{x}\left[\frac{1}{2}\Phi(3/4-\Lambda,3/2;x)+\frac{1}{2x}\Phi(3/4-\Lambda,3/2;x)\right.\right.$$

$$\left.+\frac{3/4-\Lambda}{3/2}\Phi(7/4+\Lambda,5/2;x)\right]V_2(y)$$

$$-vU_2(x)e^{y/2}\sqrt{y}\left[\frac{1}{2}\Phi(3/4+\Lambda,3/2;y)+\frac{1}{2y}\Phi(3/4+\Lambda,3/2;y)\right.$$

$$\left.\left.+\frac{3/4+\Lambda}{3/2}\Phi(7/4+\Lambda,5/2;y)\right]\right\},$$

$$g_{22} = A(-iu\frac{\partial}{\partial x}+v\frac{\partial}{\partial y})\,U_2(x)V_2(y)$$

$$
= A \Bigg\{ -iue^{x/2}\sqrt{x} \left[\frac{1}{2}\Phi(3/4-\Lambda,3/2;x) + \frac{1}{2x}\Phi(3/4-\Lambda,3/2;x) \right.
$$

$$
\left. + \frac{3/4-\Lambda}{3/2}\Phi(7/4+\Lambda,5/2;x) \right] V_2(y)
$$

$$
+ v \cdot U_2(x) e^{y/2}\sqrt{y} \left[\frac{1}{2}\Phi(3/4+\Lambda,3/2;y) + \frac{1}{2y}\Phi(3/4+\Lambda,3/2;y) \right.
$$

$$
\left. \left. + \frac{3/4+\Lambda}{3/2}\Phi(7/4+\Lambda,5/2;y) \right] \right\} .
$$

Recall the formulas which introduce the parabolic cylindric coordinates

$$
x = \frac{u^2 - v^2}{2} , \quad y = u\, v , \quad z = z . \tag{3.9}
$$

In order to get the inverse transformation, from the first equation (3.9) we exclude the variable u, and then the variable v:

$$
2x = -v^2 + \frac{1}{v^2}\, y^2 , \; 2x = +u^2 - \frac{1}{u^2}\, y^2 \quad \Longrightarrow \quad v^4 + 2x\, v^2 - y^2 = 0 , \; u^4 - 2x\, u^2 - y^2 = 0 .
$$

We further find

$$
v^2 = -x + \sqrt{x^2+y^2} , \quad u^2 = +x + \sqrt{x^2+y^2} . \tag{3.10}
$$

For the parametrization of the ordinary Cartesian space (x, y, z), it suffices to choose any of these four possibilities:

$$
\begin{aligned}
v &= +\sqrt{-x + \sqrt{x^2+y^2}} , \quad u = \pm\sqrt{+x + \sqrt{x^2+y^2}} , \\
v &= -\sqrt{-x + \sqrt{x^2+y^2}} , \quad u = \pm\sqrt{+x + \sqrt{x^2+y^2}} , \\
v &= \pm\sqrt{-x + \sqrt{x^2+y^2}} , \quad u = +\sqrt{+x + \sqrt{x^2+y^2}} , \\
v &= \pm\sqrt{-x + \sqrt{x^2+y^2}} , \quad u = -\sqrt{+x + \sqrt{x^2+y^2}} .
\end{aligned} \tag{3.11}
$$

For definiteness, let us use the first one:

$$
v = +\sqrt{-x + \sqrt{x^2+y^2}} , \quad u = \pm\sqrt{+x + \sqrt{x^2+y^2}} . \tag{3.12}
$$

The correspondence between (x, y) and (u, v) is clarified by the formulas

$$
\begin{aligned}
u = k\ \cos\phi , \quad v = k\ \sin\phi , \quad \phi \in [\ 0,\ \pi\ ] ; \\
x = (k^2/2)\ \cos 2\phi , \quad y = (k^2/2)\ \sin 2\phi , \quad 2\phi \in [0,\ 2\pi] ,
\end{aligned} \tag{3.13}
$$

and by Figures 1 and 2. In fact, there exists a peculiarity in parameterizing the half-line $x > 0$ by the coordinates $(u, v = 0)$ (see Fig. 3).

If one takes in mind the space models with spinor structure (a relevant motivation and references see in [4]–[7]), then the symmetry between coordinates u and v becomes complete: they relate to Cartesian coordinates $(x, y, z) \oplus (x, y, z)$ by the formulas

$$
v = \pm\sqrt{-x + \sqrt{x^2+y^2}} , \quad u = \pm\sqrt{+x + \sqrt{x^2+y^2}} . \tag{3.14}
$$

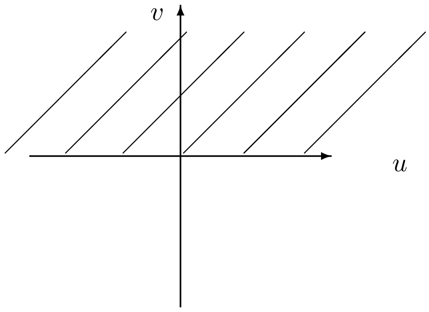

Figure 1: The domain $G(u,v)$ for vector space

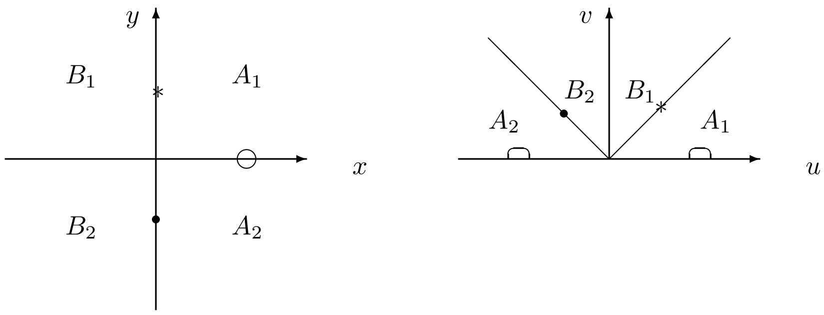

Figure 2: The map $G(x,y) \Longrightarrow G(u,v)$

To the spinor space model there corresponds the domain $\tilde{G}(u,v)$, symmetrical with respect to coordinates u and v, and without any special identification rules for the boundary points in the plane (u,v).

