# Null trapping horizons

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**Abstract.** The purpose of the present work is to study (marginally) trapped submanifolds lying in a null hypersurface. Let $x : \Sigma \to M(c)$ be a null hypersurface of a space-time with constant sectional curvature $c$, endowed with a Screen Integrable and Conformal rigging $k$. The (Marginally) Trapped Submanifolds we are interested with are particular leaves of the screen distribution according to the sign of their expansions. We prove that the cross-sections of a marginally outer trapped tube are Riemann manifolds with the same constant sectional curvature $c$. In the case $M = \mathbb{R}_1^{n+2}$ and the rigging is closed with unitary conformal screen distribution, we show that if $\Sigma$ is totally umbilical with factor $\rho$ then, each (connected) leaf of the screen distribution is a Riemannian manifold with positive constant sectional curvature $\kappa = 2\rho^2$.

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## 1 Introduction and main results

### 1.1 Literature review

Let $(M, g)$ be a proper semi-Riemannian manifold and $x : \Sigma \to M$ be an embedded hypersurface of $M$. The pull-back metric $x^*g$ can be either degenerate or non-degenerate at a given point $p \in \Sigma$. When $x^*g$ is non-degenerate on $\Sigma$ one says that $(\Sigma, x^*g)$ is a semi-Riemannian hypersurface of $M$, and when the pull-back metric is degenerate on $\Sigma$, $(\Sigma, x^*g)$ is called a null (or degenerate, or lightlike) hypersurface of $M$. Since any semi-Riemannian hypersurface has a natural transversal vector field, namely the Gauss map which is everywhere orthogonal to the hypersurface, there is a standard way to study the extrinsic geometry of such a hypersurface. Geometrical objects of the ambient manifold $M$ are projected orthogonally on $\Sigma$ and give new objects which are used to study the extrinsic geometry of the hypersurface.

On the contrary, for a null hypersurface $\Sigma$, the normal bundle $T\Sigma^{\perp}$ is not transversal but rather tangent to $\Sigma$. Therefore, the orthogonal projection is not possible and

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we need other approaches to study the extrinsic geometry of null hypersurfaces. In [13], it is proven that for every choice of a complementary distribution $S(T\Sigma)$ (called a screen distribution) of $T\Sigma^{\perp}$ in $T\Sigma$, and for every choice of a null section $\xi$ of $T\Sigma^{\perp}$, there exists a unique rank one bundle $tr(T\Sigma)$ transverse to $\Sigma$ and a locally defined null section $N$ of $tr(T\Sigma)$ such that $\langle \xi, N \rangle = 1$. To study the extrinsic geometry of $\Sigma$, geometrical objects of $M$ are projected on $\Sigma$ parallely to $tr(T\Sigma)$. Two difficulties with this method that we can mention are the large number of arbitrary elections needed and the fact that induced objects are locally defined.

In [18, 14], the authors consider a vector field $\zeta$ defined on an open subset containing $\Sigma$ and everywhere transversal to $\Sigma$ (called a rigging vector field). This vector $\zeta$ fixes a unique transversal bundle and a unique screen distribution. Notice that, a rigging vector field may not exist for some null hypersurfaces. However for a spacetime (time-orientable Lorentzian manifold) $M$, there exists a timelike vector field globally defined on $M$. This timelike vector field is a rigging for any null hypersurface of $M$, since a timelike vector field cannot be tangent to a null hypersurface in a Lorentzian ambient.

Introduced by Penrose in [31], the concept of *trapped surfaces* plays an important role in general relativity. A spacelike surface $S$ is said to be a trapped surface if all light rays emitted from the surface locally converge. Nothing can escape, not even the light. It is believed that there will be a marginally trapped surface separating the trapped surfaces from the untrapped ones where the outgoing light rays are instantaneously parallel. For example in stationary spacetimes, the event horizon of a black hole is a marginally trapped surface.

Galileo's principle according to which all bodies fall equally fast is the equivalent to the Newtonian principle saying that the initial mass (the $m$ in the fundamental Newton formula $F = ma$) and the passive gravitational mass (the mass acted on by a gravitational field) are equal for a given body [12]. Hence for these two theories, gravity is a field present in the universe and which affects all bodies. In general relativity, the gravitational field is the manifestation of the curvature of the spacetime which is the consequence of the presence of the matter and no notion of an intense gravitational field can be attached to one single spacetime point: a local notion becomes necessary. A normal bundle of a spacelike surface $S$ can be spanned by two future-directed null vector fields, say $k$ and $\ell$. (We set $\ell$ to be in the outgoing direction.) Since trajectories of light are null geodesics, $S$ can be taken as an initial event for sending two pulses of light: one toward one side of the surface (say inward) and the other toward the other side (say outward). When the gravitational field is weak, the pulse of light sent outward will increase its area, while the pulse of light sent inward will have decreasing area. If the gravitational field near the surface is intense and directed inward, it is possible that the outward light geodesics may bend inward sufficiently so that the area of the light fronts decreases. This geometric fact is taken as an indicator of the presence of a strong gravitational field. Spacelike surfaces where this behavior occurs are called trapped surfaces and the ones with a behavior borderline between the "normal" situation and the strong gravitational field situation are called marginally trapped. [17, 25, 26, 29, For more physical comment on (marginally) trapped surfaces.].

