Spin 1 particle with polarizability in the external Coulomb field, nonrelativistic description

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Abstract. The quantum mechanical equation for a spin 1 particle with an additional electromagnetic characteristic – the polarizability – is investigated in presence of the external Coulomb field. By diagonalizing the spatial inversion operator, the system of 15 radial equations splits into two subsystems related to different parities, of 5 and 10 equations respectively. The system of 5 equations reduces to a 2-nd order equation which is known within the theory of the ordinary spin 1 particle. In this case, the polarizability does not manifest itself in the energy spectrum. In the system of 10 equations, the non-relativistic approximation is performed, and two linked 2-nd order equations for two functions are derived, from which there follows a 4-th order differential equation, which has two irregular points $r = 0$ and $r = \infty$, both of the rank 3. The special case of minimal value of conserved total angular momentum $j = 0$ is considered in the nonrelativistic approximation as well, where the problem reduces to a single equation of 2-nd equation with two irregular points, $r = 0$ and $r = \infty$, both of the rank 2. Further, the Frobenius solutions of the 2-nd and 4-th order radial equations are constructed, and the solutions which might describe bound states in the system are pointed out.

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1 Spin 1 particle with polarizability, non-relativistic limit

The relativistic wave equations for spin 0 and spin 1 particles with an additional electromagnetic characteristic, the polarizability, were extensively studied [1]–[10]. The relativistic problem for a spin 1 particle in external Coulomb field turns out to

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be very difficult, and has not been solved even for ordinary particle without additional characteristic. However, the non-relativistic problem for the ordinary spin 1 particle is solvable [11]–[12].

In the present paper, we first develop the general approach to the non-relativistic limit for a spin 1 particle with polarizabity, by deriving from relativistic 15-component system a non-relativistic Pauli-like equation for a 3-component wave function. It is convenient to separate the variables in the relativistic first order system. By diagonalizing the operator of spatial reflection, we split the system of 15 radial equations into two sub-systems, of 5 and 10 equations respectively. The system of 5 equation straightforwardly reduces to the known and exactly solvable 2-nd order equation, for which the polarizability does not manifest itself anyhow in the energy spectrum. For this case we have no need to consider a non-relativistic approximation. The radial system of 10 equations is complex, hence to simplify the task we perform the non-relativistic approximation, which leads to a system of linked of 2-nd order equations, from which a 4-th order equation is derived. This has two irregular singularities, both of the rank 3. The case of minimal value $j = 0$ should be studied separately, where the problem reduces to a 2-nd order equation with two singular points $r = 0$ and $r = \infty$, both of the rank 2. We have constructed the Frobenius solutions of 2-nd and 4-th order equations. Among all solutions we separate those which might describe bound states in the system.

We start with the generalized Proca tensor equations for vector particle with polarizability [12] (by using the notation $\mu = e\sigma/m^2$):

$$i\mu D_b(\frac{1}{2}F^{kl}\Phi_{kl}) + D^a\Phi_{ba} = m\Phi_b \ , \quad D_a\Phi_b - D_b\Phi_a = m\Phi_{ab} \ . \tag{1.1}$$

The next step is to perform a (3+1)-splitting,

$$\begin{gathered} i\mu D_0(F^{0l}\Phi_{0l} + \frac{1}{2}F^{lk}\Phi_{lk}) + D^l\Phi_{0l} = m\Phi_0 \ , \\ i\mu D_n(F^{0l}\Phi_{0l} + \frac{1}{2}F^{lk}\Phi_{lk}) + D_0\Phi_{n0} + D^l\Phi_{nl} = m\Phi_n \ , \\ D_0\Phi_l - D_l\Phi_0 = m\Phi_{0l} \ , \qquad D_l\Phi_k - D_k\Phi_l = m\Phi_{lk} \ . \end{gathered} \tag{1.2}$$

After that, we exclude the non-dynamical components Φ_0, Φ_{lk}, hence producing

$$m\Phi_n = -D_0\Phi_{0n} + \frac{1}{m}(-D^lD_l\Phi_n + D^lD_n\Phi_l) + i\mu D_n(F^{0l}\Phi_{0l} + \frac{1}{m}F^{lk}D_l\Phi_k) \ , \tag{1.3}$$

$$m\Phi_{0n} = D_0\Phi_n - \frac{1}{m}D_n\ D^l\Phi_{0l} - i\frac{\mu}{m}D_nD_0(F^{0l}\Phi_{0l} + \frac{1}{m}F^{lk}D_l\Phi_k) \ . \tag{1.4}$$

Now we introduce big Ψ_n and small ψ_n components: $\Phi_n = \Psi_n + \psi_n$, $i\Phi_{0n} = \Psi_n - \psi_k$. From (1.3)–(1.4), there follow the equations

$$\begin{gathered} 2m\Psi_n = 2iD_0\Psi_n + \frac{1}{m}[-D^lD_l(\Psi_n + \psi_n) + D^lD_n(\Psi_l + \psi_l)] - \frac{1}{m}D_nD^l(\Psi_l - \psi_l) \\ +\mu D_n[F^{0l}(\Psi_l - \psi_l) + \frac{i}{m}F^{lk}D_l(\Psi_k + \psi_k)] \\ -i\frac{\mu}{m}D_nD_0[F^{0l}(\Psi_l - \psi_l) + \frac{i}{m}F^{lk}D_l(\Psi_k + \psi_k)] \ , \end{gathered}$$

$$2m\psi_n = -2iD_0\psi_n + \frac{1}{m}[-D^l D_l(\Psi_n + \psi_n) + D^l D_n(\Psi_l + \psi_l)] + \frac{1}{m} D_n D^l(\Psi_l - \psi_l)$$

$$+\mu D_n[F^{0l}(\Psi_l - \psi_l) + \frac{i}{m} F^{lk} D_l(\Psi_k + \psi_k)] + i\frac{\mu}{m} D_n D_0[F^{0l}(\Psi_l - \psi_l) + \frac{i}{m} F^{lk} D_l(\Psi_k + \psi_k)] \; .$$

The next step is to separate the rest energy by using the formal change $iD_0 \rightsquigarrow (iD_0 + m)$; besides we neglect the small components $\psi_n(x)$ in comparison with the big ones $\Psi_n(x)$, and also we neglect the term $iD_0\Psi_n(x)$ in comparison with $m\Psi_n$. In this way, we arrive at

$$0 = 2iD_0\Psi_n + \frac{1}{m}[-D^l D_l \Psi_n + (D^l D_n - D_n D^l)\Psi_l] - \frac{\mu}{m} D_n iD_0(F^{0l}\Psi_l + \frac{i}{m} F^{lk} D_l \Psi_k) \; ,$$

$$4m\psi_n = +\frac{1}{m}[-D^l D_l \Psi_n + (D^l D_n + D_n D^l)\Psi_l]$$

$$+2\mu D_n(F^{0l}\Psi_l + \frac{i}{m} F^{lk} D_l \Psi_k) + i\frac{\mu}{m} D_n D_0(F^{0l}\Psi_l + \frac{i}{m} F^{lk} D_l \Psi_k) \; .$$

Therefore, the needed nonrelativistic Pauli-like equation for the vector particle with polarizability has the form

$$iD_0\Psi_n = \frac{1}{2m}\left(D^l D_l \Psi_n - [D_l, D_n]_- \Psi^l\right)$$

$$+\frac{\mu}{2m} D_n iD_0 \left(F^{0l}\Psi_l + \frac{i}{m} F^{lk} D_l \Psi_k\right) . \tag{1.5}$$

In absence of magnetic field, this becomes simpler:

$$iD_0\Psi_n = \frac{1}{2m}\partial^l \partial_l \Psi_n + \frac{\mu}{2m}\partial_n iD_0(F^{0l}\Psi_l) \; . \tag{1.6}$$

2 Matrix form of relativistic equation and tetrad formalism

To separate the variables, the first order relativistic system and tetrad formalism are the most convenient ones. In the beginning, we consider the free particle case, where an extension to the presence of external Coulomb field will be done in the radial system.

The matrix form of the 15-component equation is specified by the relations [13]

$$(\Gamma^a \partial_a - m)\Psi = 0 \; , \quad \Psi = \begin{vmatrix} C \\ C_l \\ \Phi_l \\ \Phi_{mn} \end{vmatrix} , \quad \Gamma^a = \begin{vmatrix} 0 & G^a & 0 & 0 \\ 0 & 0 & 0 & K^a \\ \sigma\Delta^a & 0 & 0 & K^a \\ 0 & 0 & \Lambda^a & 0 \end{vmatrix} , \tag{2.1}$$

where the following block-matrices are of dimensions 1×4, 4×1, 4×6 and 6×4:

$$(G^a)_{(0)}^{\;\;k} = g^{ak} \; , \qquad (\Delta^a)^{(0)}_{\;\;n} = \delta^a_n \; ,$$

$$(K_a)_n^{\;\;kl} = -g^{ak}\,\delta^l_n + g^{al}\,\delta^k_n \; , \;\; (\Lambda^a)^k_{\;\;nb} = \delta^a_n\,\delta^k_b - \delta^k_n\,\delta^a_b \; ,$$

and σ stands for a real-valued parameter. In accordance with the retrad method, eq. (2.1) is extended to the generally covariant case as follows [13] (the metric of the Riemannian space-time is $g_{\alpha\beta}(x)$, and some related tetrad is $e^{\alpha}_{(a)}(x)$):

$$[\,\Gamma^{\alpha}(x)\,(\partial_{\alpha} + B_{\alpha}(x)) - m\,]\,\Psi(x) = 0\,, \tag{2.2}$$

$$\Gamma^{\alpha}(x) = \Gamma^{a} e^{\alpha}_{(a)}(x)\,, \quad B_{\alpha}(x) = \frac{1}{2}\, J^{ab} e^{\beta}_{(a)} \nabla_{\alpha}(e_{(b)\beta})\,. \tag{2.3}$$

We specify eq. (2.2) in Minkowski space-time, using the spherical coordinates and related tetrad [14]

$$\left(\Gamma^{0}\partial_{0} + \Gamma^{3}\partial_{r} + \frac{\Gamma^{1}J^{31} + \Gamma^{2}J^{32}}{r} + \frac{1}{r}\Sigma_{\theta,\phi} - m\right)\Psi(x) = 0\,, \tag{2.4}$$

where

$$\Sigma_{\theta,\phi} = \Gamma^{1}\partial_{\theta} + \Gamma^{2}\frac{\partial_{\phi} + \cos\theta J^{12}}{\sin\theta}\,.$$