This is an important matter, since the expected solutions must be single-valued in the whole space. Evidently, in different space models, we should expect different single-valued solutions.

To clarify this point it suffices to follow the behavior of constructed solutions (see (3.3) when $v \to 0$:

$$v \to 0, \quad \xi_0 = \left| \begin{array}{cc} \frac{(u-iv)}{\sqrt{u^2+v^2}} f & h \\ h & \frac{(u+iv)}{\sqrt{u^2+v^2}} g \end{array} \right| \to \left| \begin{array}{cc} \frac{u}{\sqrt{u^2}} f & h \\ h & \frac{u}{\sqrt{u^2}} g \end{array} \right| . \tag{3.15}$$

In the vector space model, the following identity must hold

$$\xi_0(u, v=0) = \xi_0(-u, v=0)\,; \tag{3.16}$$

however in the spinor space model, this requirement may be ignored.

Allowing for the above expressions the relations for h_{ij}, f_{ij}, g_{ij}, we find the following behavior of the four solutions at $v \to 0$ (the plus (+) or minus (-) means

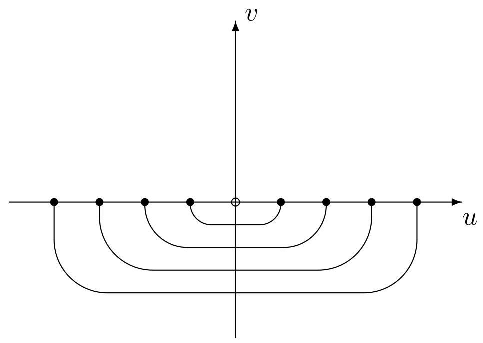

Figure 3: Identification for boundary points in $G(u, v)$

appropriateness and non-appropriateness of functions as single-valued in vector space model; see (3.15)):

$$\begin{array}{llll} f_{11} \sim u \cdot \text{function}(u^2), & g_{11} \sim u \cdot \text{function}(u^2), & (+); \\ f_{12} \sim \text{function}(u^2), & g_{12} \sim \text{function}(u^2), & (-); \\ f_{21} \sim u^2 \cdot \text{function}(u^2), & g_{21} \sim u^2 \cdot \text{function}(u^2), & (-); \\ f_{22} \sim u \cdot \text{function}(u^2), & g_{11} \sim u \cdot \text{function}(u^2) & (+). \end{array} \tag{3.17}$$

Similarly, from (3.8), it follows

$$\begin{array}{ll} h_{11} = U_1(x)V_1(y) \sim \text{function}(u^2), & (+); \\ h_{12} = U_1(x)V_2(y) \sim 0\,, & (+,-); \\ h_{21} = U_2(x)V_1(y) \sim u \cdot \text{function}(u^2), & (-); \\ h_{22} = U_1(x)V_2(y) \sim 0\,, & (+,-). \end{array} \tag{3.18}$$

Thus, we conclude that in vector space we have single valued solutions only for the variants (11) and (22); in spinor space all the four variants (11), (12), (21), (22) provide us with single-valued solutions.

4 The helicity operator

First, let us solve a subsidiary task: to find consequences of diagonalization of the helicity operator for the electromagnetic spinor (related to plane waves)

$$\Sigma = -\frac{i}{2}\,[\partial_1(\sigma_1 \otimes I + I \otimes \sigma_1) + \partial_2(\sigma_2 \otimes I + I \otimes \sigma_2) + \partial_3(\sigma_3 \otimes I + I \otimes \sigma_3)]$$

$$= -\frac{i}{2}\,[\,\partial_1\Sigma_1 + \partial_2\Sigma_2 + \partial_3\Sigma_3\,]\,. \tag{4.1}$$

Taking in mind the substitution for ξ:

$$\xi = e^{-i\omega t} e^{ik_1 x} e^{ik_2 y} e^{ik_3 z} \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| ,$$

we have the following expression for Σ:

$$\Sigma = \frac{1}{2}[k_1(\sigma_1\xi + \xi\tilde{\sigma}_1) + (\sigma_2\xi + \xi\tilde{\sigma}_2) + k_2 + k_3(\sigma_3\xi + \xi\tilde{\sigma}_3)]$$

$$= \frac{1}{2}\left\{ k_1 \left| \begin{array}{cc} 2h & f+g \\ f+g & 2h \end{array} \right| + k_2 \left| \begin{array}{cc} -2ih & i(f-g) \\ i(f-g) & 2ih \end{array} \right| + k_3 \left| \begin{array}{cc} 2f & 0 \\ 0 & -2g \end{array} \right| \right\} .$$

Therefore, from the eigenvalue equation $\Sigma\,\xi = \sigma\,\xi$, we get four equations

$$\begin{array}{ll} 11 & (k_1 - ik_2)\;h + (k_3-\sigma)\;f = 0\,, \\ 22 & (k_1 + ik_2)\;h - (k_3+\sigma)\;g = 0\,, \\ 12 & \frac{1}{2}(k_1+ik_2)\;f + \frac{1}{2}(k_1-ik_2)\;g - \sigma\;h = 0\,, \\ 21 & \frac{1}{2}(k_1+ik_2)\;f + \frac{1}{2}(k_1-ik_2)\;g - \sigma\;h = 0\,; \end{array} \qquad (4.2)$$

where the last two equations coincide. In (4.2), we have the system of three equations

$$\left| \begin{array}{ccc} (k_3-\sigma) & 0 & (k_1-ik_2) \\ 0 & -(k_3+\sigma) & (k_1+ik_2) \\ \frac{1}{2}(k_1+ik_2) & \frac{1}{2}(k_1-ik_2) & -\sigma \end{array} \right| \left| \begin{array}{c} f \\ g \\ h \end{array} \right| = 0\,. \qquad (4.3)$$

