

# GENERALIZED FINSLER SUPERSPACES

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## Abstract

The theory of locally anisotropic superspaces (supersymmetric generalizations of various types of Kaluza–Klein, Lagrange and Finsler spaces) is laid down. In this framework we perform the analysis of construction of the supervector bundles provided with nonlinear and distinguished connections and metric structures. Two models of locally anisotropic supergravity are proposed and studied in details.

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## 1 Introduction

Differential geometric techniques plays an important role in formulation and mathematical formalization of models of fundamental interactions of physical fields. In the last twenty years there has been a substantial interest in the construction of differential supergeometry with the aim of getting a framework for the supersymmetric field theories (the theory of graded manifolds [1-4] and the theory of supermanifolds [5-9]). Detailed considerations of geometric and topological aspects of supermanifolds and formulation of superanalysis are contained in [10-16].

Spaces with local anisotropy are used in some divisions of theoretical and mathematical physics [17-20] (recent applications in physics and biology are summarized in [21,22]). The first models of locally anisotropic (la) spaces (la-spaces) have been proposed by P.Finsler [23] and E.Cartan [24]. Early approaches and modern treatments of Finsler geometry and its extensions can be found in [25-30]. We shall use the general approach to the geometry of la-spaces, developed by R.Miron and M.Anastasiu [26,27], as a starting point for our definition of superspaces with local anisotropy and formulation of la-supergravitational models.

In different models of la-spaces one considers nonlinear and linear connections and metric structures in vector and tangent bundles on locally isotropic space-times ((pseudo)-Riemannian, Einstein-Cartan and more general types of curved spaces with torsion and nonmetricity). It seems likely that la-spaces make up a more convenient geometric background for developing in a selfconsistent manner classical and quantum statistical and field theories in non homogeneous, dispersive media with radiational, turbulent and random processes. In [31-35] some variants of Yang-Mills, gauge gravity and the definition of spinors on la-spaces have been proposed. In connection with the above mentioned the formulation of supersymmetric extensions of classical and quantum field theories on la-spaces presents a certain interest

In works [36-38] a new viewpoint on differential geometry of supermanifolds is discussed. The author introduced the nonlinear connection (N-connection) structure and developed a corresponding distinguished by N-connection supertensor covariant differential calculus in the frame of De Witt [5] approach to supermanifolds, by considering the particular case of superbundles with typical fibres parametrized by noncommutative coordinates. This is the first example of superspace with local anisotropy. But up to the present we have not a general, rigorous mathematical, definition of locally anisotropic superspaces (la-superspaces).

In this paper we intend to give some contributions to the theory of vector and tangent superbundles provided with nonlinear and distinguished connections and metric structures (a generalized model of la-superspaces). Such superbundles contain as particular cases the supersymmetric extensions of Lagrange and Finsler spaces. We shall also formulate and analyze two models of locally anisotropic supergravity.

The plan of the work is the following: After giving in Sec. 2 the basic terminology on supermanifolds and superbundles, in Sec. 3 we introduce nonlinear and linear distinguished connections in vector superbundles. The geometry of the total space of vector superbundles will be studied in Sec. 4 by considering distinguished connections and their structure equations. Generalized Lagrange and Finsler superspaces will

be defined in Sec. 5. In Sec. 6 the Einstein equations on the la-superspaces are written and analyzed. A version of gauge like la-supergravity will be also proposed. Concluding remarks and discussion are contained in Sec. 7.

## 2 Supermanifolds and Superbundles

In this section we outline some necessary definitions, concepts and results on the theory of supermanifolds (s-manifolds) [5–14].

The basic structures for building up s-manifolds (see [6,9,14]) are Grassmann algebra and Banach space. Grassmann algebra is considered a real associative algebra  $\Lambda$  (with unity) possessing a finite (canonical) set of anticommutative generators  $\beta_{\hat{A}}$ ,  $[\beta_{\hat{A}}, \beta_{\hat{B}}]_+ = \beta_{\hat{A}}\beta_{\hat{B}} + \beta_{\hat{B}}\beta_{\hat{A}} = 0$ , where  $\hat{A}, \hat{B}, \dots = 1, 2, \dots, \hat{L}$ . This way it is defined a  $Z_2$ -graded commutative algebra  $\Lambda_0 + \Lambda_1$ , whose even part  $\Lambda_0$  (odd part  $\Lambda_1$ ) represents a  $2^{\hat{L}-1}$ -dimensional real vector space of even (odd) products of generators  $\beta_{\hat{A}}$ . After setting  $\Lambda_0 = \mathcal{R} + \Lambda_0'$ , where  $\mathcal{R}$  is the real number field and  $\Lambda_0'$  is the subspace of  $\Lambda$  consisting of nilpotent elements, the projections  $\sigma : \Lambda \rightarrow \mathcal{R}$  and  $s : \Lambda \rightarrow \Lambda_0'$  are called, respectively, the body and soul maps.

A Grassmann algebra can be provided with both structures of a Banach algebra and Euclidean topological space by the norm [6]

$$\|\xi\| = \sum_{\hat{A}_i} |a^{\hat{A}_1 \dots \hat{A}_k}|, \xi = \sum_{r=0}^{\hat{L}} a^{\hat{A}_1 \dots \hat{A}_r} \beta_{\hat{A}_1} \dots \beta_{\hat{A}_r}.$$

A superspace is defined as a product

$$\Lambda^{n,k} = \underbrace{\Lambda_0 \times \dots \times \Lambda_0}_n \times \underbrace{\Lambda_1 \times \dots \times \Lambda_1}_k.$$

This represents the  $\Lambda$ -envelope of a  $Z_2$ -graded vector space  $V^{n,k} = V_0 \otimes V_1 = \mathcal{R}^n \oplus \mathcal{R}^k$ , which is obtained by multiplication of even (odd) vectors of  $V$  by even (odd) elements of  $\Lambda$ . The superspace (as the  $\Lambda$ -envelope) posses  $(n + k)$  basis vectors  $\{\hat{\beta}_i, i = 0, 1, \dots, n - 1, \text{ and } \beta_{\hat{i}}, \hat{i} = 1, 2, \dots, k\}$ . Coordinates of even (odd) elements of  $V^{n,k}$  are even (odd) elements of  $\Lambda$ . On the other hand, a superspace  $V^{n,k}$  forms a  $(2^{\hat{L}-1})(n + k)$ -dimensional real vector spaces with a basis  $\{\hat{\beta}_{i(\Lambda)}, \beta_{\hat{i}(\Lambda)}\}$ .