Below we need the explicit form of the matrices Γ^{a}; their blocks $G^{a}, \Delta^{a}, \Lambda^{a}, K^{a}$ are given by the relations

$$G^{0} = (+1,0,0,0)\,,\ G^{1} = (0,-1,0,0)\,,$$

$$G^{2} = (0,0,-1,0)\,,\ G^{3} = (0,0,0,-1)\,,$$

$$\Delta^{0} = \left|\begin{array}{c} 1\\0\\0\\0 \end{array}\right|, \quad \Delta^{1} = \left|\begin{array}{c} 0\\1\\0\\0 \end{array}\right|, \quad \Delta^{2} = \left|\begin{array}{c} 0\\0\\1\\0 \end{array}\right|, \quad \Delta^{3} = \left|\begin{array}{c} 0\\0\\0\\1 \end{array}\right|,$$

$$\Lambda^{0} = \left|\begin{array}{cccc} 0 & +1 & 0 & 0\\ 0 & 0 & +1 & 0\\ 0 & 0 & 0 & +1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right|, \ \Lambda^{1} = \left|\begin{array}{cccc} -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & +1 & 0 \end{array}\right|,$$

$$\Lambda^{2} = \left|\begin{array}{cccc} 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & +1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{array}\right|, \ \Lambda^{3} = \left|\begin{array}{cccc} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & +1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right|,$$

$$K^{0} = \left|\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right|, \ K^{1} = \left|\begin{array}{cccccc} -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & +1\\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right|,$$

$$K^{2} = \left|\begin{array}{cccccc} 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & +1 & 0 & 0 \end{array}\right|, \ K^{3} = \left|\begin{array}{cccccc} 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & +1 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right|.$$

By using the notations

$$\vec{e}_1 = \left(\begin{array}{ccc} 1 & 0 & 0 \end{array} \right), \; \vec{e}_2 = \left(\begin{array}{ccc} 0 & 1 & 0 \end{array} \right), \; \vec{e}_3 = \left(\begin{array}{ccc} 0 & 0 & 1 \end{array} \right) ,$$

$$\vec{e}_1^{\;t} = \left| \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right| , \qquad \vec{e}_2^{\;t} = \left| \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right| , \qquad \vec{e}_3^{\;t} = \left| \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right| ,$$

$$\tau_1 = \left| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right| , \; \tau_2 = \left| \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right| , \; \tau_3 = \left| \begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right| ,$$

the above block-matrices are presented shorter,

$$\Delta^0 = \left| \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right| , \; G^0 = (1,0,0,0) \; , \; \Lambda^0 = \left| \begin{array}{cc} 0 & I \\ 0 & 0 \end{array} \right| \; , \; K^0 = \left| \begin{array}{cc} 0 & 0 \\ -I & 0 \end{array} \right| ,$$

$$\Delta^i = \left| \begin{array}{c} 0 \\ \vec{e}_i^{\;t} \end{array} \right| , \; G^i = \left| \begin{array}{cc} 0 & -\vec{e}_i \end{array} \right| , \; \Lambda^i = \left| \begin{array}{cc} -\vec{e}_i^{\;t} & 0 \\ \vec{0}^{\;t} & \tau_i \end{array} \right| , \; K^i = \left| \begin{array}{cc} \vec{e}_i & \vec{0} \\ 0 & \tau_i \end{array} \right| .$$

Also, we need the explicit form of three generators of the total angular momentum in spherical tetrad basis [14]:

$$J_1 = l_1 + \frac{\cos\phi}{\sin\theta} S_3 \; , \qquad J_2 = l_2 + \frac{\sin\phi}{\sin\theta} S_3 \; , \qquad J_3 = l_3 \; ,$$

$$\text{(2.5)} \qquad \vec{J}^2 = -\frac{1}{\sin\theta}\partial_\theta \, \sin\theta \, \partial_\theta \; + \; \frac{-\partial^2_\phi \; + \; 2\, i\, \partial_\phi \, S_3 \cos\theta \; + \; S_3^2}{\sin^2\theta} \; .$$

We note that $S_3 = \text{diag}\{0, \tau_3, ..\}$ and

$$\tau_1 = \frac{1}{\sqrt{2}} \left| \begin{array}{ccc} 0 & -i & 0 \\ -i & 0 & -i \\ 0 & -i & 0 \end{array} \right| , \; \tau_2 = \frac{1}{\sqrt{2}} \left| \begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right| , \; \tau_3 = -i \left| \begin{array}{ccc} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right| .$$

The most convenient is the so called cyclic representation, in which the matrix S_3 is diagonal, and the vectors $\vec{e}_i$ and $\vec{e}_i^{\;t}$ are given by the relations

$$\vec{e}_1 = (-\frac{1}{\sqrt{2}}, \; 0, \; \frac{1}{\sqrt{2}}) \; , \; \vec{e}_2 = (-\frac{i}{\sqrt{2}}, \; 0, \; -\frac{i}{\sqrt{2}}) \; , \; \vec{e}_3 = (0,1,0) \; ,$$

$$\vec{e}_1^{\;t} = \left| \begin{array}{c} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{array} \right| , \qquad \vec{e}_2^{\;t} = \left| \begin{array}{c} \frac{i}{\sqrt{2}} \\ 0 \\ \frac{i}{\sqrt{2}} \end{array} \right| , \qquad \vec{e}_3^{\;t} = \left| \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right| .$$

3 Separating the variables, the radial equations

The general structure of the 15-component wave function with the quantum numbers ϵ, j, m is [14]

$$\text{(3.1)} \qquad \Psi(x) = \{ \; C(x), \; C_0(x), \; \vec{C}(x), \; \Phi_0(x), \; \vec{\Phi}(x), \; \vec{E}(x), \vec{H}(x) \; \} ,$$

where [14]

$$C(x) = e^{-i\epsilon t} C(r) D_0 \; , \;\; C_0(x) = e^{-i\epsilon t} C_0(r) D_0 \; , \;\; \Phi_0(x) = e^{-i\epsilon t} \Phi_0(x) D_0 \; ,$$

(3.2)
$$\vec{C}(x) = e^{-i\epsilon t} \left| \begin{array}{l} C_1(r) D_{-1} \\ C_2(r) D_0 \\ C_3(r) D_{+1} \end{array} \right| , \; \vec{\Phi}(x) = e^{-i\epsilon t} \left| \begin{array}{l} \Phi_1(r) D_{-1} \\ \Phi_2(r) D_0 \\ \Phi_3(r) D_{+1} \end{array} \right| ,$$
$$\vec{E}(x) = e^{-i\epsilon t} \left| \begin{array}{l} E_1(r) D_{-1} \\ E_2(r) D_0 \\ E_3(r) D_{+1} \end{array} \right| , \;\; \vec{H}(x) = e^{-i\epsilon t} \left| \begin{array}{l} H_1(r) D_{-1} \\ H_2(r) D_0 \\ H_3(r) D_{+1} \end{array} \right| .$$

We will use the recurrent formulas [14] for Wigner functions $D_\sigma = D^j_{-m,\sigma}(\phi, \theta, 0)$:

$$\partial_\theta D_{-1} = (1/2)(a D_{-2} - \nu D_0) \; , \quad \frac{-m + \cos\theta}{\sin\theta} D_{-1} = (1/2)(-a D_{-2} - \nu D_0) \; ,$$

(3.3)
$$\partial_\theta D_0 = (1/2)(\nu D_{-1} - \nu D_{+1}) \; , \quad \frac{-m}{\sin\theta} D_0 = (1/2)(-\nu D_{-1} - \nu D_{+1}) \; ,$$
$$\partial_\theta D_{+1} = (1/2)(\nu D_0 - a D_{+2}) \; , \quad \frac{-m - \cos\theta}{\sin\theta} D_{+1} = (1/2)(-\nu D_0 - \; a D_{+2}) \; ,$$

where $\nu = \sqrt{j(j+1)}$, $a = \sqrt{(j-1)(j+2)}, j = 1, 2, 3, \ldots$.

As a final result, we produce the following radial system (let $\nu = \sqrt{j(j+1)}/\sqrt{2}$; together with the radial equations we write down the related tensor equations):

$$\underline{\partial^a C_a = mC} \; , \; -i\epsilon C_0 - (\frac{d}{dr} + \frac{2}{r}) C_2 - \frac{\nu}{r}(C_1 + C_3) = mC \; ;$$

$$\underline{\partial^a \Phi_{ba} = m \; C_a} \; ,$$

$$-(\frac{d}{dr} + \frac{2}{r}) E_2 - \frac{\nu}{r}(E_1 + E_3) = mC_0 \; , \; +i\epsilon E_1 + i(\frac{d}{dr} + \frac{1}{r}) H_1 + i\frac{\nu}{r} H_2 = mC_1 \; ,$$

$$+i\epsilon E_2 - i\frac{\nu}{r}(H_1 - H_3) = mC_2 \; , \; +i\epsilon E_3 - i(\frac{d}{dr} + \frac{1}{r}) H_3 - i\frac{\nu}{r} H_2 = mC_3 \; ;$$

$$\underline{\sigma \partial_a C + \partial^a \Phi_{ba} = m\Phi_a} \; ,$$

$$-i;\epsilon\sigma C - (\frac{d}{dr} + \frac{2}{r}) E_2 - \frac{\nu}{r}(E_1 + E_3) = m\Phi_0 \; , \; +i\epsilon E_1 + i(\frac{d}{dr} + \frac{1}{r}) H_1 + i\frac{\nu}{r} H_2 - \sigma\frac{\nu}{r} C = m\Phi_1 \; ,$$

$$+i\epsilon E_2 + \sigma\frac{d}{dr} C - i\frac{\nu}{r}(H_1 - H_3) = m\Phi_2 \; , \; +i\epsilon E_3 - i(\frac{d}{dr} + \frac{1}{r}) H_3 - i\frac{\nu}{r} H_2 - \sigma\frac{\nu}{r} C = m\Phi_3 \; ;$$

$$\underline{\partial_a \Phi_b - \partial_b \Phi_a = m\Phi_{ab}} \; ,$$

$$-i\epsilon\Phi_1 + \frac{\nu}{r}\Phi_0 = mE_1 \; , \; -i\epsilon\Phi_2 - \frac{d}{dr}\Phi_0 = mE_2 \; , \; -i\epsilon\Phi_3 + \frac{\nu}{r}\Phi_0 = mE_3 \; ,$$

$$-i(\frac{d}{dr} + \frac{1}{r})\Phi_1 - i\frac{\nu}{r}\Phi_2 = mH_1 \; , \; +i\frac{\nu}{r}(\Phi_1 - \Phi_3) = mH_2 \; , \; i(\frac{d}{dr} + \frac{1}{r}) Phi_3 + i\frac{\nu}{r}\Phi_2 = mH_3 \; .$$