For the eigenvalues σ we have a cubic equation

$$(k_3-\sigma)(k_3+\sigma)\sigma + \frac{1}{2}(k_1+ik_2)(k_1-ik_2)(k_3+\sigma) - \frac{1}{2}(k_1+ik_2)(k_1-ik_2)(k_3-\sigma) = 0$$

or $(k_1^2+k_2^2+k_3^2-\sigma^2) = 0$, so we obtain three roots

$$\sigma = 0, +k, -k; \;\; k = \sqrt{k_1^2+k_2^2+k_3^2} = \omega\,. \qquad (4.4)$$

Let us find the solutions of the system (4.3) at $\sigma = 0$:

$$\left| \begin{array}{ccc} k_3 & 0 & (k_1-ik_2) \\ 0 & -k_3 & (k_1+ik_2) \\ \frac{1}{2}(k_1+ik_2) & \frac{1}{2}(k_1-ik_2) & 0 \end{array} \right| \left| \begin{array}{c} f \\ g \\ h \end{array} \right| = 0\,,$$

or

$$k_3 f + (k_1-ik_2)h = 0\,,\; -k_3 g + (k_1+ik_2)h = 0,\; (k_1+ik_2)f + (k_1-ik_2)g = 0$$

as first two equations, the third one being an identity $0 \equiv 0$; therefore we get

$$\sigma = 0, \quad f = -\frac{k_1-ik_2}{k_3}h\,, \quad g = +\frac{k_1+ik_2}{k_3}h\,. \qquad (4.5)$$

Let us find the solutions of the system (4.3) at $\sigma = -k$:

$$\left| \begin{array}{ccc} (k_3+k) & 0 & (k_1-ik_2) \\ 0 & -(k_3-k) & (k_1+ik_2) \\ \frac{1}{2}(k_1+ik_2) & \frac{1}{2}(k_1-ik_2) & k \end{array} \right| \left| \begin{array}{c} f \\ g \\ h \end{array} \right| = 0\,,$$

that is

$$(k_3+k)f + (k_1-ik_2)h = 0\,, -(k_3-k)g + (k_1+ik_2)h = 0\,,$$
$$\frac{1}{2}(k_1+ik_2)f + \frac{1}{2}(k_1-ik_2)g + kh = 0\,;$$

with the first two equations, the third one turns out to be an identity $0 \equiv 0$; therefore we get

$$\sigma = -k, \qquad f = -\frac{k_1 - ik_2}{k_3 + k} h, \quad g = +\frac{k_1 + ik_2}{k_3 - k} h\,. \tag{4.6}$$

Let us find the solutions of the system (4.3) at $\sigma = +k$:

$$\left| \begin{array}{ccc} (k_3 - k) & 0 & (k_1 - ik_2) \\ 0 & -(k_3 + k) & (k_1 + ik_2) \\ \frac{1}{2}(k_1 + ik_2) & \frac{1}{2}(k_1 - ik_2) & -k \end{array} \right| \left| \begin{array}{c} f \\ g \\ h \end{array} \right| = 0\,,$$

or

$$(k_3 - k)f + (k_1 - ik_2)h = 0\,, \quad -(k_3 + k)g + (k_1 + ik_2)h = 0\,,$$
$$\frac{1}{2}(k_1 + ik_2)f + \frac{1}{2}(k_1 - ik_2)g - kh = 0\,;$$

with the first two equations, the third one becomes an identity, so we arrive at

$$\sigma = +k, \qquad f = -\frac{k_1 - ik_2}{k_3 - k} h, \quad g = +\frac{k_1 + ik_2}{k_3 + k} h\,. \tag{4.7}$$

Now we are to compare (4.5)–(4.7) with the solutions of the Maxwell equations in spinor form

$$\begin{array}{l} \left\{ \begin{array}{l} (k_3 - \omega)f + (k_1 - ik_2)h = 0\,, \\ (k_1 + ik_2)f - (k_3 + \omega)h = 0\,, \end{array} \right. \qquad \omega = +k, \quad f = -\dfrac{k_1 - ik_2}{k_3 - k} h\,; \\ \left\{ \begin{array}{l} (k_1 - ik_2)g + (k_3 - \omega)h = 0\,, \\ -(k_3 + \omega)g + (k_1 + ik_2)h = 0\,, \end{array} \right. \qquad \omega = +k, \quad g = +\dfrac{k_1 + ik_2}{k_3 + k} h\,. \end{array} \tag{4.8}$$

We conclude that the solution of the spinor Maxwell equations is the eigenstate with $\sigma = +1$.