Functions of superspaces, differentiation with respect to Grassmann coordinates, supersmooth (superanalytic) functions and mappings are defined by analogy with the ordinary case, but with a glance to certain specificity caused by changing of real (or complex) number field into Grassmann algebra  $\Lambda$ . Here we remark that functions on a superspace  $\Lambda^{n,k}$  which takes values in Grassmann algebra can be considered as mappings of the space  $\mathcal{R}^{(2^{\hat{L}-1})(n+k)}$  into the space  $\mathcal{R}^{2^{\hat{L}}}$ . Functions being differentiable with regard to Grassmann coordinates can be rewritten via derivatives on real coordinates, which obey a generalized version of Cauchy-Riemann conditions.

A  $(n, k)$ -dimensional s-manifold  $M$  is defined as a Banach manifold (see, for example, [39]) modelled on  $\Lambda^{n,k}$  endowed with an atlas  $\psi = \{U_{(i)}, \psi_{(i)} : U_{(i)} \rightarrow \Lambda^{n,k}, (i) \in J\}$  whose transition functions  $\psi_{(i)}$  are supersmooth [6,9]. Instead of supersmooth

functions we can use  $G^\infty$ -functions [6] and define  $G^\infty$ -supermanifolds ( $G^\infty$  denotes the class of superdifferentiable functions). The local structure of a  $G^\infty$ -supermanifold can be built very much as on a  $C^\infty$ -manifold. Just as a vector field on a  $n$ -dimensional  $C^\infty$ -manifold can be expressed locally as

$$\sum_{i=0}^{n-1} f_i(x^j) \frac{\partial}{\partial x^i},$$

where  $f_i$  are  $C^\infty$ -functions, a vector field on an  $(n, k)$ -dimensional  $G^\infty$ -supermanifold  $M$  can be expressed locally on an open region  $U \subset M$  as

$$\begin{aligned} \sum_{I=0}^{n-1+k} f_I(x^J) \frac{\partial}{\partial x^I} = \\ \sum_{i=0}^{n-1} f_i(x^j, \theta^{\hat{j}}) \frac{\partial}{\partial x^i} + \sum_{i=1}^k f_i(x^j, \theta^{\hat{j}}) \frac{\partial}{\partial \theta^{\hat{i}}}, \end{aligned}$$

where  $x = (\hat{x}, \theta) = \{x^I = (\hat{x}^i, \theta^{\hat{i}})\}$  are local (even, odd) coordinates. We shall use indices  $I = (i, \hat{i}), J = (j, \hat{j}), K = (k, \hat{k}), \dots$  for geometric objects on  $M$ . A vector field on  $U$  is an element  $X \in \text{End}[G^\infty(U)]$  (we can also consider supersmooth functions instead of  $G^\infty$ -functions) such that

$$X(fg) = (Xf)g + (-)^{|f||X|} fXg,$$

for all  $f, g$  in  $G^\infty(U)$ , and

$$X(af) = (-)^{|X||a|} aXf,$$

where  $|X|$  and  $|a|$  denote correspondingly the parity ( $= 0, 1$ ) of values  $X$  and  $a$  and for simplicity in this work we shall write  $(-)^{|f||X|}$  instead of  $(-1)^{|f||X|}$ .

A super Lie group (sl-group) [7] is both an abstract group and a s-manifold, provided that the group composition law fulfils a suitable smoothness condition (i.e. to be superanalytic, for short, *sa* [9]).

In our further considerations we shall use the group of automorphisms of  $\Lambda^{(n,k)}$ , denoted as  $GL(n, k, \Lambda)$ , which can be parametrized as the super Lie group of invertible matrices

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where  $A$  and  $D$  are respectively  $(n \times n)$  and  $(k \times k)$  matrices consisting of even Grassmann elements and  $B$  and  $C$  are rectangular matrices consisting of odd Grassmann elements. A matrix  $Q$  is invertible as soon as maps  $\sigma A$  and  $\sigma D$  are invertible matrices. A sl-group represents an ordinary Lie group included in the group of linear transforms  $GL(2^{\hat{L}-1}(n+k), \mathcal{R})$ . For matrices of type  $Q$  one defines [1-3] the superdeterminant,  $sdetQ$ , supertrace,  $strQ$ , and superrank,  $srankQ$ .

One calls Lie superalgebra (sl-algebra) any  $Z_2$ -graded algebra  $A = A_0 \oplus A_1$  endowed with product  $\{, \}$  satisfying the following properties:

$$[I, I'] = -(-)^{|I||I'|} [I', I],$$

$$[I, [I', I'']] = [[I, I'], I''] + (-)^{|I||I'|} [I' [I, I'']],$$

$I \in A_{|I|}, I' \in A_{|I'|}$ , where  $|I|, |I'| = 0, 1$  enumerates, respectively, the possible parity of elements  $I, I'$ . The even part  $A_0$  of a sl-algebra is a usual Lie algebra and the odd part  $A_1$  is a representation of this Lie algebra. This enables us to classify sl-algebras following the Lie algebra classification [40]. We also point out that irreducible linear representations of Lie superalgebra  $A$  are realized in  $Z_2$ -graded vector spaces by matrices  $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$  for even elements and  $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$  for odd elements and that, roughly speaking,  $A$  is a superalgebra of generators of a sl-group.

An sl-module  $W$  (graded Lie module) [7] is a  $Z_2$ -graded left  $\Lambda$ -module endowed with a product  $[\cdot, \cdot]$  which satisfies the graded Jacobi identity and makes  $W$  into a graded-anticommutative Banach algebra over  $\Lambda$ . One calls the Lie module  $G$  the set of the left-invariant derivatives of a sl-group  $G$ .