To simplify the system of equations, in addition to $\vec{J}^{\,2}, J_3$, let us diagonalize the operator of spatial reflection $\hat{\Pi}$. In Cartesian tetrad basis and Cartesian representation for the matrices Γ^a, this is given by the known expression

$$(3.4) \qquad \hat{\Pi} = \left| \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +I \end{array} \right| \hat{P}\;, \quad \hat{P}\Psi(\vec{r}) = \Psi(-\vec{r})\;.$$

After transition to tetrad basis and to cyclic representation for Γ^a, we get [14]

$$(3.5) \; \hat{\Pi}' = \left| \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Pi_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Pi_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Pi_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\Pi_3 \end{array} \right| \hat{P}\;, \quad \Pi_3 = \left| \begin{array}{ccc} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{array} \right| .$$

The eigenvalue equation $\hat{\Pi}\Psi = P\Psi$ (note the property of Wigner functions $\hat{P}D_\sigma = (-1)^j D_{-\sigma}$) gives two solutions:

$$(3.6) \quad \begin{array}{c} \underline{P = (-1)^{j+1}}\;, \qquad C = 0\;, \;\; C_0 = 0\;, \;\; C_3 = -C_1\;, \;\; C_2 = 0\;, \\ \Phi_0 = 0\;, \;\; \Phi_3 = -\Phi_1\;, \;\; \Phi_2 = 0\;, \quad E_3 = -E_1\;, \;\; E_2 = 0\;, \;\; H_3 = H_1\;\; ; \end{array}$$

$$(3.7) \quad \underline{P = (-1)^j}, \; C_3 = +C_1, \; \Phi_3 = +\Phi_1, \; E_3 = +E_1, \; H_3 = -H_1, \; H_2 = 0.$$

It is readily verified that these restrictions are consistent with the above 15 radial equations, and further we derive two more simple sub-systems:

$\underline{P = (-1)^{j+1}}$,

$$(3.8) \quad \begin{array}{c} +i\,\epsilon\,E_1 + i\,(\dfrac{d}{dr} + \dfrac{1}{r})\,H_1 + i\,\dfrac{\nu}{r}H_2 = m\,C_1\;, \\ +i\,\epsilon\,E_1 + i\,(\dfrac{d}{dr} + \dfrac{1}{r})\,H_1 + i\,\dfrac{\nu}{r}\,H_2 = m\,\Phi_1\;, \quad -i\,\epsilon\,\Phi_1 = m\,E_1\;, \\ -i\,(\dfrac{d}{dr} + \dfrac{1}{r})\,\Phi_1 = m\,H_1\;, \quad 2i\,\dfrac{\nu}{r}\,\Phi_1 = m\,H_2\;; \end{array}$$

$\underline{P = (-1)^j}$,

$$
\begin{aligned}
&-i\epsilon C_0 - (\frac{d}{dr} + \frac{2}{r})C_2 - 2\frac{\nu}{r}C_1 = mC, -(\frac{d}{dr} + \frac{2}{r})E_2 - 2\frac{\nu}{r}E_1 = mC_0,\\
&+i\epsilon E_1 + i(\frac{d}{dr} + \frac{1}{r})H_1 = mC_1, i\epsilon E_2 - 2i\frac{\nu}{r}H_1 = mC_2,\\
&-i\epsilon\sigma C - (\frac{d}{dr} + \frac{2}{r})E_2 - 2\frac{\nu}{r}E_1 = m\Phi_0,\\
&i\epsilon E_1 + i(\frac{d}{dr} + \frac{1}{r})H_1 - \sigma\frac{\nu}{r}C = m\Phi_1,\\
&i\epsilon E_2 - 2i\frac{\nu}{r}H_1 + \sigma\frac{d}{dr}C = m\Phi_2, -i\epsilon\Phi_1 + \frac{\nu}{r}\Phi_0 = mE_1,\\
&-i\epsilon\Phi_2 - \frac{d}{dr}\Phi_0 = mE_2, -i(\frac{d}{dr} + \frac{1}{r})\Phi_1 - i\frac{\nu}{r}\Phi_2 = mH_1.
\end{aligned}
\tag{3.9}
$$

4 Taking into account the Coulomb field, non-relativistic limit

Now let us take into account the presence of the Coulomb field (note the notation $e^2 = \alpha$). From the sub-systems (3.7) and (3.9), we readily obtain
$\underline{P = (-1)^{j+1}}$,

$$
\begin{aligned}
&+i(\epsilon + \frac{\alpha}{r})E_1 + i(\frac{d}{dr} + \frac{1}{r})H_1 + i\frac{\nu}{r}H_2 = mC_1 ,\\
&+i(\epsilon + \frac{\alpha}{r})E_1 + i(\frac{d}{dr} + \frac{1}{r})H_1 + i\frac{\nu}{r}H_2 = m\Phi_1 ,\\
&-i(\epsilon + \frac{\alpha}{r})\Phi_1 = mE_1 , \ -i(\frac{d}{dr} + \frac{1}{r})\Phi_1 = mH_1 , \ 2i\frac{\nu}{r}\Phi_1 = mH_2 ;
\end{aligned}
\tag{4.1}
$$

$\underline{P = (-1)^j}$,

$$
\begin{aligned}
&-i(\epsilon + \frac{\alpha}{r})C_0 - (\frac{d}{dr} + \frac{2}{r})C_2 - 2\frac{\nu}{r}C_1 = mC , \ -(\frac{d}{dr} + \frac{2}{r})E_2 - 2\frac{\nu}{r}E_1 = mC_0 ,\\
&+i(\epsilon + \frac{\alpha}{r})E_1 + i(\frac{d}{dr} + \frac{1}{r})H_1 = mC_1 , \ +i(\epsilon + \frac{\alpha}{r})E_2 - 2i\frac{\nu}{r}H_1 = mC_2 ;\\
&-i(\epsilon + \frac{\alpha}{r})\sigma C - (\frac{d}{dr} + \frac{2}{r})E_2 - 2\frac{\nu}{r}E_1 = m\Phi_0 ,\\
&+i(\epsilon + \frac{\alpha}{r})E_1 + i(\frac{d}{dr} + \frac{1}{r})H_1 - \sigma\frac{\nu}{r}C = m\Phi_1 ,\\
&+i(\epsilon + \frac{\alpha}{r})E_2 - 2i\frac{\nu}{r}H_1 + \sigma\frac{d}{dr}C = m\Phi_2 ,\\
&-i(\epsilon + \frac{\alpha}{r})\Phi_1 + \frac{\nu}{r}\Phi_0 = mE_1 , \ -i(\epsilon + \frac{\alpha}{r})\Phi_2 - \frac{d}{dr}\Phi_0 = mE_2 ,\\
&-i(\frac{d}{dr} + \frac{1}{r})\Phi_1 - i\frac{\nu}{r}\Phi_2 = mH_1 .
\end{aligned}
\tag{4.2}
$$

It should be noticed that equations in the system (4.1) does not contain the polarizability parameter σ; these equations may be easily solved. Indeed, from (4.1) it follows

$$C_1(x) = \Phi_1(x), mH_2 = 2i\frac{\nu}{r}\Phi_1, m\ E_1 = -i(\epsilon + \frac{\alpha}{r})\Phi_1, mH_1 = -i(\frac{d}{dr} + \frac{1}{r})\Phi_1.$$

By excluding the variables E_1, H_1, H_2, we derive a second order equation for $\Phi_1(r)$:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + (\epsilon + \frac{\alpha}{r})^2 - m^2 - \frac{j(j+1)}{r^2} \right] \Phi_1 = 0\ . \tag{4.3}$$

Equation (4.3) is a well known one – it arises in describing the ordinary scalar particle (or ordinary vector particle with parity $P = (-1)^{j+1}$) in the Coulomb field; its solutions and energy levels are known.

Now let us consider the equations for states with parity $P = (-1)^j$. Using the second, third and fourth equations from (4.2), we get the following expression for $C(x)$:

$$C = i\frac{\alpha}{m^2 r^2}\ E_2\ . \tag{4.4}$$

By excluding $C(x)$ from the remaining equations in (4.2), we get

$$\begin{gathered} -i(\epsilon + \frac{\alpha}{r})\sigma i \frac{\alpha}{m^2 r^2} E_2 - (\frac{d}{dr} + \frac{2}{r})E_2 - 2\frac{\nu}{r}E_1 = m\Phi_0\ , \\ +i(\epsilon + \frac{\alpha}{r})E_1 + i(\frac{d}{dr} + \frac{1}{r})H_1 - \sigma\frac{\nu}{r}i\ \frac{\alpha}{m^2 r^2}E_2 = m\Phi_1\ , \\ +i(\epsilon + \frac{\alpha}{r})E_2 - 2i\frac{\nu}{r}H_1 + \sigma\frac{d}{dr}i\frac{\alpha}{m^2 r^2}E_2 = m\Phi_2\ , \\ -i(\epsilon + \frac{\alpha}{r})\Phi_1 + \frac{\nu}{r}\Phi_0 = mE_1\ ,\ -i(\epsilon + \frac{\alpha}{r})\Phi_2 - \frac{d}{dr}\Phi_0 = mE_2\ , \\ -i(\frac{d}{dr} + \frac{1}{r})\Phi_1 - \frac{i\nu}{r}\ \Phi_2 = mH_1\ . \end{gathered} \tag{4.5}$$