Now, let us consider the similar problem in cylindric parabolic coordinates. First, we transform the helicity operator Σ (see (4.1)) to cylindric parabolic coordinates and tetrad. From the eigenstate equation in Cartesian basis $-\frac{i}{2}[\partial_1 \Sigma_1 + \partial_2 \Sigma_2 + \partial_z \Sigma_3]\xi_0 = \sigma \xi_0$ it follows (see (3.1) and (3.2))

$$\begin{array}{l} -\dfrac{i}{2} \Big[S\Sigma_1 S^{-1} (\dfrac{\partial}{\partial x^1} + S \dfrac{\partial S^{-1}}{\partial x^1}) + S\Sigma_2 S^{-1} (\dfrac{\partial}{\partial x^2} + S \dfrac{\partial S^{-1}}{\partial x^2}) \\ \qquad\qquad + S\Sigma_3 S^{-1} (\dfrac{\partial}{\partial x^2} + S \dfrac{\partial S^{-1}}{\partial x^3}) \Big] = \sigma \xi\,, \end{array} \tag{4.9}$$

the matrix S depends only on (x^1, x^2) (or (u, v)). There exist the identities

$$S\Sigma_1 S^{-1} = (s \otimes s)[\sigma_1 \otimes I + I \otimes \sigma_1](s^{-1} \otimes s^{-1}) = s\sigma_1 s^{-1} \otimes I + I \otimes s\sigma_1 s^{-1}\,,$$

$$S\Sigma_2 S^{-1} = (s \otimes s)[\sigma_2 \otimes I + I \otimes \sigma_2](s^{-1} \otimes s^{-1}) = s\sigma_2 s^{-1} \otimes I + I \otimes s\sigma_2 s^{-1}\,,$$

$$S\Sigma_3 S^{-1} = (s \otimes s)[\sigma_3 \otimes I + I \otimes \sigma_3](s^{-1} \otimes s^{-1}) = s\sigma_3 s^{-1} \otimes I + I \otimes s\sigma_3 s^{-1}\,,$$

and also the identities

$$S\frac{\partial S^{-1}}{\partial x^1} = (s\otimes s)\frac{\partial (s^{-1}\otimes s^{-1})}{\partial x^1} = s\frac{\partial s^{-1}}{\partial x^1}\otimes I + I\otimes s\frac{\partial s^{-1}}{\partial x^1}\,,$$

$$S\frac{\partial S^{-1}}{\partial x^2} = (s\otimes s)\frac{\partial (s^{-1}\otimes s^{-1})}{\partial x^2} = s\frac{\partial s^{-1}}{\partial x^2}\otimes I + I\otimes s\frac{\partial s^{-1}}{\partial x^2}\,,\quad S\frac{\partial S^{-1}}{\partial x^3} = 0\,.$$

Taking these in mind, we obtain (see (4.9))

$$S\Sigma_1 S^{-1}(\frac{\partial}{\partial x^1} + S\frac{\partial S^{-1}}{\partial x^1})$$

$$= \left(s\sigma_1 s^{-1}\otimes I + I\otimes s\sigma_1 s^{-1}\right)\left(\frac{\partial}{\partial x^1} + s\frac{\partial s^{-1}}{\partial x^1}\otimes I + I\otimes s\frac{\partial s^{-1}}{\partial x^1}\right),$$

that is

$$S\Sigma_1 S^{-1}(\frac{\partial}{\partial x^1} + S\frac{\partial S^{-1}}{\partial x^1}) = (s\sigma_1 s^{-1}\otimes I + I\otimes s\sigma_1 s^{-1})\,\frac{\partial}{\partial x^1}$$

$$+s\sigma_1\frac{\partial s^{-1}}{\partial x^1}\otimes I + I\otimes s\sigma_1\frac{\partial s^{-1}}{\partial x^1} + s\sigma_1 s^{-1}\otimes s\frac{\partial s^{-1}}{\partial x^1} + s\frac{\partial s^{-1}}{\partial x^1}\otimes s\sigma_1 s^{-1}\,.$$

Similarly, we derive

$$S\Sigma_2 S^{-1}(\frac{\partial}{\partial x^2} + S\frac{\partial S^{-1}}{\partial x^2}) = (s\sigma_2 s^{-1}\otimes I + I\otimes s\sigma_2 s^{-1})\,\frac{\partial}{\partial x^2}$$

$$+s\sigma_2\frac{\partial s^{-1}}{\partial x^2}\otimes I + I\otimes s\sigma_2\frac{\partial s^{-1}}{\partial x^2} + s\sigma_2 s^{-1}\otimes s\frac{\partial s^{-1}}{\partial x^2} + s\frac{\partial s^{-1}}{\partial x^2}\otimes s\sigma_2 s^{-1}\,,$$

and

$$S\Sigma_3 S^{-1}(\frac{\partial}{\partial x^3} + S\frac{\partial S^{-1}}{\partial x^3}) = S\Sigma_3 S^{-1}\frac{\partial}{\partial x^3} = (s\sigma_3 s^{-1}\otimes I + I\otimes s\sigma_3 s_3)\frac{\partial}{\partial x^3}\,.$$

Let us write down a general structure for the operator Σ:

$$\Sigma = -\frac{i}{2}\left\{\left(s\sigma_1 s^{-1}\otimes I + I\otimes s\sigma_1 s^{-1}\right)\,\frac{\partial}{\partial x^1} + \left(s\sigma_2 s^{-1}\otimes I + I\otimes s\sigma_2 s^{-1}\right)\,\frac{\partial}{\partial x^2}\right.$$

$$+\left(s\sigma_3 s^{-1}\otimes I + I\otimes s\sigma_3 s_3\right)\,\frac{\partial}{\partial x^3} + \left(s\sigma_1 s^{-1}s\frac{\partial s^{-1}}{\partial x^1}\otimes I + I\otimes s\sigma_1 s^{-1}s\frac{\partial s^{-1}}{\partial x^1}\right)$$

$$+\left(s\sigma_2 s^{-1}s\frac{\partial s^{-1}}{\partial x^2}\otimes I + s\sigma_2 s^{-1}s\frac{\partial s^{-1}}{\partial x^2}\otimes I\right)$$

$$\left.+\left(s\sigma_1 s^{-1}\otimes s\frac{\partial s^{-1}}{\partial x^1} + s\frac{\partial s^{-1}}{\partial x^1}\otimes s\sigma_1 s^{-1}\right) + \left(s\sigma_2 s^{-1}\otimes s\frac{\partial s^{-1}}{\partial x^2} + s\frac{\partial s^{-1}}{\partial x^2}\otimes s\sigma_2 s^{-1}\right)\right\}.$$