One constructs the supertangent bundle (st-bundle)  $TM$  over a s-manifold  $M$ ,  $\pi : TM \rightarrow M$  in a usual manner (see, for instance, [39]) by taking as the typical fibre the superspace  $\Lambda^{n,k}$  and as the structure group the group of automorphisms, i.e. the sl-group  $GL(n, k, \Lambda)$ .

A s-manifold and a st-bundle  $TM$  may be represented as a certain  $2^{\hat{L}-1}(n+k)$ -dimensional real manifold and the tangent bundle over it whose transition function obey the special conditions of Cauchy-Riemann type.

Let us denote  $\hat{\mathcal{E}}$  a vector superspace (vs-space) of dimension  $(m, l)$  (with respect to a chosen base we parametrize an element  $y \in \hat{\mathcal{E}}$  as  $y = (\hat{y}, \zeta) = \{y^A = (\hat{y}^a, \zeta^{\hat{a}})\}$ , where  $a = 1, 2, \dots, m$  and  $\hat{a} = 1, 2, \dots, l$ ). We shall use indices  $A = (a, \hat{a}), B = (b, \hat{b}), \dots$  for objects on vs-spaces. A vector superbundle (vs-bundle)  $\mathcal{E}$  over base  $M$  with total superspace  $E$ , standard fibre  $\hat{\mathcal{F}}$  and surjective projection  $\pi_E : E \rightarrow M$  is defined (see details and variants in [11,16]) as in the case of ordinary manifolds (see, for instance, [39,26,27]). A section of  $\mathcal{E}$  is a supersmooth map  $s : U \rightarrow E$  such that  $\pi_E \circ s = id_U$ .

A subbundle of  $\hat{\mathcal{E}}$  is a triple  $(B, f, f')$ , where  $B$  is a vs-bundle on  $M$ , maps  $f : B \rightarrow E$  and  $f' : M \rightarrow M$  are supersmooth, and (i)  $\pi_E \circ f = f' \circ \pi_B$ ; (ii)  $f : \pi_B^{-1}(x) \rightarrow \pi_E^{-1} \circ f'(x)$  is a vs-space homomorphism.

We denote by  $u = (x, y) = (\hat{x}, \theta, \hat{y}, \zeta) = \{u^\alpha = (x^I, y^A) = (\hat{x}^i, \theta^{\hat{i}}, \hat{y}^a, \zeta^{\hat{a}}) = (\hat{x}^i, x^{\hat{i}}, \hat{y}^a, y^{\hat{a}})\}$  the local coordinates in  $\hat{\mathcal{E}}$  and write their transformations as

$$x^{I'} = x^{I'}(x^I), \quad srank\left(\frac{\partial x^{I'}}{\partial x^I}\right) = (n, k), \tag{1}$$

$y^{A'} = M_A^{A'}(x)y^A$ , where  $M_A^{A'}(x) \in G(m, l, \Lambda)$ .

For local coordinates and geometric objects on ts-bundle  $TM$  we shall not distinguish indices of coordinates on the base and in the fibre and write, for instance,  $u = (x, y) = (\hat{x}, \theta, \hat{y}, \zeta) = \{u^\alpha = (x^I, y^I) = (\hat{x}^i, \theta^{\hat{i}}, \hat{y}^i, \zeta^{\hat{i}}) = (\hat{x}^i, x^{\hat{i}}, \hat{y}^i, y^{\hat{i}})\}$ .

Finally, in this section, we remark that to simplify considerations in this work we shall consider only locally trivial super fibre bundles.

### 3 Nonlinear Connections in Vector Superbundles

The concept of nonlinear connection (N-connection) was introduced in the framework of Finsler geometry [24,41,42]. The global definition of N-connection is given in [43]. In works [26,27] nonlinear connection structures are studied in details. In this section we shall present the notion of nonlinear connection in vs-bundles and its main properties in a way necessary for our further considerations.

Let us consider a vs-bundle  $\mathcal{E} = (E, \pi_E, M)$  whose type fibre is  $\hat{\mathcal{F}}$  and  $\pi^T : T\mathcal{E} \rightarrow TM$  is the superdifferential of the map  $\pi_E$  ( $\pi^T$  is a fibre-preserving morphism of the st-bundle  $(T\mathcal{E}, \tau_E, M)$  to  $E$  and of st-bundle  $(TM, \tau, M)$  to  $M$ ). The kernel of this vs-bundle morphism being a subbundle of  $(TE, \tau_E, E)$  is called the vertical subbundle over  $\mathcal{E}$  and denoted by  $V\mathcal{E} = (VE, \tau_V, E)$ . Its total space is  $V\mathcal{E} = \bigcup_{u \in \mathcal{E}} V_u$ , where  $V_u = \ker \pi^T$ ,  $u \in \mathcal{E}$ . A vector

$$Y = Y^\alpha \frac{\partial}{\partial u^\alpha} = Y^I \frac{\partial}{\partial x^I} + Y^A \frac{\partial}{\partial y^A} = Y^i \frac{\partial}{\partial x^i} + Y^{\hat{i}} \frac{\partial}{\partial \theta^{\hat{i}}} + Y^a \frac{\partial}{\partial y^a} + Y^{\hat{a}} \frac{\partial}{\partial \zeta^{\hat{a}}}$$

tangent to  $\mathcal{E}$  in the point  $u \in \mathcal{E}$  is locally represented as

$$(u, Y) = (u^\alpha, Y^\alpha) = (x^I, y^A, Y^I, Y^A) = (\hat{x}^i, \theta^{\hat{i}}, \hat{y}^a, \zeta^{\hat{a}}, \hat{Y}^i, Y^{\hat{i}}, \hat{Y}^a, Y^{\hat{a}}).$$

**Definition 1** A *nonlinear connection, N-connection*, in sv-bundle  $\mathcal{E}$  is a splitting on the left of the exact sequence

$$0 \longrightarrow V\mathcal{E} \xrightarrow{i} T\mathcal{E} \longrightarrow T\mathcal{E}/V\mathcal{E} \longrightarrow 0, \tag{2}$$

i.e. a morphism of vs-bundles  $N : T\mathcal{E} \in V\mathcal{E}$  such that  $N \circ i$  is the identity on  $V\mathcal{E}$ .