After excluding the non-dynamical variables Φ_0 and H_1, we get four equations:

$$\begin{gathered} (\epsilon + \frac{\alpha}{r})\ iE_1 + \frac{1}{m}(\frac{d}{dr} + \frac{1}{r})^2\Phi_1 + (\frac{d}{dr} + \frac{1}{r})\frac{\nu}{mr}\Phi_2 = m\ \Phi_1 + \frac{\sigma\nu\alpha}{m^2 r^3}\ iE_2\ , \\ (\epsilon + \frac{\alpha}{r})\ \Phi_1\ - \frac{\nu}{mr}(\frac{d}{dr} + \frac{2}{r})iE_2 - \frac{2\nu^2}{mr^2}iE_1 = m\ iE_1 - (\epsilon + \frac{\alpha}{r})\frac{\sigma\alpha\nu}{m^3 r^3}iE_2\ ; \end{gathered} \tag{4.6}$$

$$\begin{gathered} (\epsilon + \frac{\alpha}{r})\ iE_2 - \frac{2\nu}{mr}(\frac{d}{dr} + \frac{1}{r})\Phi_1 - \frac{2\nu^2}{mr^2}\Phi_2 = m\ \Phi_2 - \frac{d}{dr}\ \frac{\sigma\alpha}{m^2 r^2}\ iE_2\ , \\ (\epsilon + \frac{\alpha}{r})\Phi_2 + \frac{d}{dr}\frac{1}{m}(\frac{d}{dr} + \frac{2}{r})iE_2 + \frac{d}{dr}\frac{2\nu}{mr}iE_1 = m\ iE_2 + \frac{d}{dr}(\epsilon + \frac{\alpha}{r})\frac{\sigma\alpha}{m^3 r^2}iE_2\ . \end{gathered} \tag{4.7}$$

Now we are to introduce big $B_{1,2}(r)$ and small $M_{1,2}(r)$ components:

$$\Phi_1 = (B_1 + M_1), \Phi_2 = (B_2 + M_2), iE_1 = (B_1 - M_1), iE_2 = (B_2 - M_2);$$

also we are to separate the rest energy with the help of formal change $\epsilon = m + E$. This provides us with more simple equations (we neglect small functions M_i in comparison with the big ones B_i, and neglect the non-relativistic energy in comparison with the rest energy m):

$$(E+\frac{\alpha}{r})B_1+\frac{1}{m}(\frac{d}{dr}+\frac{1}{r})^2B_1+(\frac{d}{dr}+\frac{1}{r})\frac{\nu}{mr}B_2=2m\ M_1+\frac{\sigma\nu\alpha}{m^2r^3}\ B_2\ ,$$

$$(E+\frac{\alpha}{r})\ B_1\ -\frac{\nu}{mr}(\frac{d}{dr}+\frac{2}{r})B_2-\frac{2\nu^2}{mr^2}B_1=-2mM_1-(m+E+\frac{\alpha}{r})\frac{\sigma\alpha\nu}{m^3r^3}B_2\ ,$$

$$(E+\frac{\alpha}{r})\ B_2-\frac{2\nu}{mr}(\frac{d}{dr}+\frac{1}{r})B_1-\frac{2\nu^2}{mr^2}B_2=2m\ M_2-\frac{d}{dr}\ \frac{\sigma\alpha}{m^2r^2}\ B_2\ ,$$

$$(E+\frac{\alpha}{r})B_2+\frac{d}{dr}\frac{1}{m}(\frac{d}{dr}+\frac{2}{r})B_2+\frac{d}{dr}\frac{2\nu}{mr}B_1=-2m\ M_2+\frac{d}{dr}(m+E+\frac{\alpha}{r})\frac{\sigma\alpha}{m^3r^2}B_2\ .$$

In order to obtain equations with respect to only big components, we shall add the equations within each pair:

$$2m(E+\frac{\alpha}{r})B_1+(\frac{d}{dr}+\frac{1}{r})^2B_1+(\frac{d}{dr}+\frac{1}{r})\frac{\nu}{r}B_2$$

$$-\frac{\nu}{r}(\frac{d}{dr}+\frac{2}{r})B_2-\frac{2\nu^2}{r^2}B_1=-(E+\frac{\alpha}{r})\frac{\sigma\alpha\nu}{m^2r^3}B_2\ ,$$

$$2m(E+\frac{\alpha}{r})\ B_2-\frac{2\nu}{r}(\frac{d}{dr}+\frac{1}{r})B_1-\frac{2\nu^2}{r^2}B_2+\frac{d}{dr}(\frac{d}{dr}+\frac{2}{r})B_2$$

$$+\frac{d}{dr}\frac{2\nu}{r}B_1=+\frac{d}{dr}(E+\frac{\alpha}{r})\frac{\sigma\alpha}{m^2r^2}B_2\ .$$

They can be re-written as follows

$$\Box B_1=\frac{2\nu}{r^2}B_2-e\mu(E+\frac{\alpha}{r})\frac{\nu}{r^3}B_2\,,\qquad \Box B_2=+\frac{3\nu}{r^2}B_1+\frac{2}{r^2}B_2+e\mu\frac{d}{dr}(E+\frac{\alpha}{r})\frac{1}{r^2}B_2\,, \tag{4.8}$$

where the following notations are used

$$e^2=\alpha,\quad \frac{e\sigma}{m^2}=\mu\ ,\qquad \Box=\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+2m(E+\frac{\alpha}{r})-\frac{2\nu^2}{r^2}\,. \tag{4.9}$$

For the more simple case of vanishing parameter $\mu=0$, the system (4.8) was studied when considering the non-relativistic approximation for the ordinary vector particle [11]. We will apply the previously used linear transformation over the functions (recall that $\nu=\sqrt{j(j+1)/2}$)

$$F_1=2\nu B_1+\lambda_1B_2,\qquad F_2=2\nu B_1+\lambda_2B_2\ , \tag{4.10}$$

where λ_1,λ_2 stand for the roots of the quadratic equation

$$\lambda^2-\lambda-j(j+1)=0,\qquad \lambda_1=j+1,\ \ \lambda_2=-j\ . \tag{4.11}$$

The inverse transformation is determined by the formulas

$$B_1=\frac{1}{2\nu}(\frac{j}{2j+1}F_1+\frac{j+1}{2j+1}F_2),\quad B_2=\frac{1}{2j+1}F_1-\frac{1}{2j+1}F_2\,. \tag{4.12}$$

By combining the equations in (4.8), we shall transform them to the known functions F_1, F_2. First, we obtain

$$\Box F_1 = \frac{4\nu^2}{r^2} B_2 - e\mu (E + \frac{\alpha}{r}) \frac{2\nu^2}{r^3} B_2$$

$$+ \frac{4\nu(j+1)}{r^2} B_1 + \frac{2(j+1)}{r^2} B_2 + e\mu (j+1) \frac{d}{dr} (E + \frac{\alpha}{r}) \frac{1}{r^2} B_2 \,,$$

$$\Box F_2 = \frac{4\nu^2}{r^2} B_2 - e\mu (E + \frac{\alpha}{r}) \frac{2\nu^2}{r^3} B_2 - \frac{4\nu j}{r^2} B_1 - \frac{2j}{r^2} B_2 - e\mu j \frac{d}{dr} (E + \frac{\alpha}{r}) \frac{1}{r^2} B_2 \,.$$

We further get

$$\Box F_1 = 2(j+1) F_1 + e\mu (j+1) \left\{ -\frac{\alpha}{r^4} + (E + \frac{\alpha}{r}) \frac{1}{r^2} (\frac{d}{dr} - \frac{j+2}{r}) \right\} \frac{F_1 - F_2}{2j+1},$$

$$\Box F_2 = -2j F_2 - e\mu\, j \left\{ -\frac{\alpha}{r^4} + (E + \frac{\alpha}{r}) \frac{1}{r^2} (\frac{d}{dr} + \frac{j-1}{r}) \right\} \frac{F_1 - F_2}{2j+1} \,.$$

It is convenient to use the following combinations:

$$G = F_1 - F_2, \quad H = F_1 + F_2 \,, \tag{4.13}$$

so from previous equations we obtain

$$\Box G = 2jH + G + H + e\mu \left\{ -\frac{\alpha}{r^4} + (E + \frac{\alpha}{r}) \frac{1}{r^2} \frac{d}{dr} - (E + \frac{\alpha}{r}) \frac{2}{r} \right\} G \,,$$

$$\Box H = 2jG + G + H + e\mu \left\{ -\frac{\alpha}{r^4} + (E + \frac{\alpha}{r}) \frac{1}{r^2} \frac{d}{dr} - (E + \frac{\alpha}{r}) \frac{2(j^2+j+1)}{r} \right\} \frac{G}{2j+1} \,.$$

The last equations may be finally re-written as

$$\left\{ \Box - 1 + e\mu \,[-\frac{\alpha}{r^4} + (E + \frac{\alpha}{r}) \frac{1}{r^2} \frac{d}{dr} - (E + \frac{\alpha}{r}) \frac{2}{r}] \right\} G = (2j+1) H \,,$$

$$(\Box - 1)\ (2j+1) H = \left\{ (2j+1)^2 + e\mu \,[-\frac{\alpha}{r^4} + (E + \frac{\alpha}{r}) \frac{1}{r^2} \frac{d}{dr} - (E + \frac{\alpha}{r}) \frac{2(j^2+j+1)}{r}] \right\} G \,. \tag{4.14}$$

From (4.14), after excluding the function $H(r)$, we can obtain (see below) a 4-th order differential equation for the function $G(r)$.