## 1.2 Main results

Let's recall the following definitions.

**Definition 1.1** ([24]). • The *trapped region* of the Lorentzian manifold $M$ is an inextensible region for which each point lies in some trapped submanifold $S \subset M$. Its boundary is called a *trapping boundary*.

- A *trapping horizon* of $M$ is (the closure of) a hypersurface $\Sigma$ foliated by closed MOTSs (see Definition 2.1), called cross sections, for which the ingoing expansion $\theta^{(k)} \neq 0$ and $\delta_k \theta^{(\ell)} \neq 0$.
- A trapping horizon $\Sigma$ is said future (respectively, past) if for each MOTS leaf of $M$, there exists $k$ and $\ell$ (as above) such that $\theta^{(k)} < 0$ (respectively, $\theta^{(k)} > 0$).
- A trapping horizon $\Sigma$ is said outer (respectively, inner) if for each MOTS leaf of $\Sigma$, there exist $k$ and $\ell$ (as above) such that $\delta_k \theta^{(\ell)} < 0$ (respectively, $\delta_k \theta^{(\ell)} > 0$).

These notions of trapped surfaces are used to capture a practical idea of black hole which can be simulated numerically. More precisely, one sees locally a black hole as a Future Outer Trapping Horizon (FOTH). In such a region of the space-time, the ingoing light rays are converging ($\theta^{(k)} < 0$) and outgoing light rays are instantaneously parallel on the horizon ($\theta^{(\ell)} = 0$), and they are converging inside the horizon ($\delta_k \theta^{(\ell)} < 0$). A *Marginally Outer Trapped Tube* (MOTT) is a hypersurface foliated by MOTSs [2]. When the MOTSs foliating a MOTT are closed and the ingoing null expansion $\theta^{(k)}$ and the variation of the outgoing null expansion $\theta^{(\ell)}$ along the ingoing direction do not vanish, the MOTT becomes a trapping horizon. We prove the following result.

**Theorem 1.1.** *In a spacetime with constant sectional curvature c, cross-sections of a MOTT are Riemannian manifolds with the same constant sectional curvature c.*

From the above theorem, one deduces the following corollaries. The first one is justified by the fact that a Riemannian manifold with non-positive sectional curvature cannot be isometric to the sphere $\mathbb{S}^2$ while the second one holds as any compact Riemannian manifold is geodesically complete and then is a space form provided the sectional curvature is positive.

**Corollary 1.2.** *A space-time with constant non-positive sectional curvature cannot contain a null non-expanding horizon.*

**Corollary 1.3.** *In an Einstein's space-time with positive constant sectional curvature and where dominant energy condition holds, any null trapping horizon is a null non-expanding horizon.*

We will prove the following complete classification of null MOTTs in the case where the null hypersuface is globally the graph of a smooth function.

**Theorem 1.4.** *A Monge null hypersurface $\Sigma \to \mathbb{R}^{n+2}_1$ graph of a function $F$ is a MOTT if and only if $F$ is harmonic.*

The following theorem gives a relationship between the umbilical factor of a totally umbilical null hypersurface and the sectional curvature of the leaves of a particular foliation of this hypersurface.

**Theorem 1.5.** *Let $x : \Sigma \to \mathbb{R}^{n+2}_1$ be a null hypersurface of the Lorentz-Minkowski space endowed with a closed rigging with unitary conformal screen distribution (i.e. $\varphi = 1$). If $\Sigma$ is totally umbilical with umbilical factor $\rho$ (i.e. $B = \rho\langle,\rangle$) then, each (connected) leaf of the screen distribution is a Riemannian manifold with positive constant sectional curvature $\kappa = 2\rho^2$.*

**Example 1.2.** Let $x : \Lambda^{n+1}_0 \to \mathbb{R}^{n+2}_1$, $p = (x^1, \dots x^{n+1}) \mapsto x = (x^0 = F(x^1, \dots x^{n+1}), x^1, \dots x^{n+1})$ be the future null cone, which is the graph of the function

$$F = \left( \sum_{a=1}^{n+1} (x^a)^2 \right)^{1/2} .$$

This is a totally umbilical null hypersurface in $\mathbb{R}^{n+2}_1$ and the generic SIC-rigging (4.2) writes

$$k = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^0} - \frac{1}{x^0\sqrt{2}} \sum_{a=1}^{n+1} (x^a) \frac{\partial}{\partial x^a},$$

and

$$\ell = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^0} + \frac{1}{x^0\sqrt{2}} \sum_{a=1}^{n+1} (x^a) \frac{\partial}{\partial x^a}.$$