The kernel of the morphism  $N$  is called the horizontal subbundle and denoted by  $(H\mathcal{E}, \tau_E, E)$ . From the exact sequence (2) one follows that N-connection structure can be equivalently defined as a distribution  $\{E_u \rightarrow H_u E, T_u E = H_u E \oplus V_u E\}$  on  $E$  defining a global decomposition, as a Whitney sum,

$$T\mathcal{E} = H\mathcal{E} + V\mathcal{E}. \tag{3}$$

To a given N-connection we can associate a covariant s-derivation on M:

$$\nabla_X Y = X^I \left\{ \frac{\partial Y^A}{\partial x^I} + N_I^A(x, Y) \right\} s_A, \tag{4}$$

where  $s_A$  are local independent sections of  $\mathcal{E}$ ,  $Y = Y^A s_A$  and  $X = X^I s_I$ .

S-differentiable functions  $N_I^A$  from (4) written as functions on  $x^I$  and  $y^A$ ,  $N_I^A(x, y)$ , are called the coefficients of the N-connection and satisfy these transformation laws under coordinate transforms (1) in  $\mathcal{E}$ :

$$N_{I'}^{A'} \frac{\partial x^{I'}}{\partial x^I} = M_{A'}^A N_I^A - \frac{\partial M_{A'}^A(x)}{\partial x^I} y^A.$$

If coefficients of a given N-connection are s- differentiable with respect to coordinates  $y^A$  we can introduce (additionally to covariant nonlinear s-derivation (4)) a linear covariant s-derivation  $\hat{D}$  (which is a generalization for sv-bundles of the Berwald connection [44]) given as follows:

$$\hat{D}_{(\frac{\partial}{\partial x^I})}(\frac{\partial}{\partial y^A}) = \hat{N}_{AI}^B(\frac{\partial}{\partial y^B}), \quad \hat{D}_{(\frac{\partial}{\partial y^A})}(\frac{\partial}{\partial y^B}) = 0,$$

where

$$\hat{N}_{BI}^A(x, y) = \frac{\partial N^A_I(x, y)}{\partial y^B} \tag{5}$$

and

$$\hat{N}_{BC}^A(x, y) = 0.$$

For a vector field on  $\mathcal{E}$   $Z = Z^I \frac{\partial}{\partial x^I} + Y^A \frac{\partial}{\partial y^A}$  and  $B = B^A(y) \frac{\partial}{\partial y^A}$  being a section in the vertical s-bundle  $(VE, \tau_V, E)$  the linear connection (5) defines s-derivation (compare with (4)):

$$\hat{D}_Z B = [Z^I (\frac{\partial B^A}{\partial x^I} + \hat{N}_{BI}^A B^B) + Y^B \frac{\partial B^A}{\partial y^B}] \frac{\partial}{\partial y^A}.$$

Another important characteristic of a N-connection is its curvature:

$$\Omega = \frac{1}{2} \Omega_{IJ}^A dx^I \wedge dx^J \otimes \frac{\partial}{\partial y^A}$$

with local coefficients

$$\Omega_{IJ}^A = \frac{\partial N_I^A}{\partial x^J} - (-)^{|IJ|} \frac{\partial N_J^A}{\partial x^I} + N_I^B \hat{N}_{BJ}^A - (-)^{|IJ|} N_J^B \hat{N}_{BI}^A,$$

where for simplicity we have written  $(-)^{|K||J|} = (-)^{|KJ|}$ .

We note that linear connections are particular cases of N-connections, when  $N_I^A(x, y)$  are parametrized as  $N_I^A(x, y) = K_{BI}^A(x) x^I y^B$ , where functions  $K_{BI}^A(x)$ , defined on M, are called the Christoffel coefficients.

## 4 Geometry of the Total Space of a Sv-Bundle

The geometry of the sv- and st-bundles is very rich. It contains a lot of geometrical objects and properties which could be of great importance in theoretical physics. In this section we shall present the main results from geometry of total spaces of sv-bundles. In order to avoid long computations and maintain the geometric meaning the notion of nonlinear connections will systematically used in a manner generalizing to s-spaces the classical results [26,27].

#### 4.1 Distinguished tensors and connections in sv-bundles

In sv-bundle  $\mathcal{E}$  we can introduce a local basis adapted to the given N-connection:

$$\delta_\alpha = \frac{\delta}{\delta u^\alpha} = (\delta_I = \frac{\delta}{\delta x^I} = \partial_I - N_I^A(x, y) \frac{\partial}{\partial y^A}, \quad \partial_A), \quad (6)$$

where  $\partial_I = \frac{\partial}{\partial x^I}$  and  $\partial_A = \frac{\partial}{\partial y^A}$  are usual partial s-derivations. The dual to (6) basis is defined as

$$\delta^\alpha = \delta u^\alpha = (\delta^I = \delta x^I = dx^I, \delta^A = \delta y^A = dy^A + N_I^A(x, y) dx^I). \quad (7)$$

By using adapted bases (6) and (7) one introduces algebra  $DT(\mathcal{E})$  of distinguished tensor s-fields (ds-fields, ds-tensors, ds-objects) on  $\mathcal{E}$ ,  $\mathcal{T} = \mathcal{T}_{qs}^{pr}$ , which is equivalent to the tensor algebra of sv-bundle  $\pi_d : H\mathcal{E} \oplus V\mathcal{E} \rightarrow \mathcal{E}$ , hereafter briefly denoted as  $\mathcal{E}_d$ .