5 The case of minimal value $j = 0$

In this section we consider the most simple case when the quantum number j takes on the minimal value $j = 0$. Here we must start with the more simple substitution

$$C(x) = e^{-i\epsilon t} C(r) D_0 \ , \quad C_0(x) = e^{-i\epsilon t} C_0(r) D_0 \ , \quad \Phi_0(x) = e^{-i\epsilon t} \Phi_0(r) D_0 \ ,$$

$$\vec{C}(x) = e^{-i\epsilon t} \left| \begin{array}{c} 0 \cdot D_{-1} \\ C_2(r) D_0 \\ 0 \cdot D_{+1} \end{array} \right| , \quad \vec{\Phi}(x) = e^{-i\epsilon t} \left| \begin{array}{c} 0 \cdot D_{-1} \\ \Phi_2(r) D_0 \\ 0 \cdot (r) D_{+1} \end{array} \right| ,$$

$$\vec{E}(x) = e^{-i\epsilon t} \left| \begin{array}{c} 0 \cdot D_{-1} \\ E_2(r) D_0 \\ 0 \cdot D_{+1} \end{array} \right| , \quad \vec{H}(x) = e^{-i\epsilon t} \left| \begin{array}{c} 0 \cdot D_{-1} \\ H_2(r) D_0 \\ 0 \cdot D_{+1} \end{array} \right| . \tag{5.1}$$

This wave function has the parity value $\Pi = (-1)^j = (-1)^0 = +1$, and the relevant radial system reads

$$\begin{gathered} -i(\epsilon + \frac{\alpha}{r}) C_0 - (\frac{d}{dr} + \frac{2}{r}) C_2 = mC \ , \\ -(\frac{d}{dr} + \frac{2}{r}) E_2 = mC_0 \ , \quad +i(\epsilon + \frac{\alpha}{r}) E_2 = mC_2 \ , \\ -i(\epsilon + \frac{\alpha}{r}) \sigma C - (\frac{d}{dr} + \frac{2}{r}) E_2 = m\Phi_0 \ , \\ i(\epsilon + \frac{\alpha}{r}) E_2 + \sigma \frac{d}{dr} C = m\Phi_2 \ , \\ -i(\epsilon + \frac{\alpha}{r}) \Phi_2 - \frac{d}{dr} \Phi_0 = mE_2 \ , \quad H_2 = 0 \ . \end{gathered} \tag{5.2}$$

The first three equations provide the following expression for $C(r)$

$$C(r) = \frac{i}{m^2} \left[(\epsilon + \frac{\alpha}{r})(\frac{d}{dr} + \frac{2}{r}) - (\frac{d}{dr} + \frac{2}{r})(\epsilon + \frac{\alpha}{r}) \right] E_2(r) = \frac{i\alpha}{m^2} \frac{1}{r^2} \ E_2(r) \ ; \tag{5.3}$$

after that we exclude the function $C(r)$ from the remaining three equations:

$$\begin{gathered} -i(\epsilon + \frac{\alpha}{r}) \sigma \frac{i\alpha}{m^2} \frac{1}{r^2} E_2(r) - (\frac{d}{dr} + \frac{2}{r}) E_2(r) = m\Phi_0(r) \ , \\ i(\epsilon + \frac{\alpha}{r}) E_2(r) + \sigma \frac{d}{dr} \frac{i\alpha}{m^2} \frac{1}{r^2} \ E_2(r) = m\Phi_2(r) \ , \\ -i(\epsilon + \frac{\alpha}{r}) \Phi_2(r) - \frac{d}{dr} \Phi_0 = mE_2(r) \ . \end{gathered} \tag{5.4}$$

Now we are to perform the non-relativistic approximation in (5.4). As the first step, we exclude the non-dynamical variable Φ_0:

$$\Phi_0 = \frac{1}{m} \left[\frac{\sigma\alpha}{m^2} (\epsilon + \frac{\alpha}{r}) \frac{1}{r^2} - (\frac{d}{dr} + \frac{2}{r}) \right] E_2 \ ,$$

so producing the following two equations

$$\begin{gathered} (\epsilon + \frac{\alpha}{r}) iE_2 + \frac{\sigma\alpha}{m^2} \frac{d}{dr} \frac{1}{r^2} \ iE_2 = m \ \Phi_2 \ , \\ (\epsilon + \frac{\alpha}{r}) \Phi_2 - \frac{d}{dr} \frac{\sigma\alpha}{m^2} \frac{1}{m} (\epsilon + \frac{\alpha}{r}) \frac{1}{r^2} iE_2 + \frac{d}{dr} \frac{1}{m} (\frac{d}{dr} + \frac{2}{r}) iE_2 = m \ iE_2 \ . \end{gathered} \tag{5.5}$$

By introducing the big and small components

$$\Phi_2 = (B_2 + M_2), \quad iE_2 = (B_2 - M_2)\,, \tag{5.6}$$

and by separating the rest energy (by the formal change $\epsilon \Longrightarrow M + \epsilon$), we derive

$$(m + \epsilon + \frac{\alpha}{r})(B_2 - M_2) + \frac{\sigma\alpha}{m^2}\frac{d}{dr}\frac{1}{r^2}\,(B_2 - M_2) = m\,(B_2 + M_2)\,,$$

$$(m + \epsilon + \frac{\alpha}{r})(B_2 + M_2) - \frac{d}{dr}\frac{\sigma\alpha}{m^2}\frac{1}{m}(\epsilon + \frac{\alpha}{r})\frac{1}{r^2}(B_2 - M_2)$$
$$+\frac{d}{dr}\frac{1}{m}(\frac{d}{dr} + \frac{2}{r})(B_2 - M_2) = m\,(B_2 - M_2)\,,$$

whence it follows

$$(\epsilon + \frac{\alpha}{r})(B_2 - M_2) + \frac{\sigma\alpha}{m^2}\frac{d}{dr}\frac{1}{r^2}\,(B_2 - M_2) = 2m\,M_2\,,$$

$$(\epsilon + \frac{\alpha}{r})(B_2 + M_2) - \frac{d}{dr}\frac{\sigma\alpha}{m^2}\frac{1}{m}\frac{1}{r}(\epsilon + \frac{\alpha}{r})\frac{1}{r^2}(B_2 - M_2) +$$
$$+\frac{d}{dr}\frac{1}{m}(\frac{d}{dr} + \frac{2}{r})(B_2 - M_2) = -2m\,M_2\,.$$

In order to obtain an equation for the big component B_2, we are to sum two last equations:

$$\left[\frac{d}{dr}(\frac{d}{dr} + \frac{2}{r}) + 2m(\epsilon + \frac{\alpha}{r}) + \frac{\sigma\alpha}{m}\frac{d}{dr}\frac{1}{r^2} - \frac{d}{dr}\frac{\sigma\alpha}{m}\frac{1}{m}(\epsilon + \frac{\alpha}{r})\frac{1}{r^2}\right] B_2 = 0\,.$$

This may be presented as follows (let $B_2(r) = F(r)$):

$$\left\{\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{-\sigma\alpha\epsilon/m^2 + \sigma\alpha/m}{r^2} - \frac{\alpha^2\sigma/m^2}{r^3}\right)\frac{d}{dr}\right.$$
$$\left. + \left(2m\epsilon + \frac{2m\alpha}{r} - \frac{2}{r^2} + \frac{-2\sigma\alpha/m + 2\sigma\alpha\epsilon/m^2}{r^3} + \frac{3\sigma\alpha^2/m^2}{r^4}\right)\right\} F = 0. \tag{5.7}$$

Here we have an equation with two irregular points: 0 and ∞, both of the rank 3. In the present time, we can construct only its formal Frobenius solutions, and we have no rule to get quantized values for energies. When σ equals to zero, from (5.7) there follows a simple equation, whose solutions for bound states may be constructed in terms of confluent hypergeometric functions.

Let us examine local Frobenius solutions of eq. (5.7) in vicinity of the point $r = 0$. In accordance with the rank of the singular point $r = 0$, we search such solutions in the form

$$F(r) = r^A e^{\frac{B}{r}} e^{\frac{C}{r^2}} f(r)\,; \tag{5.8}$$

which leads to an equations for $f(r)$:

$$\frac{d^2 f}{dr^2} + \left(\frac{2A+2}{r} + \frac{1}{r^2}\left(\frac{\sigma\alpha}{m} - \frac{\sigma\alpha\epsilon}{m^2} - 2B\right) - \frac{\sigma\alpha^2/m^2 + 4C}{r^3}\right)\frac{df}{dr}$$

$$+\left(2\,m\epsilon+2\,\frac{m\alpha}{r}+\frac{A^2+A-2}{r^2}+\frac{1}{r^3}\left(-2\frac{\sigma\alpha}{m}-\frac{\sigma\alpha\epsilon A}{m^2}-2\,AB+\frac{\sigma\alpha A}{m}+2\frac{\sigma\alpha\epsilon}{m^2}\right)\right.$$
$$+\frac{1}{r^4}\left(2C-4AC-\frac{\alpha^2\sigma A}{m^2}-\frac{\sigma\alpha B}{m}+B^2+\frac{\sigma\alpha\epsilon B}{m^2}+3\,\frac{\sigma\,\alpha^2}{m^2}\right)$$
$$\left.+\frac{4BC+\alpha^2\sigma\,B/m^2-2\sigma\alpha C/m+2\alpha\sigma\epsilon C/m^2}{r^5}+\frac{4C^2+2\alpha^2\sigma C/m^2}{r^6}\right)f=0.$$

Let the coefficient at $\frac{1}{r^6}$ vanish

$$4C^2+2\alpha^2\sigma C/m^2=0 \quad\Longrightarrow\quad C_1=0,\quad C_2=-\frac{\alpha^2\sigma}{2m^2}\,. \tag{5.9}$$

Also let the coefficient at $\frac{1}{r^5}$ vanish,

$$B=\frac{2\alpha\sigma C(m-\epsilon)}{4Cm^2+\alpha^2\sigma}=0 \quad\Longrightarrow\quad B_1=0,\quad B_2=\frac{\alpha\sigma(m-\epsilon)}{m^2}\,, \tag{5.10}$$

and finally, let the coefficient at $\frac{1}{r^4}$ vanish

$$A=\frac{B^2m^2-Bm\alpha\sigma+B\alpha\epsilon\sigma+2Cm^2+3\alpha^2\sigma}{4Cm^2+\alpha^2\sigma}\Longrightarrow A_1=3, A_2=-2\,. \tag{5.11}$$

Thus, we have two types of solutions, respectively functions $f_1(r)$ and $f_2(r)$. For the case

$$A_1=3, B_1=0, C_1=0 \tag{5.12}$$

we have the equation

$$\frac{d^2f_1}{dr^2}+\left(\frac{8}{r}+\frac{\sigma\,\alpha/m-\sigma\,\alpha\,\epsilon/m^2}{r^2}-\frac{\sigma\,\alpha^2/m^2}{r^3}\right)\frac{df_1}{dr}$$
$$+\left(2\,m\epsilon+\frac{2m\alpha}{r}+\frac{10}{r^2}+\frac{\sigma\,\alpha/m-\sigma\,\alpha\,\epsilon/m^2}{r^3}\right)f_1=0\,.$$

For the case

$$A_2=-2, B_2=\frac{\alpha\sigma(m-\epsilon)}{m^2}, C_2=-\frac{\alpha^2\sigma}{2m^2} \tag{5.13}$$

we have the equation

$$\frac{d^2f_2}{dr^2}+\left(-\frac{2}{r}-\frac{\sigma\,\alpha(m-\epsilon)/m^2}{r^2}+\frac{\sigma\,\alpha^2/m^2}{r^3}\right)\frac{df_2}{dr}+\left(2\,m\epsilon+\frac{2m\alpha}{r}\right)f_2=0\,.$$