We readily find the formulas

$$s\sigma_3 s^{-1} = \sigma_3\,,$$

$$s\sigma_1 s^{-1} = \frac{1}{\sqrt{u^2+v^2}}\begin{vmatrix} 0 & u+iv \\ u-iv & 0 \end{vmatrix} = \frac{1}{\sqrt{u^2+v^2}}(u\sigma_1 - v\sigma_2)\,, \tag{4.10}$$

$$s\sigma_2 s^{-1} = \frac{1}{\sqrt{u^2+v^2}}\begin{vmatrix} 0 & v-iu \\ v+iu & 0 \end{vmatrix} = \frac{1}{\sqrt{u^2+v^2}}(v\sigma_1 + u\sigma_2)\,.$$

Taking in mind the relations

$$\frac{\partial u}{\partial x^1} = \frac{u}{u^2+v^2}\,, \qquad \frac{\partial v}{\partial x^1} = \frac{-v}{u^2+v^2}\,, \qquad \frac{\partial u}{\partial x^2} = \frac{v}{u^2+v^2}\,, \qquad \frac{\partial v}{\partial x^2} = \frac{u}{u^2+v^2}\,,$$

we derive the formulas

$$\frac{\partial}{\partial x^1} = \frac{1}{u^2+v^2}(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v})\,, \qquad \frac{\partial}{\partial x^2} = \frac{1}{u^2+v^2}(v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v})\,. \tag{4.11}$$

Let us consider the term

$$(s\sigma_3 s^{-1}\otimes I + I\otimes s\sigma_3 s_3)\frac{\partial}{\partial x^3} + [s\sigma_1 s^{-1}\otimes I + I\otimes s\sigma_1 s^{-1}]\;\frac{\partial}{\partial x^1}$$

$$+[s\sigma_2 s^{-1}\otimes I + I\otimes s\sigma_2 s^{-1}]\;\frac{\partial}{\partial x^2} = (\sigma_3\otimes + I\otimes\sigma_3)\frac{\partial}{\partial x^3} + \frac{1}{\sqrt{u^2+v^2}}\frac{1}{u^2+v^2}$$

$$\times\Big\{[(u\sigma_1 - v\sigma_2)\otimes I + I\otimes(u\sigma_1 - v\sigma_2)](u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v})$$

$$+[(v\sigma_1 + u\sigma_2)\otimes I + I\otimes(v\sigma_1 + u\sigma_2)](v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v})\Big\}$$

$$= (\sigma_3\otimes + I\otimes\sigma_3)\frac{\partial}{\partial x^3} + \frac{1}{\sqrt{u^2+v^2}}\left[(\sigma_1\frac{\partial}{\partial u} + \sigma_2\frac{\partial}{\partial v})\otimes I + I\otimes(\sigma_1\frac{\partial}{\partial u} + \sigma_2\frac{\partial}{\partial v})\right]\,.$$

We further consider the term

$$s\frac{\partial s^{-1}}{\partial x^1} = \frac{1}{u^2+v^2}s\left\{u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right\}s^{-1}\frac{1}{u^2+v^2}\frac{1}{(u^2+v^2)^{1/4}}\left|\begin{array}{cc}\sqrt{u+iv} & 0\\ 0 & \sqrt{u-iv}\end{array}\right|$$

$$\times\left\{-\frac{u^2}{2(u^2+v^2)(u^2+v^2)^{1/4}}\left|\begin{array}{cc}\sqrt{u-iv} & 0\\ 0 & \sqrt{u+iv}\end{array}\right| + \frac{1}{2(u^2+v^2)^{1/4}}\left|\begin{array}{cc}\frac{u}{\sqrt{u-iv}} & 0\\ 0 & \frac{u}{\sqrt{u+iv}}\end{array}\right|\right.$$

$$\left.+\frac{v^2}{2(u^2+v^2)(u^2+v^2)^{1/4}}\left|\begin{array}{cc}\sqrt{u-iv} & 0\\ 0 & \sqrt{u+iv}\end{array}\right| + \frac{1}{2(u^2+v^2)^{1/4}}\left|\begin{array}{cc}\frac{iv}{\sqrt{u-iv}} & 0\\ 0 & \frac{-iv}{\sqrt{u+iv}}\end{array}\right|\right\}$$

$$= \frac{1}{2}\frac{1}{u^2+v^2}\frac{1}{(u^2+v^2)^{1/2}}\left|\begin{array}{cc}\sqrt{u+iv} & 0\\ 0 & \sqrt{u-iv}\end{array}\right|$$

$$\times\left\{-\frac{u^2}{(u^2+v^2)}\left|\begin{array}{cc}\sqrt{u-iv} & 0\\ 0 & \sqrt{u+iv}\end{array}\right| + \left|\begin{array}{cc}\frac{u}{\sqrt{u-iv}} & 0\\ 0 & \frac{u}{\sqrt{u+iv}}\end{array}\right|\right.$$

$$\left.+\frac{v^2}{(u^2+v^2)}\left|\begin{array}{cc}\sqrt{u-iv} & 0\\ 0 & \sqrt{u+iv}\end{array}\right| + \left|\begin{array}{cc}\frac{iv}{\sqrt{u-iv}} & 0\\ 0 & \frac{-iv}{\sqrt{u+iv}}\end{array}\right|\right\}$$

$$= \frac{1}{2}\frac{v^2-u^2}{(u^2+v^2)^2} + \frac{1}{2}\frac{\sqrt{u+iv}\sqrt{u-iv}}{(u^2+v^2)^2}\left|\begin{array}{cc}\frac{(u+iv)\sqrt{u+iv}}{\sqrt{u-iv}} & 0\\ 0 & \frac{(u-iv)\sqrt{u-iv}}{\sqrt{u+iv}}\end{array}\right|$$

$$= \frac{1}{2}\frac{v^2-u^2}{(u^2+v^2)^2} + \frac{1}{2}\frac{1}{(u^2+v^2)^2}\left|\begin{array}{cc}(u+iv)^2 & 0\\ 0 & (u-iv)^2\end{array}\right|,$$

so, arriving at a simple result

$$s\frac{\partial s^{-1}}{\partial x^1} = \frac{2i\,uv}{2(u^2+v^2)^2}\left| \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right| = \frac{2i\,uv}{2(u^2+v^2)^2}\,\sigma_3\,. \tag{4.12}$$