An element  $Q \in \mathcal{T}_{qs}^{pr}$ , ds-field of type  $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ , can be written in local form as

$$Q = Q_{J_1 \dots J_q B_1 \dots B_s}^{I_1 \dots I_p A_1 \dots A_r}(x, y) \delta_{I_1} \otimes \dots \otimes \delta_{I_p} \otimes dx^{J_1} \otimes \dots \otimes dx^{J_q} \otimes \partial_{A_1} \otimes \dots \otimes \partial_{A_r} \otimes \delta y^{B_1} \otimes \dots \otimes \delta y^{B_s}. \quad (8)$$

In addition to ds-tensors we can introduce ds-objects with various s-group and coordinate transforms adapted to global splitting (3).

**Definition 2** A *linear distinguished connection, d-connection*, in sv- bundle  $\mathcal{E}$  is a linear connection  $D$  on  $\mathcal{E}$  which preserves by parallelism the horizontal and vertical distributions in  $\mathcal{E}$ .

By a linear connection of a s-manifold we understand a linear connection in its tangent bundle.

Let denote by  $\Xi(M)$  and  $\Xi(\mathcal{E})$ , respectively, the modules of vector fields on s-manifold  $M$  and sv-bundle  $\mathcal{E}$  and by  $\mathcal{F}(M)$  and  $\mathcal{F}(\mathcal{E})$ , respectively, the s-modules of functions on  $M$  and on  $\mathcal{E}$ .

It is clear that for a given global splitting into horizontal and vertical s-subbundles (3) we can associate operators of horizontal and vertical covariant derivations (h- and v-derivations, denoted respectively as  $D^{(h)}$  and  $D^{(v)}$ ) with properties:

$$D_X Y = (XD)Y = D_{hX} Y + D_{vX} Y,$$

where

$$D_X^{(h)} Y = D_{hX} Y, \quad D_X^{(h)} f = (hX)f$$

and

$$D_X^{(v)} Y = D_{vX} Y, \quad D_X^{(v)} f = (vX)f,$$

for every  $f \in \mathcal{F}(M)$  with decomposition of vectors  $X, Y \in \Xi(\mathcal{E})$  into horizontal and vertical parts,  $X = hX + vX$  and  $Y = hY + vY$ .



The local coefficients of a d-connection  $D$  in  $\mathcal{E}$  with respect to the local adapted frame (6) separate into four groups. We introduce local coefficients  $(L^I{}_{JK}(u), L^A{}_{BK}(u))$  of  $D^{(h)}$  such that

$$\begin{aligned} D^{(h)}_{\left(\frac{\delta}{\delta x^K}\right)} \frac{\delta}{\delta x^J} &= L^I{}_{JK}(u) \frac{\delta}{\delta x^I}, \\ D^{(h)}_{\left(\frac{\delta}{\delta x^K}\right)} \frac{\partial}{\partial y^B} &= L^A{}_{BK}(u) \frac{\partial}{\partial y^A}, \\ D^{(h)}_{\left(\frac{\delta}{\delta x^K}\right)} f &= \frac{\delta f}{\delta x^K} = \frac{\partial f}{\partial x^K} - N^K{}_A(u) \frac{\partial f}{\partial y^A}, \end{aligned}$$

and local coefficients  $(C^I{}_{JC}(u), C^A{}_{BC}(u))$  such that

$$\begin{aligned} D^{(v)}_{\left(\frac{\partial}{\partial y^C}\right)} \frac{\delta}{\delta x^J} &= C^I{}_{JC}(u) \frac{\delta}{\delta x^I}, \quad D^{(v)}_{\left(\frac{\partial}{\partial y^C}\right)} \frac{\partial}{\partial y^B} = C^A{}_{BC} \frac{\partial}{\partial y^A}, \\ D^{(v)}_{\left(\frac{\partial}{\partial y^C}\right)} f &= \frac{\partial f}{\partial y^C}, \end{aligned}$$

where  $f \in \mathcal{F}(\mathcal{E})$ . The covariant d-derivation along vector  $X = X^I \frac{\delta}{\delta x^I} + Y^A \frac{\partial}{\partial y^A}$  of a ds-tensor field  $Q$  of type  $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ , see (8), can be written as

$$D_X Q = D_X^{(h)} Q + D_X^{(v)} Q,$$

where h-covariant derivative is defined as

$$D_X^{(h)} Q = X^K Q^I A_{JB|K} \delta_I \otimes \partial_A \otimes dx^I \otimes \delta y^A,$$

with components

$$Q^I A_{JB|K} = \frac{\delta Q^I A_{JB}}{\delta x^K} + L^I{}_{HK} Q^H A_{JB} + L^A{}_{CK} W^I C_{JB} - L^H{}_{JK} W^I A_{HB} - L^C{}_{BK} W^I A_{JC},$$

and v-covariant derivative is defined as

$$D_X^{(v)} Q = X^C Q^I A_{JB\perp C} \delta_I \otimes \partial_A \otimes dx^I \otimes \delta y^B,$$

with components

$$Q^I A_{JB\perp C} = \frac{\partial Q^I A_{JB}}{\partial y^C} + C^I{}_{HC} Q^H A_{JB} + C^A{}_{DC} Q^I D_{JB} - C^H{}_{JC} Q^I A_{HB} - C^D{}_{BC} Q^I A_{JD}.$$