These two equations have a similar structure. Let us follow both of them, using symbolical notations

$$f''+\left(\frac{a_1}{r}+\frac{a_2}{r^2}+\frac{a_3}{r^3}\right)f'+\left(b+\frac{b_1}{r}+\frac{b_2}{r^2}+\frac{b_3}{r^3}\right)f=0\,. \tag{5.14}$$

The solutions of the last equation can be constructed in the form of power series $f = \sum_{n=0}^{\infty} c_n r^n$; we arrive at the 5-term recurrent relations

$$bc_{n-3}+b_1 c_{n-2}+[(n-1)(n-2)+a_1(n-1)+b_2]c_{n-1}+[a_2 n+b_3]c_n+[a_3(n+1)]c_{n+1} = 0\,.$$

In accordance with the Poincaré-Perrone method, let us divide the last equation by c_{n-3}:

$$b + b_1 \frac{c_{n-2}}{c_{n-3}} + [(n-1)(n-2) + a_1(n-1) + b_2]\frac{c_{n-1}}{c_{n-2}}\frac{c_{n-2}}{c_{n-3}}$$
$$+[a_2 n + b_3]\frac{c_n}{c_{n-1}}\frac{c_{n-1}}{c_{n-2}}\frac{c_{n-2}}{c_{n-3}} + [a_3(n+1)]\frac{c_{n+1}}{c_n}\frac{c_n}{c_{n-1}}\frac{c_{n-1}}{c_{n-2}}\frac{c_{n-2}}{c_{n-3}} = 0\,.$$

Then let us multiply this result by n^{-2} and tend $n \to \infty$, so we get a simple algebraic equation for the quantity R, which determines the possible convergence radius R_{conv} of the series:

$$R = \lim_{n\to\infty} \frac{c_n}{c_{n-1}}, \qquad R^2 = 0, \qquad R_{conv} = \frac{1}{|R|} = \infty.$$

Thus, we have constructed two linearly independent solutions

$$F_1(r) = r^3\, f(r)\,,$$
$$F_2(r) = \frac{1}{r^2}\ \exp(+\frac{\alpha\sigma(m-\epsilon)}{m^2 r})\ \exp(-\frac{\sigma\alpha^2/2}{m^2 r^2})\ f_2(r)\,. \tag{5.15}$$

It should be stressed that the sign of the parameter σ substantially influences the behavior of the exponential multipliers near the point $r = 0$. The main candidates for solutions related to bound states are the functions of the type $F_2(r)$ at $\sigma > 0$.

6 Frobenius solutions of the 4-th order equation

Now we turn to the system (4.14). After excluding the function $H(r)$, we obtain the following 4-th order differential equation for $G(r)$:

$$\frac{d^4G}{dr^4} + \left[\frac{4}{r} + \frac{\mu\,E}{r^2} + \frac{\mu\,\alpha}{r^3}\right]\frac{d^3G}{dr^3}$$
$$+\left[\frac{-2\,\mu\,E + 4\,m\alpha}{r} + \frac{-2\,\mu\,\alpha - 2\,j\,(j+1)}{r^2} - \frac{2\,\mu\,E}{r^3} - \frac{5\,\mu\,\alpha}{r^4} + 4\,mE - 2\right]\frac{d^2G}{dr^2}$$
$$+\left[\frac{-4 + 8\,mE}{r} + \frac{2\,mE^2\mu - 2\,\mu\,E + 4\,m\alpha}{r^2} + \frac{2\,(2\,mE+1)\,\mu\,\alpha}{r^3}\right.$$
$$\left.-\frac{\big(Ej\,(j+1) - 2\,m\alpha^2 - 2\,E\big)\,\mu}{r^4} - \frac{\big(j^2 + j - 12\big)\,\mu\,\alpha}{r^5}\right]\frac{dG}{dr}$$
$$+\left[\frac{2\,j^2\mu\,E + 2\,j\mu\,E - 4\,mE^2\mu + 4\,\mu\,E + 8\,m^2E\alpha - 4\,m\alpha}{r}\right.$$
$$+\frac{j\,(j+1)\,(-4\,mE + 2 + 2\,\mu\,\alpha) + 4\,m^2\alpha^2 + (-8\,mE + 4)\,\mu\,\alpha}{r^2}$$

$$+\frac{2\,j^2\mu\,E-4\,m\alpha\,j^2-4\,jm\alpha+2\,j\mu\,E-4\,m\mu\,\alpha^2}{r^3}$$
$$+\frac{j^4+2\,j^3-j^2-2\,j+2\,j\mu\,\alpha\,(j+1)-2\,\mu\,\alpha\,(mE+1)}{r^4}$$
$$-\frac{2\,m\mu\,\alpha^2}{r^5}+\frac{\left(j^2+j-12\right)\mu\,\alpha}{r^6}-4\,j\,(j+1)-4\,mE+4\,m^2E^2\Big]\,G=0\,.$$

Let us take the substitution

$$G=r^Ae^{B/r}e^{L/r^2}f(r)\,; \tag{6.1}$$

this results in an equation for $f(r)$:

$$\frac{d^4f}{dr^4}+(\frac{E\mu-4\,B}{r^2}+\frac{\mu\,\alpha-8\,L}{r^3}+\frac{4\,A+4}{r})\frac{d^3f}{dr^3}$$
$$+(-2+4\,mE+\frac{-2\,E\mu+4\,m\alpha}{r}+\frac{6\,A^2-2\,j^2-2\,\mu\,\alpha+6\,A-2\,j}{r^2}+\frac{3\,\mu\,EA-12\,AB-2\,E\mu}{r^3}$$
$$+\frac{3A\mu\alpha-3BE\mu-24AL+6B^2-5\mu\alpha+12L}{r^4}+\frac{-3\mu\alpha B-6EL\mu+24BL}{r^5}+\frac{-6L\mu\alpha+24L^2}{r^6})\frac{d^2f}{dr^2}$$
$$+(\frac{8AEm+8\,mE-4A-4}{r}+\frac{2E^2m\mu-4\mu EA+8Am\alpha-8BEm-2E\mu+4m\alpha+4\,B}{r^2}$$
$$+\frac{4Em\mu\alpha+4A^3-4Aj^2-4\,A\mu\alpha+4BE\mu-8Bm\alpha-16ELm-4Aj+2\mu\alpha-4A+8L}{r^3}$$
$$+\frac{1}{r^4}[3\mu EA^2-Ej^2\mu+2m\mu\alpha^2-12A^2B-7\mu EA+4j^2B+4\mu\alpha B+8EL\mu-Ej\mu-16m\alpha L$$
$$+12\,AB+4\,jB+2\,E\mu]$$
$$+\frac{1}{r^5}[3\mu\alpha An^2-6ABE\mu-j^2\mu\alpha-24A^2L+12AB^2-13A\mu\alpha+10BE\mu+8Lj^2+8L\mu\alpha-j\mu\alpha$$
$$+48\,AL-12\,B^2+8\,Lj+12\,\mu\,\alpha-24L]$$
$$+\frac{-6AB\mu\alpha-12AEL\mu+3\,B^2E\mu+48ABL-4B^3+16\mu\alpha B+26EL\mu-72BL}{r^6}$$
$$+\frac{-12AL\mu\alpha+3B^2\mu\alpha+12BEL\mu+48AL^2-24B^2L+38L\mu\alpha-96L^2}{r^7}$$
$$+\frac{12BL\mu\alpha+12EL^2\mu-48BL^2}{r^8}+\frac{12L^2\mu\alpha-32L^3}{r^9})\frac{df}{dr}$$
$$+(-4mE+4m^2E^2-4j-4j^2+\frac{-4E^2m\mu+2Ej^2\mu+8Em^2\alpha+2Ej\mu+4E\mu-4m\alpha}{r}$$
$$+\frac{4A^2Em-4Ej^2m-8Em\mu\alpha+2j^2\mu\alpha+4m^2\alpha^2+4AEm-4Ejm+2j\mu\alpha-2A^2+2j^2+4\mu\alpha-2A+2j}{r^2}$$
$$+\frac{2E^2m\mu A-2\mu\,EA^2+4m\alpha A^2-8mEAB+2Ej^2\mu-4j^2m\alpha-4m\mu\alpha^2+2Ej\mu-4jm\alpha+4AB}{r^3}$$

$$+\frac{1}{r^4}[4AEm\mu\alpha - 2E^2m\mu B + A^4 - 2A^2j^2 - 2\mu\alpha A^2 + 4ABE\mu - 8m\alpha AB - 16mEAL + 4B^2Em$$
$$-2Em\mu\alpha + j^4 + 2j^2\mu\alpha - 2A^3 - 2A^2j + 2Aj^2 + 4A\mu\alpha - 2BE\mu + 4Bm\alpha + 8ELm + 2j^3 + 2j\mu\alpha$$
$$-A^2 + 8\,AL + 2\,Aj - 2\,B^2 - j^2 - 2\,\mu\,\alpha + 2\,A - 4\,L - 2\,j]$$
$$+\frac{1}{r^5}[A^3E\mu - AEj^2\mu + 2Am\mu\alpha^2 - 4BEm\mu\alpha - 4E^2Lm\mu - 4A^3B - 5\mu EA^2 + 4ABj^2 + 4AB\mu\alpha$$
$$+8AEL\mu - AEj\mu - 16ALm\alpha - 2B^2E\mu + 4B^2m\alpha + 16BELm - 2m\mu\alpha^2 + 12A^2B + 4ABj$$
$$+6\mu EA - 4j^2B - 6\mu\alpha B - 8EL\mu + 16m\alpha L - 8AB - 8BL - 4jB]$$
$$+\frac{1}{r^6}[A^3\mu\alpha - 3A^2BE\mu - Aj^2\mu\alpha + BEj^2\mu - 2Bm\mu\alpha^2 - 8ELm\mu\alpha - 8A^3L + 6A^2B^2 - 8\mu\alpha A^2$$
$$+13ABE\mu + 8ALj^2 + 8AL\mu\,\alpha - Aj\mu\alpha - 2B^2j^2 - 2B^2\mu\alpha - 8BEL\mu + BEj\mu + 16BLm\alpha + 16EL^2m$$
$$+j^2\mu\alpha + 36A^2L - 18AB^2 + 8ALj + 19A\mu\alpha - 2B^2j - 12BE\mu - 12Lj^2 - 16L\mu\alpha + j\mu\alpha - 52AL$$
$$+12\,B^2 - 8\,L^2 - 12\,Lj - 12\,\mu\,\alpha + 24\,L]$$
$$+\frac{1}{r^7}[-3A^2B\mu\alpha - 6A^2EL\mu + 3AB^2E\mu + Bj^2\mu\alpha + 2ELj^2\mu - 4Lm\mu\alpha^2 + 24A^2BL - 4\,AB^3$$
$$+19AB\mu\alpha + 32AEL\mu - 8B^2E\mu - 8BLj^2 - 8BL\mu\alpha + Bj\mu\alpha - 8EL^2\mu + 2ELj\mu + 16L^2m\alpha - 96ABL$$
$$+8B^3 - 8BLj - 28\mu\alpha\,B - 40EL\mu + 96\,BL]$$
$$+\frac{1}{r^8}[-6A^2L\mu\alpha + 3AB^2\mu\alpha + 12ABEL\mu - B^3E\mu + 2Lj^2\mu\alpha + 24A^2L^2 - 24AB^2L + 44AL\mu\alpha$$
$$+B^4 - 11B^2\mu\alpha - 38BEL\mu - 8L^2j^2 - 8L^2\mu\alpha$$
$$+2Lj\mu\alpha - 120AL^2 + 60B^2L - 8L^2j - 78L\mu\alpha + 156L^2]$$
$$+\frac{12ABL\mu\alpha + 12AEL^2\mu - B^3\mu\alpha - 6B^2EL\mu - 48ABL^2 + 8B^3L - 50BL\mu\alpha - 44EL^2\mu + 144BL^2}{r^9}$$
$$+\frac{12AL^2\mu\alpha - 6B^2L\mu\alpha - 12BEL^2\mu - 32AL^3 + 24B^2L^2 - 56L^2\mu\alpha + 112L^3}{r^{10}}$$
$$+\frac{-12\,BL^2\mu\,\alpha - 8\,EL^3\mu + 32\,BL^3}{r^{11}} + \frac{-8\,L^3\mu\,\alpha + 16\,L^4}{r^{12}})f = 0.$$