For this rigging, the screen distribution is integrable and the leaves are the sections of the future lightcone by the hyperplanes $x^0 = cste$. These are spheres of radius $x^0$ centered at $(x^0, 0, \dots, 0) \in \mathbb{R}^{n+2}$. (See [7] for a proof.) All the principal curvatures are given by

$$\rho = \frac{1}{x^0\sqrt{2}},$$

which are constants on each leaf of the screen distribution, and this agrees with Lemma 4.2. By Theorem 1.5, each leaf of the screen distribution has positive constant sectional curvature $\kappa = 2\rho^2 = \frac{1}{(x^0)^2}$, which is really the sectional curvature of a sphere of radius $x^0$.

The rest of the paper is organized as follows: In Section 2 we recall some notions of trapped submanifold. Section 3 is devoted to the characterization of (marginally) trapped submanifold lying in a null hypersurface $\Sigma$ with geometrical objects of $\Sigma$. In Section 4 we prove some results for SIC-normalized null trapping horizons.

## 2 Marginally (outer) trapped submanifolds

In what follows, $(M, \langle,\rangle)$ is an $(n+2)-$dimensional spacetime, i.e a time-orientable Lorentzian manifold. Let $S$ be a spacelike codimension-two submanifold of the spacetime $M$. (Some authors call $S$ a surface.) Let $\ell$ and $k$ be two future-directed lightlike vector fields of $M$ normalized by $\langle \ell, k \rangle = -1$ and such that the normal bundle

$TS^{\perp} = span\{\ell, k\}$. We assume that $\ell$ is in the outgoing direction and $k$ ingoing direction. Let $\nabla$ (resp. $\overset{\star}{\nabla}$) be the Levi-Civita connection of $M$ (resp. $S$). Then for all sections $X, Y$ of the tangent space $TS$ one can write

$$\nabla_X Y = \overset{\star}{\nabla}_X Y + \Pi(X, Y) \tag{2.1}$$

where $\Pi$ is the second fundamental form of the immersion of $S$ into $M$. The extrinsic mean curvature vector $\vec{H}$ can be written as

$$\vec{H} = tr(\Pi) = \theta^{(k)}\ell + \theta^{(\ell)}k$$

where $\theta^{(\ell)}$ and $\theta^{(k)}$ are the expansions of $S$ with respect to $\ell$ and $k$ respectively. Let $N$ be a compactly supported normal vector to $S$ and $(\phi_\epsilon^N)_{\epsilon \in I}$ the associated one parameter group of diffeomorphisms of $M$. For each $\epsilon$, $S_\epsilon = \phi_\epsilon^N(S)$ is called the *Lie dragging* of $S$ along $N$. Hence $S_0 = S$ and $\phi_\epsilon^N$ can be viewed as the variation of the immersion $\phi_0^N : S \to M$ of $S$ into $M$ with the velocity vector of the variation $N = \frac{d\phi_\epsilon^N}{d\epsilon}|_{\epsilon=0}$. Let $|S_\epsilon|$ denote the area of $S_\epsilon$. From the first variation formula one gets

$$\frac{d}{d\epsilon}|S_\epsilon|_{|\epsilon=0} = -\int_S \langle \vec{H}, N \rangle \eta_g, \tag{2.2}$$

where $\eta_g$ is the metric volume form on $S$. The first order variation of the area of $S$ with respect to the deformations along $N$ is $\delta_N|S| := \frac{d}{d\epsilon}\left[|S_\epsilon|\right]_{|\epsilon=0}$. It follows that

$$\delta_k|S| = \int_S \theta^{(k)}\eta_g \qquad \text{and} \qquad \delta_\ell|S| = \int_S \theta^{(\ell)}\eta_g.$$

Hence for $\theta^{(\ell)} < 0$, the area of $S$ decrease when $S$ is dragging along $\ell$; this is taken as a clear signal of the presence of a strong gravitational field which that tends to drag things and even light toward it [21], and $S$ is called a *weakly future trapped surface* [30].

**Definition 2.1.** A codimension-two spacelike submanifold of a Lorentzian manifold is called a future

- *Trapped Submanifold (TS)* if $\theta^{(\ell)} < 0$ and $\theta^{(k)} < 0$.
- *Marginally Trapped Submanifold (MTS)* if $\theta^{(\ell)} = 0$ and $\theta^{(k)} \leq 0$.
- *Trapped Outer Submanifold (TOS)* if $\theta^{(\ell)} < 0$.
- *Marginally Outer Trapped Submanifold (MOTS)* if its mean curvature vector is lightlike or zero.