The above presented formulas show that

$$\begin{aligned} D\Gamma &= (L, \tilde{L}, \tilde{C}, C) = \\ &= (L^A{}_{JK}(u), L^A{}_{BK}(u), C^I{}_{JA}(u), C^A{}_{BC}(u)) \end{aligned}$$

are the local coefficients of the d-connection  $D$  with respect to the local frame  $(\frac{\delta}{\delta x^I}, \frac{\partial}{\partial y^a})$ . If a change (1) of local coordinates on  $\mathcal{E}$  is performed, by using the law of transformation of local frames under it

$$(\delta_\alpha = (\delta_I, \partial_A) \mapsto \delta_{\alpha'} = (\delta_{I'}, \partial_{A'}), \quad (9)$$

where

$$\delta_{I'} = \frac{\partial x^I}{\partial x^{I'}} \delta_I, \quad \partial_{A'} = M_{A'}^A(x) \partial_A \quad ),$$

we obtain the following transformation laws of the local coefficients of a d-connection:

$$L^{I' J' K'} = \frac{\partial x^{I'}}{\partial x^I} \frac{\partial x^J}{\partial x^{J'}} \frac{\partial x^K}{\partial x^{K'}} L^I{}_{JK} + \frac{\partial x^{I'}}{\partial x^K} \frac{\partial^2 x^K}{\partial x^{J'} \partial x^{K'}}, \quad (10)$$

$$L^A{}_{B'K'} = M_A^{A'} M_{B'}^B \frac{\partial x^K}{\partial x^{K'}} L^A{}_{BK} + M_C^{A'} \frac{\partial M_{B'}^C}{\partial x^{K'}},$$

and

$$C_{J'C'}^{I'} = \frac{\partial x^{I'}}{\partial x^I} \frac{\partial x^J}{\partial x^{J'}} M_{C'}^C C^I{}_{JC}, \quad C_{B'C'}^{A'} = M_A^{A'} M_{B'}^B M_{C'}^C C^A{}_{BC}.$$

As in the usual case of tensor calculus on locally isotropic spaces the transformation laws (10) for d-connections differ from those for ds-tensors, which are written (for instance, we consider transformation laws for ds-tensor (8)) as

$$Q_{J'_1 \dots B'_1 \dots}^{I'_1 \dots A'_1 \dots} = \frac{\partial x^{I'_1}}{\partial x^{I_1}} \dots M_{A'_1}^{A_1} \dots \frac{\partial x^{J_1}}{\partial x^{J'_1}} \dots M_{B'_1}^{B_1} \dots Q_{J_1 \dots B_1 \dots}^{I_1 \dots A_1 \dots}.$$

We note that defined distinguished s-tensor algebra and d-covariant calculus in sv-bundles provided with N-connection structure is a supersymmetric generalization of the corresponding formalism for usual vector bundles presented in [26,27]. To obtain Miron and Anastasiei local formulas we have to restrict us with even components of geometric objects by changing, formally, capital indices  $(I, J, K, \dots)$  into  $(i, j, k, a, \dots)$  and s-derivation and s-commutation rules into those for real number fields on usual manifolds. For brevity, in this work we shall omit proofs and cumbersome computations if they will be simple supersymmetric generalizations of those presented in the just cited monographs.

## 4.2 Torsion and curvature of the distinguished connection in sv-bundle

Let  $\mathcal{E} = (E, \pi_E, M)$  be a sv-bundle endowed with N-connection and d-connection structures. The torsion of d-connection is introduced into usual manner:

$$T(X, Y) = [X, DY] - [X, Y], \quad X, Y \in \Xi(M).$$

The following decomposition is possible by using h- and v-projections (associated to N):

$$T(X, Y) = T(hX, hY) + T(hX, vY) + T(vX, hX) + T(vX, vY).$$

Taking into account the skewsupersymmetry of  $T$  and the equation  $h[vX, vY] = 0$  we can verify that the torsion of a d-connection is completely determined by the following ds-tensor fields:

$$\begin{aligned} hT(hX, hY) &= [X(D^{(h)}h)Y] - h[hX, hY], \\ vT(hX, hY) &= -v[hX, hY], \\ hT(hX, vY) &= -D_Y^{(v)}hX - h[hX, vY], \\ vT(hX, vY) &= D_X^{(h)}vY - v[hX, vY], \\ vT(vX, xY) &= [X(D^{(v)}v)Y] - v[vX, vY], \end{aligned}$$

where  $X, Y \in \Xi(\mathcal{E})$ . In order to get the local form of the ds-tensor fields which determine the torsion of d-connection  $D\Gamma$  (the torsions of  $D\Gamma$ ) we use equations

$$\left[ \frac{\delta}{\delta x^J}, \frac{\delta}{\delta x^K} \right] = R^A{}_{JK} \frac{\partial}{\partial y^A},$$

where

$$\begin{aligned} R^A{}_{JK} &= \frac{\delta N_J^A}{\delta x^K} - (-)^{|KJ|} \frac{\delta N_K^A}{\delta x^J}, \\ \left[ \frac{\delta}{\delta x^J}, \frac{\partial}{\partial y^A} \right] &= \frac{\partial N_J^A}{\partial y^B} \frac{\partial}{\partial y^A}, \end{aligned}$$

and introduce notations

$$\begin{aligned} hT\left(\frac{\delta}{\delta x^K}, \frac{\delta}{\delta x^J}\right) &= T^I{}_{JK} \frac{\delta}{\delta x^I}, & vT\left(\frac{\delta}{\delta x^K}, \frac{\delta}{\delta x^J}\right) &= \tilde{T}_{JK}^A \frac{\partial}{\partial y^A}, \\ hT\left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial x^J}\right) &= \tilde{P}_{JB}^I \frac{\delta}{\delta x^I}, & vT\left(\frac{\partial}{\partial y^B}, \frac{\delta}{\delta x^J}\right) &= P^A{}_{JB} \frac{\partial}{\partial y^A}, \\ vT\left(\frac{\partial}{\partial y^B}, \frac{\partial}{\partial y^C}\right) &= S^A{}_{BC} \frac{\partial}{\partial y^A}. \end{aligned} \quad (11)$$

Now we can compute the local components of the torsions, introduced in (11), with respect to the frame  $(\frac{\delta}{\delta x}, \frac{\partial}{\partial y})$ , of a d-connection  $D\Gamma = (L, \tilde{L}, \tilde{C}, C)$ :

$$\begin{aligned} T^I{}_{JK} &= L^I{}_{JK} - (-)^{|JK|} L^I{}_{KJ}, \quad \tilde{T}_{JK}^A = R^A{}_{JK}, \quad \tilde{P}_{JB}^I = C^I{}_{JB}, \\ P^A{}_{JB} &= \frac{\partial N_J^A}{\partial y^B} - L^A{}_{BJ}, \quad S^A{}_{BC} = C^A{}_{BC} - (-)^{|BC|} C^A{}_{CB}. \end{aligned} \quad (12)$$

The even and odd components of torsions (12) can be specified in explicit form by using decompositions of indices into even and odd parts ( $I = (i, \hat{i}), J = (j, \hat{j}), \dots$ ), for instance,

$$\begin{aligned} T^i{}_{jk} &= L^i{}_{jk} - L^i{}_{kj}, & T^i{}_{j\hat{k}} &= L^i{}_{j\hat{k}} + L^i{}_{\hat{k}j}, \\ T^{\hat{i}}{}_{jk} &= L^{\hat{i}}{}_{jk} - L^{\hat{i}}{}_{kj}, & \dots & \end{aligned}$$

and so on.