Similarly, we consider the term

$$s\frac{\partial s^{-1}}{\partial x^2} = \frac{1}{u^2+v^2} s \left\{ v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} \right\} s^{-1} = \frac{1}{u^2+v^2}\frac{1}{(u^2+v^2)^{1/4}} \left| \begin{array}{cc} \sqrt{u+iv} & 0 \\ 0 & \sqrt{u-iv} \end{array} \right|$$

$$\times \left\{ -\frac{uv}{2(u^2+v^2)(u^2+v^2)^{1/4}} \left| \begin{array}{cc} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{array} \right| + \frac{1}{2(u^2+v^2)^{1/4}} \left| \begin{array}{cc} \frac{v}{\sqrt{u-iv}} & 0 \\ 0 & \frac{v}{\sqrt{u+iv}} \end{array} \right| \right.$$

$$\left. -\frac{uv}{2(u^2+v^2)(u^2+v^2)^{1/4}} \left| \begin{array}{cc} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{array} \right| + \frac{1}{2(u^2+v^2)^{1/4}} \left| \begin{array}{cc} \frac{-iu}{\sqrt{u-iv}} & 0 \\ 0 & \frac{+iu}{\sqrt{u+iv}} \end{array} \right| \right\}$$

$$= \frac{1}{2}\frac{1}{u^2+v^2}\frac{1}{(u^2+v^2)^{1/2}} \left| \begin{array}{cc} \sqrt{u+iv} & 0 \\ 0 & \sqrt{u-iv} \end{array} \right|$$

$$\times \left\{ -\frac{uv}{(u^2+v^2)} \left| \begin{array}{cc} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{array} \right| + \left| \begin{array}{cc} \frac{v}{\sqrt{u-iv}} & 0 \\ 0 & \frac{v}{\sqrt{u+iv}} \end{array} \right| \right.$$

$$\left. -\frac{uv}{(u^2+v^2)} \left| \begin{array}{cc} \sqrt{u-iv} & 0 \\ 0 & \sqrt{u+iv} \end{array} \right| + \left| \begin{array}{cc} \frac{-iu}{\sqrt{u-iv}} & 0 \\ 0 & \frac{+iu}{\sqrt{u+iv}} \end{array} \right| \right\}$$

$$= -\frac{1}{2}\frac{2uv}{(u^2+v^2)^2} + \frac{1}{2}\frac{\sqrt{u+iv}\sqrt{u-iv}}{(u^2+v^2)^2} \left| \begin{array}{cc} \frac{(v-iu)\sqrt{u+iv}}{\sqrt{u-iv}} & 0 \\ 0 & \frac{(v+iu)\sqrt{u-iv}}{\sqrt{u+iv}} \end{array} \right|$$

$$= -\frac{1}{2}\frac{2uv}{(u^2+v^2)^2} + \frac{1}{2}\frac{1}{(u^2+v^2)^2} \left| \begin{array}{cc} (u+iv)(v-iu) & 0 \\ 0 & (u-iv)(v+iu) \end{array} \right|,$$

so that

$$s\frac{\partial s^{-1}}{\partial x^2} = \frac{i\,(v^2-u^2)}{2(u^2+v^2)^2}\left| \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right| = \frac{i\,(v^2-u^2)}{2(u^2+v^2)^2}\,\sigma_3\,. \tag{4.13}$$

Taking into account the formulas (4.10)–(4.13, we transform the above expression for Σ to the following form

$$\Sigma = -\frac{i}{2}\left\{ (\sigma_3 \otimes + I \otimes \sigma_3)\frac{\partial}{\partial x^3} + \frac{1}{\sqrt{u^2+v^2}}\left[(\sigma_1\frac{\partial}{\partial u} + \sigma_2\frac{\partial}{\partial v}) \otimes I + I \otimes (\sigma_1\frac{\partial}{\partial u} + \sigma_2\frac{\partial}{\partial v}) \right] \right.$$

$$+\frac{1}{\sqrt{u^2+v^2}}\frac{2i\,uv}{2(u^2+v^2)^2}\ [(u\sigma_1 - v\sigma_2)\sigma_3 \otimes I + I \otimes (u\sigma_1 - v\sigma_2)\sigma_3]$$

$$+\frac{1}{\sqrt{u^2+v^2}}\frac{i\,(v^2-u^2)}{2(u^2+v^2)^2}\,[(v\sigma_1 + u\sigma_2)\,\sigma_3 \times I + I \otimes (v\sigma_1 + u\sigma_2)\,\sigma_3]$$

$$+\frac{1}{\sqrt{u^2+v^2}}\frac{2i\,uv}{2(u^2+v^2)^2}\,[(u\sigma_1 - v\sigma_2) \otimes \sigma_3 + \sigma_3 \otimes (u\sigma_1 - v\sigma_2)]$$

$$\left. +\frac{1}{\sqrt{u^2+v^2}}\frac{i\,(v^2-u^2)}{2(u^2+v^2)^2}\,[(v\sigma_1 + u\sigma_2) \otimes\ \sigma_3 + \sigma_3 \otimes (v\sigma_1 + u\sigma_2)] \right\}.$$