The outgoing direction depends on an arbitrary choice and when the expansion in one direction is zero one takes this direction as the outgoing direction. In other words, a MOTS is a codimension-two spacelike submanifold for which the expansion in one normal null direction vanishes.

# 3 Null trapping horizon

Let $x : \Sigma \to M$ be a null hypersurface of a spacetime $M$, meaning that the pullback $x^*\langle , \rangle$ is a degenerate metric on $\Sigma$ . So, the radical distribution $Rad(T\Sigma) := T\Sigma \cap T\Sigma^\perp$ is nontrivial and is locally spanned by a null vector field, say $\xi$ (we assume $\xi$ to be past directed). Let $S(T\Sigma)$ be a complementary bundle of $Rad(T\Sigma)$ in $T\Sigma$. Such a bundle $S(T\Sigma)$ is called a screen distribution. We assume that $S(T\Sigma)$ is integrable (this is not always guaranteed), then leaves of $S(T\Sigma)$ are spacelike codimension-two submanifold of the spacetime $M$. Let $S$ be one of these leaves, then $TS = S(T\Sigma)|_S$. The future directed null vector field $\ell = -\xi$ is orthogonal to $S$ and there exists a second future directed null vector field $k$ such that $TS^\perp$ is locally spanned by $\{\ell, k\}$. We keep the notations of the previous section.

Let $\nabla^\Sigma$ be the connection induced on $\Sigma$ through the projection along $k$, thus

$$\nabla_X Y = \nabla^\Sigma_X Y + B(X,Y)k = \overset{\star}{\nabla}_X Y + B(X,Y)k + C(X,Y)\ell \tag{3.1}$$

for all sections $X, Y$ of $S(T\Sigma)$. The local second fundamental forms $B$ and $C$ are given by

$$B(X,Y) = \langle \nabla_X \ell, Y\rangle \qquad \text{and} \qquad C(X,Y) = \langle \nabla_X k, Y\rangle.$$

The expansion $\theta^{(\ell)}$ (resp. $\theta^{(k)}$) is the trace of $B$ (resp. $C$) on $TS$. $\nabla^\Sigma$ is a linear connection on $\Sigma$ but is not compatible with the pullback metric of the spacetime on $\Sigma$ and depend on the chosen screen distribution.

**Definition 3.1.** We say that the null hypersurface $\Sigma$ is totally geodesic (resp. is minimal) when $B \equiv 0$ (resp. $\theta^{(\ell)} \equiv 0$). When there exists a function $\rho$ on $\Sigma$ such that $B(X,Y) = \rho\langle X, Y\rangle$ for all section of $T\Sigma$ we say that $\Sigma$ is totally umbilical.

All those notions do not depend on $\ell$ (and on the chosen screen distribution) in $Rad(T\Sigma)$. In fact, $Rad(T\Sigma)$ being of dimension 1, another section of $Rad(T\Sigma)$ is as $\tilde{\ell} = \alpha\ell$. From which we deduce $\tilde{B} = \alpha B$. One shows that $\Sigma$ is totally geodesic if and only if every geodesic curve in $\Sigma$ is also geodesic in $M$ [13].

Note that $\ell$ is necessarily pre-geodesic (and integral curves of $\ell$ are called null generators of $\Sigma$), i.e $\nabla_\ell \ell = \tau(\ell)\ell$. $\tau(\ell)$ is called the surface gravity of $\Sigma$. This terminology comes from general relativity when $\Sigma$ is the horizon of a Kerr black hole, see [16] for more details. It is nothing to see that for any section $X$ of $T\Sigma$, the vector field $\nabla_X \ell$ is also a section of $T\Sigma$. We extend $\tau$ on $\Sigma$ and call it the rotation $1-$form :

$$\tau(X) := -\langle \nabla_X \ell, k\rangle$$

A normalized or rigged null hypersurface is a null hypersurface $\Sigma$ on which we have fixed either a screen distribution $S(T\Sigma)$ or a null rigging $k$.

**Remark 3.2.** When $\Sigma$ is an isolated horizon of a dimension 4 black hole $M$, the two first $\Psi_0$ and $\Psi_1$ of Weyl components vanish on $\Sigma$ and the only probably non-vanishing component $\Psi_2$ is gauge invariant and related to $\tau$ by $d\tau = (Im\Psi_2)\varepsilon$, being $\varepsilon$ the natural area $2-$form on $\Sigma$ and $Im\Psi_2$ the imaginary part of $\Psi_2$. When $Im\Psi_2$ vanishes, all angular momentum multipoles vanish and this horizon is said to be non-rotating, see [4, 3] for more details. This explains why $\tau$ is called rotation $1-$form of an isolated horizon. However, $\tau$ is called by Hájicek [19, 20] *a gravimagnetic field* and by Damour [10, 11] *a surface momentum density.*