Let the coefficient at r^{-12} vanish; then the 4-th order algebraic equation leads to two different roots L:

$$-8\,L^3\mu\,\alpha + 16\,L^4 = 0 \quad \Longrightarrow \quad L_1 = \frac{1}{2}\mu\alpha, \quad L_{2,3,4} = 0. \tag{6.2}$$

For the case $L_1 = \frac{1}{2}\mu\alpha$, let the coefficient at r^{-11} vanish; this yields

$$-12\,BL^2\mu\,\alpha - 8\,EL^3\mu + 32\,BL^3 = 0 \quad \Longrightarrow \quad B_1 = E\mu\,. \tag{6.3}$$

When

$$L_1 = \frac{1}{2}\mu\alpha, \qquad B_1 = E\mu\,,$$

let the coefficient at r^{-10} vanish:

$$-A\mu^3\alpha^3 = 0 \Rightarrow A_1 = 0\,. \tag{6.4}$$

For the variant L_1, B_1, A_1, the previous equation takes the form

$$\frac{d^4 f}{dr^4} + (-3\frac{\mu\alpha}{r^3} - 3\frac{E\mu}{r^2} + \frac{4}{r})\frac{d^3 f}{dr^3}$$

$$+(-2+\frac{-2E\mu+4m\alpha}{r}+\frac{-2j^2-2\mu\alpha-2j}{r^2}-2\frac{E\mu}{r^3}+\frac{3E^2\mu^2+\mu\alpha}{r^4}+4mE+6\frac{E\mu^2\alpha}{r^5}+3\frac{\mu^2\alpha^2}{r^6})\frac{d^2 f}{dr^2}$$

$$+(\frac{8\,mE-4}{r} + \frac{-6\,E^2 m\mu + 2\,E\mu + 4\,m\alpha}{r^2} + \frac{4\,E^2\mu^2 - 12\,Em\mu\,\alpha + 6\,\mu\,\alpha}{r^3}$$

$$+\frac{3Ej^2\mu + 8E\mu^2\alpha - 6mu\,\alpha^2 + 3Ej\mu + 2E\mu}{r^4} + \frac{-2E^2\mu^2 + 3j^2\mu\alpha + 4\mu^2\alpha^2 + 3j\mu\alpha}{r^5}$$

$$+\frac{-E^3\mu^3 - 7E\mu^2\alpha}{r^6} + \frac{-3E^2\mu^3\alpha - 5\mu^2\alpha^2}{r^7} - 3\frac{E\mu^3\alpha^2}{r^8} - \frac{\mu^3\alpha^3}{r^9})\frac{df}{dr}$$

$$+(-4mE + \frac{2\,Ej^2\mu - 4j^2m\alpha - 4m\mu\alpha^2 + 2Ej\mu - 4jm\alpha}{r^3}$$

$$+4\,m^2E^2 + \frac{2E^3\mu^2 m - 4E^2\mu^2 + 6Em\mu\alpha + j^4 + 2j^2\mu\alpha + 2j^3 + 2j\mu\alpha - j^2 - 4\mu\alpha - 2j}{r^4}$$

$$+\frac{-2E^3\mu^3 + 6E^2\mu^2 m\alpha - 4\,Ej^2\mu - 14E\mu^2\alpha + 6m\mu\alpha^2 - 4Ej\mu}{r^5} - 4j$$

$$+\frac{-E^2\mu^2 j^2 - 6E^2\mu^3\alpha + 6E\mu^2 m\alpha^2 - E^2\mu^2 j - 5j^2\mu\alpha - 10\,\mu^2\alpha^2 - 5j\mu\alpha}{r^6}$$

$$-4j^2 + \frac{-2E\mu^2 j^2\alpha - 6E\mu^3\alpha^2 + 2\mu^2\alpha^3 m - 2E\mu^2 j\alpha}{r^7}$$

$$+\frac{-4E^2 m\mu + 2Ej^2\mu + 8Em^2\alpha + 2Ej\mu + 4E\mu - 4m\alpha}{r}$$

$$+\frac{-\mu^2\alpha^2 j^2 - 2\,\mu^3\alpha^3 - \mu^2\alpha^2 j}{r^8}$$

$$+\frac{-4Ej^2m - 8Em\mu\alpha + 2j^2\mu\alpha + 4\,m^2\alpha^2 - 4Ejm + 2j\mu\alpha + 2\,j^2 + 4\mu\alpha + 2j}{r^2})f = 0\,.$$

Its structure shortly may be presented as follows

$$f'''' + (\frac{a_1}{r} + ... + \frac{a_3}{r^3})f''' + (b + \frac{b_1}{r} + ... + \frac{b_6}{r^6})f''$$
$$+(\frac{c_1}{r} + ... + \frac{c_8}{r^9})f' + (d + \frac{d_1}{r} + ... + \frac{d_8}{r^8})f = 0\,. \tag{6.5}$$

Now, we consider the case when (see (6.2))

$$L_{2,3,4} = 0\,; \tag{6.6}$$

it is readily checked that the coefficients at r^{-11}, r^{-10} are equal to zero. Let the coefficient at r^{-9} vanish; this yields

$$-B^3\mu\alpha = 0 \quad \Longrightarrow \quad B_{2,3,4} = 0\,. \tag{6.7}$$

We readily verify that coefficient at r^8, r^{-7} also vanish. Let the coefficient at r^{-6} be zero; this results in the 3-rd order equation with respect to parameter A:

$$A^3\mu\alpha - Aj^2\mu\alpha - 8A^2\mu\alpha - Aj\mu\alpha + j^2\mu\alpha + 19A\mu\alpha + j\mu\alpha - 12\mu\alpha = 0\,,$$

its roots are $A_{2,3,4}$:

$$A_2 = 1\,, \quad A_3 = 3 - j\,, \quad A_4 = 4 + j\,. \tag{6.8}$$

For the description of bound states, the cases $A_2 = 1, A_4 = 4 + j$ seem to be suitable. Explicitly, the equations for these most interesting cases look as follows:

$$L_2 = 0, \quad B_2 = 0, \quad A_2 = 1, \quad \frac{d^4 f}{dr^4} + (\frac{E\mu}{r^2} + \frac{\mu\,\alpha}{r^3} + 8\,r^{-1})\frac{d^3 f}{dr^3} +$$
$$+(-2 - 2\frac{\mu\alpha}{r^4} + \frac{E\mu}{r^3} + \frac{-2E\mu + 4m\alpha}{r} + \frac{-2j^2 - 2\mu\alpha - 2j + 12}{r^2} + 4mE)\frac{d^2 f}{dr^2}$$
$$+(\frac{-Ej^2\mu + 2m\mu\alpha^2 - Ej\mu - 2E\mu}{r^4} + \frac{16mE - 8}{r} + \frac{2E^2 m\mu - 6E\mu + 12m\alpha}{r^2}$$
$$+\frac{4\,Em\mu\alpha - 4j^2 - 2\mu\alpha - 4j}{r^3} + \frac{-j^2\mu\alpha - j\mu\alpha + 2\mu\alpha}{r^5})\frac{df}{dr}$$
$$+(-4mE + \frac{2E^2 m\mu + 2Ej^2\mu - 4j^2 m\alpha - 4m\mu\alpha^2 + 2Ej\mu - 4jm\alpha - 2E\mu + 4m\alpha}{r^3} + 4m^2E^2$$
$$+\frac{2\,Em\mu\alpha + j^4 + 2j^2\mu\alpha + 2j^3 + 2j\mu\,\alpha - j^2 - 2j}{r^4} + \frac{-Ej^2\mu - Ej\mu + 2E\mu}{r^5} - 4j - 4\,j^2$$
$$+\frac{-4E^2 m\mu + 2Ej^2\mu + 8Em^2\alpha + 2Ej\mu + 4E\mu - 4m\alpha}{r}$$
$$+\frac{-4Ej^2 m - 8Em\mu\alpha + 2j^2\mu\alpha + 4m^2\alpha^2 - 4Ejm + 2j\mu\alpha + 8mE + 2j^2 + 4\mu\alpha + 2j - 4}{r^2})f = 0\,;$$