Another important characteristic of a d-connection  $D\Gamma$  is its curvature:

$$R(X, Y)Z = D_{[X}D_{Y]}Z - D_{[X, Y]}Z,$$

where  $X, Y, Z \in \Xi(E)$ . By using h- and v-projections we can prove that

$$vR(X, Y)hZ = 0, \quad hR(X, Y)vZ = 0 \quad (13)$$

and

$$R(X, Y)Z = hR(X, Y)hZ + vR(X, Y)vZ,$$

where  $X, Y, Z \in \Xi(E)$ . Taking into account properties (13) and the equation  $R(X, Y) = -(-)^{|XY|}R(Y, X)$  we prove that the curvature of a d-connection  $D$  in the total space of a sv-bundle  $\mathcal{E}$  is completely determined by the following six ds-tensor fields:

$$\begin{aligned} R(hX, hY)hZ &= (D_{[X}^{(h)}D_{Y]}^{(h)} - D_{[hX, hY]}^{(h)} - D_{[hX, hY]}^{(v)})hZ, \\ R(hX, hY)vZ &= (D_{[X}^{(h)}D_{Y]}^{(h)} - D_{[hX, hY]}^{(h)} - D_{[hX, hY]}^{(v)})vZ, \\ R(vX, hY)hZ &= (D_{[X}^{(v)}D_{Y]}^{(h)} - D_{[vX, hY]}^{(h)} - D_{[vX, hY]}^{(v)})hZ, \\ R(vX, hY)vZ &= (D_{[X}^{(v)}D_{Y]}^{(h)} - D_{[vX, hY]}^{(h)} - D_{[vX, hY]}^{(v)})vZ, \\ R(vX, vY)hZ &= (D_{[X}^{(v)}D_{Y]}^{(v)} - D_{[vX, vY]}^{(v)})hZ, \\ R(vX, vY)vZ &= (D_{[X}^{(v)}D_{Y]}^{(v)} - D_{[vX, vY]}^{(v)})vZ, \end{aligned} \quad (14)$$

where

$$\begin{aligned} D_{[X}^{(h)}D_{Y]}^{(h)} &= D_X^{(h)}D_Y^{(h)} - (-)^{|XY|}D_Y^{(h)}D_X^{(h)}, \\ D_{[X}^{(h)}D_{Y]}^{(v)} &= D_X^{(h)}D_Y^{(v)} - (-)^{|XY|}D_Y^{(v)}D_X^{(h)} \end{aligned}$$

and

$$D_{[X}^{(v)}D_{Y]}^{(h)} = D_X^{(v)}D_Y^{(h)} - (-)^{|XY|}D_Y^{(h)}D_X^{(v)}.$$

We introduce the local components of ds-tensor fields (14) as follows:

$$\begin{aligned} R(\delta_K, \delta_J)\delta_H &= R_H^I{}_{JK}\delta_I, & R(\delta_K, \delta_J)\partial_B &= \tilde{R}_{\tilde{B}.JK}^A\partial_A, \\ R(\partial_C, \delta_K)\delta_J &= \tilde{P}_{J.KC}^I\delta_I, & R(\partial_C, \delta_K)\partial_B &= P_B^A{}_{KC}\partial_A, \\ R(\partial_C, \partial_B)\delta_J &= \tilde{S}_{J.BC}^I\delta_I, & R(\partial_D, \partial_C)\partial_B &= S_B^A{}_{CD}\partial_A. \end{aligned} \quad (15)$$

Putting the components of a d-connection  $D\Gamma = (L, \tilde{L}, \tilde{C}, C)$  in (15), by a direct computation, we obtain these locally adapted components of the curvature (curvatures):

$$\begin{aligned} R_H^I{}_{JK} &= \delta_K L^I{}_{HJ} - (-)^{|KJ|}\delta_J L^I{}_{HK} + L^M{}_{HJ}L^I{}_{MK} - (-)^{|KJ|}L^M{}_{HK}L^I{}_{MJ} + C^I{}_{HA}R^A{}_{JK}, \\ \tilde{R}_{\tilde{B}.JK}^A &= \delta_K L^A{}_{BJ} - (-)^{|KJ|}\delta_J L^A{}_{BK} + L^C{}_{BJ}L^A{}_{CK} - (-)^{|KJ|}L^C{}_{BK}L^A{}_{KJ} + C^A{}_{BC}R^C{}_{JK}, \end{aligned}$$

$$\begin{aligned}
 \tilde{P}_{J.KA}^I &= \partial_A L^I_{JK} - C^I_{JA|K} + C^I_{JB} P^B_{KA}, \\
 P_B^A{}_{KC} &= \partial_C L^A_{BK} - C^A_{BC|K} + C^A_{BD} P^D_{KC}, \\
 \tilde{S}_{J.BC}^I &= \partial_C C^I_{JB} - (-)^{|BC|} \partial_B C^I_{JC} + C^H_{JB} C^I_{HC} - (-)^{|BC|} C^H_{JC} C^I_{HB}, \\
 S_B^A{}_{CD} &= \partial_D C^A_{BC} - (-)^{|CD|} \partial_C C^A_{BD} + C^E_{BC} C^A_{ED} - (-)^{|CD|} C^E_{BD} C^A_{EC}.
 \end{aligned} \tag{16}$$

We can also compute even and odd components of curvatures (16) by splitting indices into even and odd parts, for instance,

$$\begin{aligned}
 R_h^i{}_{jk} &= \delta_k L^i_{hj} - \delta_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} + C^i_{ha} R^a_{jk}, \\
 R_h^i{}_{j\hat{k}} &= \delta_{\hat{k}} L^i_{hj} + \delta_j L^i_{h\hat{k}} + L^m_{hj} L^i_{m\hat{k}} + L^m_{h\hat{k}} L^i_{mj} + C^i_{ha} R^a_{j\hat{k}} \quad , \dots
 \end{aligned}$$

(we omit the formulas for the rest of even-odd components of curvatures because we shall not use them in this work).