**Theorem 3.1** ([6])**.** *We assume that $M$ has constant sectional curvature. A null hypersurface $\Sigma$ is totally geodesic if and only if $\Sigma$ is minimal.*

**Example 3.3.** We consider the 6−dimensional spacetime $M$ endowed with the metric

$$\langle , \rangle := -(dx^0)^2 + (dx^1)^2 + \exp 2x^0[(dx^2)^2 + (dx^3)^2] + \exp 2x^1[(dx^4)^2 + (dx^5)^2],$$

$(x^0, ..., x^5)$ being the usual Cartesian coordinates on $\mathbb{R}^6$. The hypersurface

$$\Sigma := \{(x^0, ..., x^5) \in \mathbb{R}^6 \,;\; x^0 + x^1 = 0\}$$

is a null (or lightlike) hypersurface of $M$. One can take $k := -\frac{1}{2}\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right)$ and $\ell = \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}$. The associated screen distribution is given by $S(T\Sigma) = span\{E_1, E_2, E_3, E_4\}$ with

$$E_1 = e^{-2x^0}\frac{\partial}{\partial x^2}, \quad E_2 = e^{-2x^0}\frac{\partial}{\partial x^3}, \quad E_3 = e^{-2x^1}\frac{\partial}{\partial x^4}, \quad E_4 = e^{-2x^1}\frac{\partial}{\partial x^5}.$$

By direct computations, one finds that $\Sigma$ is not totally geodesic but is minimal (hence $M$ does not have constant sectional curvature) and $\theta^{(k)} = 2 > 0$. Hence leaves of the screen distribution given by $S = \{x^0 = cste, x^1 = cste\}$ are marginally trapped submanifolds of $M$.

**Definition 3.4.** We say that the screen distribution $S(T\Sigma)$ is totally geodesic (resp. is minimal) when $C \equiv 0$ (resp. $\theta^{(k)} \equiv 0$). When there exists a function $\lambda$ on $\Sigma$ such that $C(X, Y) = \lambda\langle X, Y\rangle$ for all section of $S(T\Sigma)$ we say that $S(T\Sigma)$ is totally umbilical.

**Definition 3.5** ([28])**.** A *Non-Expanding Horizon* (NEH) in a 4−dimensional spacetime $M$ is a rigged null hypersurface $\Sigma$ such that:

- $M$ has topology $\mathbb{S}^2 \times \mathbb{R}$;
- the ingoing null expansion $\theta^{(\ell)}$ vanishes on $M$;
- Einstein's equation hold, and the matter stress-energy tensor $\mathbf{T}_{ab}$ is such that for any future directed null-normal $-\mathbf{T}^a_b\ell^b$ is future causal (dominant energy condition).

**Example 3.6.** Let us consider the Schwarzschild spacetime, whose the metric is given in the ingoing Eddington-Finkelstein coordinates $(t, r, \theta, \varphi)$ by

$$\langle , \rangle = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{4m}{r}dtdr + \left(1 + \frac{2m}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

It is nothing to see that the hypersurface

$$\Sigma : \; r = 2m,$$

is a null hypersurface, called the event horizon. One can rig this lightlike hypersurface by the following future directed null vector fields

$$k = \frac{r}{2m}\partial_t - \frac{r}{2m}\partial_r, \quad \ell = \partial_t.$$

The corresponding screen distribution is given by $S(T\Sigma) = span(\overset{\star}{E}_1, \overset{\star}{E}_2)$ with

$$\overset{\star}{E}_1 = \partial_\theta, \quad \overset{\star}{E}_2 = \partial_\varphi.$$

This screen is integrable and leaves are the spheres $\{t = cste, r = 2m\}$, which foliate $\Sigma$. A direct computation gives

$$\theta^{(\ell)} = 0, \quad C = -\frac{1}{2m}\langle,\rangle|_{S(T\Sigma)}, \quad \tau|_{S(T\Sigma)} = 0, \quad \tau(\ell) = \frac{1}{4m}, \quad \theta^{(k)} = -\frac{1}{m} \leq 0.$$

Hence, $\Sigma$ is totally geodesic, the screen distribution is totally umbilical with $\lambda = -1/2m$. Also, the spheres $t = cste, r = 2m$ are marginally future trapped surfaces. Hence, $\Sigma$ is a null MOTT. Also, $\Sigma$ is a non-expanding horizon and spheres $\{t = cste, r = 2m\}$ are the cross sections.