(6.9)

and

$$L_4 = 0, \qquad B_4 = 0, A_4 = 4 + j, \quad \frac{d^4 f}{dr^4} + \left(\frac{E\mu}{r^2} + \frac{\mu\,\alpha}{r^3} + \frac{20 + 4\,j}{r}\right)\frac{d^3 f}{dr^3}$$
$$+(-2 + \frac{3(4 + j)\mu\alpha - 5\mu\alpha}{r^4} + \frac{3(4 + j)E\mu - 2E\mu}{r^3} + \frac{-2E\mu + 4m\alpha}{r}$$
$$+\frac{6(4 + j)^2 - 2j^2 - 2\mu\alpha + 24 + 4j}{r^2} + 4mE)\frac{d^2}{dr^2}$$

$$+(\frac{3(4+j)^2E\mu - Ej^2\mu + 2m\mu\alpha^2 - 7(4+j)E\mu - Ej\mu + 2E\mu}{r^4} + \frac{8(4+j)Em + 8mE - 20 - 4j}{r}$$

$$+\frac{2E^2m\mu - 4(4+j)E\mu + 8\,(4+j)\,m\alpha - 2E\mu + 4m\alpha}{r^2}$$

$$+\frac{4Em\mu\alpha + 4(4+j)^3 - 4(4+j)j^2 - 4(4+j)\mu\alpha - 16 - 4(4+j)j + 2\mu\alpha - 4j}{r^3}$$

$$+\frac{3(4+j)^2\mu\alpha - j^2\mu\alpha - 13(4+j)\mu\alpha - j\mu\alpha + 12\mu\alpha}{r^5})\frac{df}{dr}$$

$$+(-4\,mE + 4\,m^2E^2$$

$$+\frac{2(4+j)E^2m\mu - 2(4+j)^2E\mu + 4(4+j)^2m\alpha + 2Ej^2\mu - 4j^2m\alpha - 4m\mu\alpha^2 + 2Ej\mu - 4jm\alpha}{r^3}$$

$$+\frac{1}{r^4}[8+4\,(4+j)Em\mu\alpha+2j^2\mu\alpha+2j\mu\alpha-2Em\mu\alpha+4(4+j)\mu\alpha-2(4+j)^2\mu alpha+(4+j)^4$$

$$-(4+j)^2 - 2(4+j)^3 - j^2 + j^4 + 2j^3 - 2(4+j)^2j^2 - 2(4+j)^2j + 2(4+j)j^2 + 2(4+j)j - 2\mu\alpha]$$

$$+\frac{-(4+j)Ej^2\mu + 2(4+j)m\mu\alpha^2 - (4+j)Ej\mu - 2m\mu\alpha^2 + 6(4+j)E\mu - 5(4+j)^2E\mu + (4+j)^3E\mu}{r^5}$$

$$-4\,j - 4\,j^2 + \frac{-4\,E^2m\mu + 2\,Ej^2\mu + 8\,Em^2\alpha + 2\,Ej\mu + 4\,E\mu - 4\,m\alpha}{r}$$

$$+\frac{1}{r^2}[44+j)^2Em - 4\,Ej^2m - 8\,Em\mu\,\alpha + 2\,j^2\mu\,\alpha + 4\,m^2\alpha^2 + 4(4+j)Em - 4Ejm$$

$$+2\,j\mu\,\alpha - 2(4+j)^2 + 2\,j^2 + 4\mu\alpha - 8])f = 0\,. \tag{6.10}$$

The two equations for the functions $f(r)$ in the cases

$$L_{2,,4} = 0\,, \quad B_{2,4} = 0, \quad A = A_2, A_4$$

have a similar general structure (to avoid misunderstandings, we change the notation $f(r) \rightsquigarrow g(r)$)

$$g'''' + (\frac{a_1}{r} + ... + \frac{a_3}{r^3})g''' + (b + \frac{b_1}{r} + ... + \frac{b_4}{r^4})g''$$

$$+(\frac{c_1}{r} + ... + \frac{c_5}{r^5})g' + (d + \frac{d_1}{r} + ... + \frac{d_5}{r^5})g = 0\,. \tag{6.11}$$

The solutions of equation (6.11) may be constructed in the form of power series. Indeed, multiplying (6.11) by r^5, we get

$$r^5g'''' + (a_1r^4 + a_2r^3 + a_3r^2)g''' + (br^5 + b_1r^4 + b_2r^3 + b_3r^2 + b_4r)g'' +$$

$$+(c_1r^4 + c_2r^3 + c_3r^2 + c_4r + c_5)g' + (dr^5 + d_1r^4 + d_2r^3 + d_3r^2 + d_4r + d_5)g = 0\,; \tag{6.12}$$

let it be

$$g = \sum_{n=0}^{\infty} c_nr^n, \qquad g' = \sum_{n=1}^{\infty} nc_nr^{n-1}, \qquad g'' = \sum_{n=2}^{\infty} n(n-1)c_nr^{n-2},$$

$$g''' = \sum_{n=3}^{\infty} n(n-1)(n-2)c_n r^{n-3}, \qquad g'''' = \sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)c_n r^{n-4}.$$

Equation (6.12) gives

$$\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)c_n r^{n+1}$$

$$+a_1 \sum_{n=3}^{\infty} n(n-1)(n-2)c_n r^{n+1} + a_2 \sum_{n=3}^{\infty} n(n-1)(n-2)c_n r^{n} + a_3 \sum_{n=3}^{\infty} n(n-1)(n-2)c_n r^{n-1}$$

$$+b \sum_{n=2}^{\infty} n(n-1)c_n r^{n+3} + b_1 \sum_{n=2}^{\infty} n(n-1)c_n r^{n+2}$$

$$+b_2 \sum_{n=2}^{\infty} n(n-1)c_n r^{n+1} + b_3 \sum_{n=2}^{\infty} n(n-1)c_n r^{n} + b_4 \sum_{n=2}^{\infty} n(n-1)c_n r^{n-1}$$

$$+c_1 \sum_{n=1}^{\infty} nc_n r^{n+3} + c_2 \sum_{n=1}^{\infty} nc_n r^{n+2} + c_3 \sum_{n=1}^{\infty} nc_n r^{n+1} + c_4 \sum_{n=1}^{\infty} nc_n r^{n} + c_5 \sum_{n=1}^{\infty} nc_n r^{n-1}$$

$$+d \sum_{n=0}^{\infty} c_n r^{n+5} + d_1 \sum_{n=0}^{\infty} c_n r^{n+4} + d_2 \sum_{n=0}^{\infty} c_n r^{n+3} + d_3 \sum_{n=0}^{\infty} c_n r^{n+2} + d_4 \sum_{n=0}^{\infty} c_n r^{n+1} + d_5 \sum_{n=0}^{\infty} c_n r^{n} = 0.$$

Further, we construct the recurrent formulas for the coefficients:

$$n = 0, \qquad c_5 c_1 + d_5 c_0 = 0;$$

$$n = 1, \qquad 2b_4 c_2 + c_4 c_1 + 2c_5 c_2 + d_4 c_0 + d_5 c_1 = 0;$$

$$n = 2, \qquad 6a_3 c_3 + 2b_3 c_2 + 6b_4 c_3 + c_3 c_1 + 2c_4 c_2 + 3c_5 c_3 + d_3 c_0 + d_4 c_1 + d_5 c_2 = 0;$$

$$n = 3, \qquad 6a_2 c_3 + 24a_3 c_4 + 2b_2 c_2 + 6b_3 c_3 + 12b_4 c_4 + c_2 c_1$$
$$+2c_3 c_2 + 3c_4 c_2 + 4c_5 c_4 + d_2 c_0 + d_3 c_1 + d_4 c_2 + d_5 c_3 = 0;$$

$$n = 4, \qquad 6a_1 c_3 + 24a_2 c_4 + 60a_3 c_5 + 2b_1 c_2 + 6b_2 c_3 + 12b_3 c_4 + 20b_4 c_5 + c_1 c_1 + 2c_2 c_2$$
$$+3c_3 c_3 + 4c_4 c_4 + 5c_5 c_5 + d_1 c_0 + d_2 c_1 + d_3 c_2 + d_4 c_3 + d_5 c_4 = 0;$$

$$n = 5, \qquad 24c_4 + 24a_1 c_4 + 60a_2 c_5 + 120a_3 c_6 + 2bc_2 + 6b_1 c_3 + 12b_2 c_4 + 20b_3 c_5 + 30b_4 c_6 +$$
$$+c_1 c_2 + 3c_2 c_3 + 4c_3 c_4 + 5c_4 c_5 + 6c_5 c_6 + dc_0 + d_1 c_1 + d_2 c_2 + d_3 c_3 + d_4 c_4 + d_5 c_5 = 0;$$

$$n = 6, 7, 8... \qquad (n-1)(n-2)(n-3)(n-4)c_{n-1}$$
$$+a_1(n-1)(n-2)(n-3)c_{n-1} + a_2 n(n-1)(n-2)c_n + a_3(n+1)n(n-1)c_{n+1}$$
$$+b(n-3)(n-4)c_{n-3} + b_1(n-2)(n-3)c_{n-2} + b_2(n-1)(n-2)c_{n-1} + b_3 n(n-1)c_n + b_4 n(n+1)c_{n+1}$$
$$+c_1(n-3)c_{n-3} + c_2(n-2)c_{n-2} + c_3(n-1)c_{n-1} + c_4 nc_n + c_5(n+1)c_{n+1}$$
$$+dc_{n-5} + d_1 c_{n-4} + d_2 c_{n-3} + d_3 c_{n-2} + d_4 c_{n-1} + d_5 c_n = 0.$$

Thus, there arise the 7-term recurrent relations ($n = 6, 7, ...$)

$$dc_{n-5} + d_1 c_{n-4} + [b(n-3)(n-4) + c_1(n-3) + d_2]c_{n-3}$$

$$+[b_1(n-2)(n-3)+c_2(n-2)+d_3]c_{n-2}$$

$$+[(n-1)(n-2)(n-3)(n-4)+a_1(n-1)(n-2)(n-3)+b_2(n-1)(n-2)+c_3(n-1)+d_4]\ c_{n-1}$$

$$+[a_2 n(n-1)(n-2)+b_3 n(n-1)+c_4 n+d_5]c_n$$

$$+[a_3(n+1)n(n-1)+b_4 n(n+1)+c_5(n+1)]c_{n+1}=0\,. \tag{6.13}$$

Multiplying eq. (6.13) by n^{-4} and tending $n \to \infty$, we derive the algebraic equation for the quantity R, which determines the possible convergence radius R_{conv} of the power series:

$$R=\lim_{n\to\infty}\frac{c_n}{c_{n-1}}, \qquad R^4=0, \qquad R_{conv}=\frac{1}{|R|}=\infty. \tag{6.14}$$

7 Conclusions

All the constructed solutions are exact, but in a sense they are formal, because in the present time we do not know the rules to get discrete energy levels which might be associated with physical bound states.

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