This may be re-written differently,

$$\Sigma = -\frac{i}{2}\Big\{(\sigma_3 \otimes + I \otimes \sigma_3)\frac{\partial}{\partial x^3} + \frac{1}{\sqrt{u^2+v^2}}\Big\{(\sigma_1\partial_u + \sigma_2\frac{\partial}{\partial v}) \otimes I + I \otimes (\sigma_1\partial_u + \sigma_2\frac{\partial}{\partial v})$$

$$+\frac{2uv}{2(u^2+v^2)^2}\,[(v\sigma_1+u\sigma_2)\otimes I + I\otimes(v\sigma_1+u\sigma_2)+(u\sigma_1-v\sigma_2)\otimes i\sigma_3 + i\sigma_3\otimes(u\sigma_1-v\sigma_2)]$$

$$+\frac{v^2-u^2}{2(u^2+v^2)^2}[(-u\sigma_1+v\sigma_2)\otimes I + I\otimes(-u\sigma_1+v\sigma_2) + +(v\sigma_1+u\sigma_2)\otimes i\sigma_3 + i\sigma_3\otimes(v\sigma_1+u\sigma_2)]\ \}\ \}\,.$$

Thus, the eigenvalue equation $\Sigma\xi = \sigma\xi$ leads to

$$\{(\sigma_1\partial_u+\sigma_2\partial_v)\otimes I + I\otimes(\sigma_1\partial_u+\sigma_2\partial_v) + \frac{2uv}{2(u^2+v^2)^2}\,[(v\sigma_1+u\sigma_2)\otimes I + I\otimes(v\sigma_1+u\sigma_2)]$$

$$+\frac{v^2-u^2}{2(u^2+v^2)^2}[(-u\sigma_1+v\sigma_2)\otimes I + I\otimes(-u\sigma_1+v\sigma_2)]$$

$$+\frac{2uv}{2(u^2+v^2)^2}\,[(u\sigma_1-v\sigma_2)\otimes i\sigma_3 + i\sigma_3\otimes(u\sigma_1-v\sigma_2)]$$

$$+\frac{v^2-u^2}{2(u^2+v^2)^2}[(v\sigma_1+u\sigma_2)\otimes i\sigma_3 + i\sigma_3\otimes(v\sigma_1+u\sigma_2)]\Big\}\,\xi$$

$$= \sqrt{u^2+v^2}\,(2i\sigma - ik\sigma_3\otimes - ikI\otimes\sigma_3)\,\xi\,. \tag{4.14}$$

By regrouping the terms within the lines 2-3, and also within the lines 3-4, we obtain a simpler form for the equation:

$$[(\sigma_1\partial_u+\sigma_2\partial_v)\otimes I + I\otimes(\sigma_1\partial_u+\sigma_2\partial_v)]\xi + \frac{1}{2(u^2+v^2)}$$

$$\times\{(u\sigma_1+v\sigma_2)\otimes I + I\otimes(u\sigma_1+v\sigma_2) + (v\sigma_1-u\sigma_2)\otimes i\sigma_3 + i\sigma_3\otimes(v\sigma_1-u\sigma_2)\}\,\xi$$

$$= \sqrt{u^2+v^2}\,(2i\sigma - ik\sigma_3\otimes I - ikI\otimes\sigma_3)\,\xi, \quad \text{where} \quad h = \begin{vmatrix} f & h \\ h & g \end{vmatrix}.$$

First we calculate

$$(\sigma_1\partial_u+\sigma_2\partial_v)\otimes I \begin{vmatrix} f & h \\ h & g \end{vmatrix} = \begin{vmatrix} (\partial_u - i\partial_v)h & (\partial_u - i\partial_v)g \\ (\partial_u + i\partial_v)f & (\partial_u + i\partial_v)h \end{vmatrix},$$

and

$$I\otimes(\sigma_1\partial_u+\sigma_2\frac{\partial}{\partial v}) \begin{vmatrix} f & h \\ h & g \end{vmatrix} = \begin{vmatrix} (\partial_u - i\partial_v)h & (\partial_u + i\partial_v)f \\ (\partial_u - i\partial_v)g & (\partial_u + i\partial_v)h \end{vmatrix}.$$

Their sum equals to

$$(\sigma_1\partial_u+\sigma_2\frac{\partial}{\partial v})\otimes I \begin{vmatrix} f & h \\ h & g \end{vmatrix} + I\otimes(\sigma_1\partial_u+\sigma_2\frac{\partial}{\partial v}) \begin{vmatrix} f & h \\ h & g \end{vmatrix}$$

$$= \begin{vmatrix} 2(\partial_u - i\partial_v)h & (\partial_u + i\partial_v)f + (\partial_u - i\partial_v)g \\ (\partial_u + i\partial_v)f + (\partial_u - i\partial_v)g & 2(\partial_u + i\partial_v)h \end{vmatrix}. \tag{4.15}$$

Then we calculates the terms

$$[(u\sigma_1 + v\sigma_2) \otimes I + I \otimes (u\sigma_1 + v\sigma_2)]\xi$$

$$= \left| \begin{array}{cc} 0 & u-iv \\ u+iv & 0 \end{array} \right| \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| + \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| \left| \begin{array}{cc} 0 & u+iv \\ u-iv & 0 \end{array} \right|$$

$$= \left| \begin{array}{cc} 2(u-iv)h & (u+iv)f + (u-iv)g \\ (u+iv)f + (u-iv)g & 2(u+iv)h \end{array} \right| ,$$

and

$$[(v\sigma_1 - u\sigma_2) \otimes i\sigma_3 + i\sigma_3 \otimes (v\sigma_1 - u\sigma_2)]\xi$$

$$= \left| \begin{array}{cc} 0 & v+iu \\ v-iu & 0 \end{array} \right| \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| \left| \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right| + \left| \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right| \left| \begin{array}{cc} f & h \\ h & g \end{array} \right| \left| \begin{array}{cc} 0 & v-iu \\ v+iu & 0 \end{array} \right|$$

$$= \left| \begin{array}{cc} 2(iv-u)h & (iv+u)f + (-iv+u)g \\ (iv+u)f + (-iv+u)g & 2(-iv-u)h \end{array} \right| ;$$

their sum equals to

$$2 \left| \begin{array}{cc} 0 & (u+iv)f + (u-iv)g \\ (u+iv)f + (u-iv)g & 0 \end{array} \right| . \tag{4.16}$$