Moreover, the one parameter group of diffeomorphisms (just the flow) of $k$ starting at $(t_0, r_0, \theta_0, \varphi_0)$ is given by

$$\phi_\epsilon = (t_0 + (1 - \exp(-\epsilon/2m))r_0, r_0 \exp(-\epsilon/2m), \theta_0, \varphi_0).$$

The image of a sphere $\mathcal{S} = \{t = t_0, r = 2m\}$ by $\phi_\epsilon$ (Lie dragging of $\mathcal{S}$ along $N$) is

$$\mathcal{S}_\epsilon := \phi_\epsilon(\mathcal{S}) = \{t = t_0 + 2m(1 - \exp(-\epsilon/2m)), r = 2m\exp(-\epsilon/2m)\}.$$

These spheres are spacelike surfaces and corresponding normalized pairs are given by

$$k_\epsilon = \frac{r\exp(-\epsilon/2m)}{2m}(1, -1, 0, 0),$$

$$\ell_\epsilon = \left(\frac{m\exp(\epsilon/2m)}{x} + \frac{2m^2\exp(\epsilon/m)}{x^2}, \frac{m\exp(\epsilon/2m)}{x} - \frac{2m^2\exp(\epsilon/m)}{x^2}, 0, 0\right).$$

By a direct calculation, one finds

$$\theta^{(\ell)}_{\epsilon 1} = \frac{2m\exp(\epsilon/2m)(-r + 2m\exp(\epsilon/2m))}{r^3}.$$

Hence,

$$\delta_k\theta^{(\ell)}_1 = \left.\frac{d\,\overset{\star}{S}_{\epsilon 1}}{d\epsilon}\right|_{\epsilon=0} = \frac{r - 4m}{r^3} \neq 0.$$

$\Sigma$ is then a null future trapping horizon.

**Lemma 3.2.** *If $\Sigma$ admits an integrable screen distribution then*

$$\Sigma \text{ is a null MOTT if and only if } \Sigma \text{ is minimal.}$$

*Proof.* If $\Sigma$ is minimal then the outgoing expansion identically vanishes and so every leaf of the screen distribution is a MOTS. Thus $\Sigma$ is a null MOTT. Conversely, if $\Sigma$ is a null MOTT then, $\Sigma$ is foliated by MOTSs. Since This MOTSs are spacelike, the distribution associated to this foliation is a screen distribution for $\Sigma$. Using this screen one shows that the outgoing expansion identically vanishes on $\Sigma$, thus $\Sigma$ is minimal. □

Let $R, \overset{\star}{R}$ be the Riemannian curvatures of $M$ and $S$ respectively. It is a straightforward computation to check that the Codazzi and Ricci equations of the immersion $S \to M$ is given by

$$\langle R(X,Y)Z,T\rangle = \langle \overset{\star}{R}(X,Y)Z,T\rangle + B(X,Z)C(Y,T) - B(Y,Z)C(X,T) + C(X,Z)B(Y,T) - C(Y,Z)B(X,T) \tag{3.2}$$

$$\langle R(X,Y)Z,\ell\rangle = \left(\nabla^{\Sigma}_{X}B\right)(Y,Z) - \left(\nabla^{\Sigma}_{Y}B\right)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y). \tag{3.3}$$

for all sections $X, Y, Z, T$ of $S(T\Sigma)$.

*Proof of Theorem 1.1.* With Lemma 3.2, such a MOTT $\Sigma$ is minimal and since the sectional curvature of the ambient is constant, by Theorem 3.1, $\Sigma$ is totally geodesic. Hence, equation (3.2) leads to

$$\langle \overline{R}(X,Y)Z,T\rangle = \langle \overset{\star}{R}(X,Y)Z,T\rangle \tag{3.4}$$

which completes the proof. □

# 4 SIC-normalized null hypersurface

We say that the screen distribution is conformal when there exists a function $\varphi$ such that $C = \varphi B$ on $S(T\Sigma)$.

**Definition 4.1.** A Screen Integrable and Conformal (SIC) rigging $k$ is one for which the screen distribution is integrable and conformal.

**Example 4.2.** The rigging $k$ corresponding to the screen distribution associated to the MOTSs foliation in a MOTT or a trapping horizon is a SIC-rigging.

**Lemma 4.1** ([7])**.** *Let $x : \Sigma \to (M, \langle\cdot,\cdot\rangle)$ be a rigged null hypersurface.*

1. *If $k$ is a closed rigging with conformal screen distribution then the rotation $1-$form $\tau$ vanishes on the screen distribution.*
2. *If $k$ is a rigging with conformal screen distribution and vanishing rotation $1-$form then, the $1-$form $\eta = x^{\star}\langle k,\cdot\rangle$ is closed.*

Let $\Sigma$ be a Monge hypersurface of the Lorentz-Minkowski space $\mathbb{R}^{n+2}_1$ given by

$$\Sigma = \{x = (x^0 = F(x^1,\dots,x^{n+1}), x^1,\dots,x^{n+1}) \in \mathbb{R}^{n+2}\},$$

where $F : D \to \mathbb{R}$ is a smooth function defined on some open subset $D$ of $\mathbb{R}^{n+1}$. For a vector field $X = X^A \dfrac{\partial}{\partial x^A} \in \mathbb{R}^{n+2}_1$ a necessary and sufficient condition to be tangent to $\Sigma$ is that $X^0 = X^1 F'_{x^1} + \cdots + X^1 F'_{x^{n+1}}$. Then $\delta = \dfrac{\partial}{\partial x^0} + \sum\limits_{a=1}^{n+1} F'_{x^a}\dfrac{\partial}{\partial x^a}$ is normal to $\Sigma$. The later is a null hypersurface if and only if $\delta$ is a null vector field. This is equivalent to

$$\sum_{a=1}^{n+1} (F'_{x^a})^2 = ||\nabla F||^2 = 1, \tag{4.1}$$

where $\nabla F$ is the gradient of $F$ with respect to the Euclidean structure $||\cdot||$ of $\mathbb{R}^{n+1}$.

Let us assume that $\Sigma$ is a Monge null hypersurface and consider the null pair

$$k = \frac{1}{\sqrt{2}}\Big[\frac{\partial}{\partial x^0} - \sum_{a=1}^{n+1} F'_{x^a}\frac{\partial}{\partial x^a}\Big] = \frac{1}{\sqrt{2}}(1, -\nabla F), \tag{4.2}$$

and

$$\ell = \frac{1}{\sqrt{2}}\Big[\frac{\partial}{\partial x^0} + \frac{1}{\sqrt{2}}\sum_{a=1}^{n+1} F'_{x^a}\frac{\partial}{\partial x^a}\Big] = \frac{1}{\sqrt{2}}(1, \nabla F). \tag{4.3}$$

Then, the corresponding screen distribution is integrable and conformal with conformal factor $\phi = 1$. Thus $k$ is a SIC-rigging. The expansions are given by,

$$-\theta^{(k)} = \theta^{(\ell)} = -\frac{1}{\sqrt{2}}\Delta F, \tag{4.4}$$

where $\Delta F$ is the Laplacian of $F$ in $\mathbb{R}^{n+1}$. The expansions have different signs and this is a general fact for all null hypersurfaces in the Lorentz-Minkowski space $\mathbb{R}_1^{n+2}$. Hence in Lorentz-Minkowski space, a null hypersurface cannot be foliated by trapped submanifolds.

*Proof of Theorem 1.4.* If $\Sigma$ is a MOTT then, there exists a foliation of $\Sigma$ by MOTSs of $\mathbb{R}_1^{n+2}$. The distribution of this foliation can be set as screen distribution on $\Sigma$. It follows that $\Sigma$ is minimal and from equality (4.4) $F$ is harmonic. Conversely, if $F$ is harmonic then endowed $\Sigma$ with the generic SIC rigging (4.2), it follows that the screen distribution is integrable and leaves are MOTSs. Thus $\Sigma$ is a MOTT. □

**Lemma 4.2.** *Let $x : \Sigma \to M(c)$ be a SIC-rigged null hypersurface of a Lorentzian manifold with constant sectional curvature $c$. If $M$ is totally umbilical (or geodesic) then, each (connected) leaf of the screen distribution has constant sectional curvature.*

*Proof.* Assume that $\Sigma$ is totally umbilical with $B = \rho g$. Then equation (3.2) gives

$$\langle \overset{\star}{R}(X,Y)Z,T\rangle = (c + 2\varphi\rho^2)\left(\langle Y,Z\rangle\langle X,T\rangle - \langle X,Z\rangle\langle Y,T\rangle\right),$$

Which show that sectional curvature of $S$ is the function $\kappa = c + 2\varphi\rho^2$ for all plane $\sigma = span(X,Y)$. It is then known that $\kappa$ is a constant function. This prove that each connected leaf of the screen distribution has constant sectional curvature

$$\kappa = c + 2\varphi\rho^2.$$

□

Hence for a SIC-rigged totally umbilical null hypersurface $\Sigma \to M(c)$ with $B = \rho g$ and $C = \varphi B$, the product $\varphi\rho^2$ is constant on each leaf of the screen distribution.

*Proof of Theorem 1.5.* By Lemma 4.1, the rotation 1−form vanishes on the screen distribution and equation (3.3) becomes

$$\left(\nabla^{\Sigma}_{X} B\right)(Y, Z) = \left(\nabla^{\Sigma}_{Y} B\right)(X, Z).$$

Now, if $\Sigma$ is totally umbilical then by using the above equation, one shows that $\rho$ is constant on each leaf of the screen distribution. It follows that $\varphi$ is also constant on each leaf of the screen distribution, since it is the case for $\varphi\rho^2$. □

## References


[1] H. S. Alan, *Black holes: new horizons,* World Scientific, 2013.