Besides, we find

$$(2i\sigma - ik\sigma_3 \otimes -ikI \otimes \sigma_3)\, \xi = 2i \left| \begin{array}{cc} (\sigma-k)f & \sigma h \\ \sigma h & (\sigma+k)g \end{array} \right| . \tag{4.17}$$

Taking into account the relations (4.15)–(4.17), we reduce the eigenvalue equation to the form

$$\left| \begin{array}{cc} 2(\partial_u - i\partial_v)h & (\partial_u + i\partial_v)f + (\partial_u - i\partial_v)g \\ (\partial_u + i\partial_v)f + (\partial_u - i\partial_v)g & 2(\partial_u + i\partial_v)h \end{array} \right|$$

$$+ \frac{1}{(u^2+v^2)^2} \left| \begin{array}{cc} 0 & (u+iv)f + (u-iv)g \\ (u+iv)f + (u-iv)g & 0 \end{array} \right|$$

$$= 2i\sqrt{u^2+v^2} \left| \begin{array}{cc} (\sigma-k)f & \sigma h \\ \sigma h & (\sigma+k)g \end{array} \right| , \tag{4.18}$$

whence we derive four equations.

First, let us consider the following ones:

$$11, \quad 2(\partial_u - i\partial_v)h = 2i\sqrt{u^2+v^2}(\sigma-k)f \, ,$$

$$22, \quad 2(\partial_u + i\partial_v)h = 2i\sqrt{u^2+v^2}(\sigma+k)g \, ,$$

or differently

$$(\partial_u - i\partial_v)h = i(\sigma-k)\bar{f} \, , \quad (\partial_u + i\partial_v)h = i(\sigma+k)\bar{g} \, ; \tag{4.19}$$

we recall the notations $\bar{f} = \sqrt{u^2+v^2} f,\ \bar{g} = \sqrt{u^2+v^2} g$. Let us introduce the new variables $\bar{f}+\bar{g} = F,\ \bar{f}-\bar{g} = G$; then, summing and subtracting the equations in (4.19), we obtain the following system of two linear equations with respect to F and G:

$$\sigma F - kG = -2i\partial_u h \qquad -kF + \sigma G = -2\partial_v h\,.$$

Its solution is

$$F = \frac{2}{\sigma^2 - k^2}[-i\sigma\partial_u - k\partial_v]\,h\,, \quad G = \frac{2}{\sigma^2-k^2}[-ik\partial_u - \sigma\partial_v]\,h\,. \tag{4.20}$$

These formulas may be compared with the similar ones resulting from the Maxwell equations (2.14):

$$F = \frac{2}{\omega^2 - k^2}[-i\omega\partial_u - k\partial_v]\,h\,, \quad G = \frac{2}{\omega^2-k^2}[-ik\partial_u - \omega\partial_v]\,h\,. \tag{4.21}$$

Relations (4.20) and (4.21) coincide when identifying σ and ω.

We consider now the two remaining equations:

$$12,\ \ (\partial_u + i\partial_v)f + (\partial_u - i\partial_v)g + \frac{1}{(u^2+v^2)}[(u+iv)f + (u-iv)g] = 2i\sqrt{u^2+v^2}\sigma h\,,$$

$$21,\ \ (\partial_u + i\partial_v)f + (\partial_u - i\partial_v)g + \frac{1}{(u^2+v^2)}[(u+iv)f + (u-iv)g] = 2i\sqrt{u^2+v^2}\sigma h\,;$$

they coincide with each other. Taking in mind the identities of the type

$$\partial_u f = \partial_u \frac{\bar{f}}{\sqrt{u^2+v^2}} = \frac{1}{\sqrt{u^2+v^2}}(\partial_u - \frac{u}{u^2+v^2})\bar{f}\,,$$

the above equation reduces to the simpler form

$$\frac{1}{\sqrt{u^2+v^2}}(\partial_u - \frac{u}{u^2+v^2})F + \frac{i}{\sqrt{u^2+v^2}}(\partial_v - \frac{v}{u^2+v^2})G\frac{1}{(u^2+v^2)}\frac{1}{\sqrt{u^2+v^2}}[uF+ivG]$$
$$= 2i\sqrt{u^2+v^2}\sigma h\,,$$

which after regrouping the terms, leads to

$$\frac{1}{u^2+v^2}(\partial_u - \frac{u}{u^2+v^2})F + \frac{i}{u^2+v^2}(\partial_v - \frac{v}{u^2+v^2})G + \frac{1}{(u^2+v^2)^2}[uF+ivG] = 2i\sigma h\,.$$

Its final form is

$$\frac{1}{u^2+v^2}\partial_u F + \frac{i}{u^2+v^2}\partial_v G = 2i\sigma h\,. \tag{4.22}$$

We may notice that the last equation (4.22) coincides with the third equation in the system (2.13)

$$-2i\omega h + \frac{1}{u^2+v^2}\partial_u F + \frac{i}{u^2+v^2}\partial_v G = 0\,,$$

having in mind the identity $\sigma = \omega$. We recall that the fourth equation in(2.13) is equivalent to the third one.

Thus, we have shown that by the diagonalization of the helicity operator for the 2-rank symmetric spinor, it follows the system of equations which coincides with the one produced by the Maxwell equation, when identifying the eigenvalue σ with the frequency ω. This fact does not depend on the choice of coordinates and tetrad, Cartesian or cylindric parabolic.

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