[2] L. Andersson, M. Mars, W. Simon, *Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes*, arXiv Prepr. arXiv0704.2889, 2007.

[3] A. Ashtekar, J. Engle, T. Pawlowski, C. Van Den Broeck, *Multipole moments of isolated horizons*, Class. Quantum Gravity, 21, 11 (2004), 25-49.

[4] A. Ashtekar, B. Krishnan, *Isolated and dynamical horizons and their applications*, Living Rev. Relativ. 7, 10 (2004). https://doi.org/10.12942/lrr-2004-10

[5] C. Atindogbe, *Blaschke Type Normalization on Light-Like Hypersurfaces*, J. Geom. Phys. 6(4) (2010), 362-382.

[6] C. Atindogbe, H. Fotsing Tetsing, *Newton transformations on null hypersurfaces*, Commun. Math., 23, 1 (2015), 57-83.

[7] C. Atindogbe, H. Fotsing Tetsing, F. Ngakeu, *Normalized null Hypersurfaces in the Lorentz-Minkowski space satisfying $L_r\psi = U\psi + b$*, African Diaspora J. Math., to appear, 2020.

[8] G. Compere, *An introduction to the mechanics of black holes*, arXiv Prepr. gr-qc/0611129, 2006.

[9] I. Chavel, *Riemannian Geometry*, Cambridge University Press, 2006.

[10] T. Damour, *Surface effects in black hole physics - Proc. Second Marcel Grossmann Meet*, Gen. Relativ. North-Holland, Amsterdam, 1982; 587.

[11] T. Damour, *Quelques propriétés mécaniques, électromagnétiques, thermodynamiques et quantiques des trous noirs*, Université Paris 6, Thèse de doctorat détat, 1979.

[12] R. H. Dicke, P. J. E. Peebles, *Evolution of the solar system and the expansion of the universe*, Phys. Rev. Lett., 12, 15 (1964), 435-437.

[13] K. L. Duggal and A. Bejancu, *lightlike submanifolds of semi-Riemannian manifolds and application*, Springer, The Netherlands, Dordrecht, 1996.

[14] H. Fotsing Tetsing, F. Ngakeu, B. Olea, *Rigging technique for 1-lightlike submanifolds and preferred rigged connections*, Mediterr. J. Math. 16, 139 (2019). https://doi.org/10.1007/s00009-019-1423-x

[15] G. J. Galloway, *Maximum Principles for null hypersurfaces and null splitting theorems*, Ann. Henri Poincaré, 1, 3 (2000), 543-567.

[16] E. Gourgoulhon, J. L. Jaramillo, *A* $3+1$ *perspective on null hypersurfaces and isolated horizons*, Physics Reports, 423 (2006), 159-294.

[17] E. Gourgoulhon, *Geometry and physics of black holes*, Institut dÁstrophysique de Paris, France, 2016.

[18] M. Gutierrez, B. Olea, *Induced Riemannian structures on null hypersurfaces*, Math. Nachr., 289 (2015), 1219-1236.

[19] P. Hájicek, *Exact models of charged black holes. I. Geometry of totally geodesic null hypersurface*, Commun. Math. Phys., 34, 1 (1973), 37-52.

[20] P. Hájicek, *Stationary electrovacuum spacetimes with bifurcate horizons*, J. Math. Phys., 16 (1975), 5-18.

[21] S. A. Hayward, *Black holes: new horizons*, arXiv Prepr. gr-qc/0008071, 2000.

[22] S. W. Hawking, G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.

[23] S. A. Hayward, *Black holes: New horizons*, arXiv Prepr. gr-qc/0008071, 2000.

[24] S. A. Hayward, *General laws of black-hole dynamics*, Phys. Rev. D, 49, 12 (1994), 6467-6474.

[25] J. L. Jaramillo, *An introduction to local black hole horizons in the* $3+1$ *approach to general relativity*, Int. J. Mod. Phys., 20(11) (2011), 2169-2204.

[26] J. L. Jaramillo, *Area inequalities for stable marginally trapped surfaces*, 26 (2012), 1-22.

[27] J. L. Jaramillo, *A Young-Laplace law for black hole horizons*, Phys. Rev., 89 (2014) 2, 021502.

[28] B. Krishnan, *Quasi-local black hole horizons*, in Springer Handbook of Spacetime, Springer, 2014; 527-555.

[29] A. Lars, M. Marc, S. W. Simon, *Local existence of dynamical and trapping horizons*, Phys. Rev. Lett., 95 (2005), 111102.

[30] M. Mars, *Stability of marginally outer trapped surfaces and geometric inequalities*, in General Relativity, Cosmology and Astrophysics, Springer, 2014; 191-208.

[31] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett., 14 (1965), 57-59.



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