### 4.3 Bianchi and Ricci Identities for d-Connections in SV-Bundles

The torsion and curvature of every linear connection  $D$  on sv-bundle satisfy the following generalized Bianchi identities:

$$\begin{aligned}
 \sum_{SC} [(D_X T)(Y, Z) - R(X, Y)Z + T(T(X, Y), Z)] &= 0, \\
 \sum_{SC} [(D_X R)(U, Y, Z) + R(T(X, Y)Z)U] &= 0,
 \end{aligned} \tag{17}$$

where  $\sum_{SC}$  means the respective supersymmetric cyclic sum over  $X, Y, Z$  and  $U$ . If  $D$  is a d-connection, then by using (13) and

$$v(D_X R)(U, Y, hZ) = 0, \quad h(D_X R)(U, Y, vZ) = 0,$$

the identities (17) become

$$\begin{aligned}
 \sum_{SC} [h(D_X T)(Y, Z) - hR(X, Y)Z + hT(hT(X, Y), Z) + hT(vT(X, Y), Z)] &= 0, \\
 \sum_{SC} [v(D_X T)(Y, Z) - vR(X, Y)Z + vT(hT(X, Y), Z) + vT(vT(X, Y), Z)] &= 0, \\
 \sum_{SC} [h(D_X R)(U, Y, Z) + hR(hT(X, Y), Z)U + hR(vT(X, Y), Z)U] &= 0, \\
 \sum_{SC} [v(D_X R)(U, Y, Z) + vR(hT(X, Y), Z)U + vR(vT(X, Y), Z)U] &= 0.
 \end{aligned} \tag{18}$$

In order to get the local adapted form of these identities we insert in (18) these necessary values of triples  $(X, Y, Z), (\delta_J, \delta_K, \delta_L)$ , or  $(\partial_D, \partial_C, \partial_B)$ , and putting

successively  $U = \delta_H$  and  $U = \partial_A$ . Taking into account (11),(12) and (14),(15) we obtain:

$$\begin{aligned}
& \sum_{SC\{L,K,J\}} [T^I_{JK|H} + T^M_{JK}T^J_{HM} + R^A_{JK}C^I_{HA} - R_{J^I_{KH}}] = 0, \\
& \sum_{SC\{L,K,J\}} [R^A_{JK|H} + T^M_{JK}R^A_{HM} + R^B_{JK}P^A_{HB}] = 0, \quad (19) \\
& C^I_{JB|K} - (-)^{|JK|}C^I_{KB|J} - T^I_{JK|B} + C^M_{JB}T^I_{KM} - (-)^{|JK|}C^M_{KB}T^I_{JM} + \\
& T^M_{JK}C^I_{MB} + P^D_{JB}C^I_{KD} - (-)^{|KJ|}P^D_{KB}C^I_{JD} + P_{J^I_{KB}} - (-)^{|KJ|}P_{K^I_{JB}} = 0, \\
& P^A_{JB|K} - (-)^{|KJ|}P^A_{KB|J} - R^A_{JK\perp B} + C^M_{JB}R^A_{KM} - (-)^{|KJ|}C^M_{KB}R^A_{JM} + \\
& T^M_{JK}P^A_{MB} + P^D_{JB}P^A_{KD} - (-)^{|KJ|}P^D_{KB}P^A_{JD} - R^D_{JK}S^A_{BD} + \tilde{R}^A_{\tilde{B}\cdot JK} = 0, \\
& C^I_{JB\perp C} - (-)^{|BC|}C^I_{JC\perp B} + C^M_{JC}C^I_{MB} - (-)^{|BC|}C^M_{JB}C^I_{MC} + \\
& S^D_{BC}C^I_{JD} - \tilde{S}^I_{J\cdot BC} = 0, \\
& P^A_{JB\perp C} - (-)^{|BC|}P^A_{JC\perp B} + S^A_{BC|J} + C^M_{JC}P^A_{MB} - (-)^{|BC|}C^M_{JB}P^A_{MC} + \\
& P^D_{JB}S^A_{CD} - (-)^{|CB|}P^D_{JC}S^A_{BD} + S^D_{BC}P^A_{JD} + P_{B^A_{JC}} - (-)^{|CB|}P_{C^A_{JB}} = 0, \\
& \sum_{SC\{B,C,D\}} [S^A_{BC\perp D} + S^F_{BC}S^A_{DF} - S_{B^A_{CD}}] = 0, \\
& \sum_{SC\{H,J,L\}} [R_{K^I_{HJ|L}} - T^M_{HJ}R_{K^I_{LM}} - R^A_{HJ}\tilde{P}_{K^I\cdot LA}] = 0, \\
& \sum_{SC\{H,J,L\}} [\tilde{R}^A_{\tilde{D}\cdot HJ|L} - T^M_{HJ}\tilde{R}^A_{\tilde{D}\cdot LM} - R^C_{HJ}P_{D^A_{LC}}] = 0, \\
& \tilde{P}_{K^I\cdot JD|L} - (-)^{|LJ|}\tilde{P}_{K^I\cdot LD|J} + R_{K^I_{LJ\perp D}} + C^M_{LD}R_{K^I_{JM}} - (-)^{|LJ|}C^M_{JD}R_{K^I_{LM}} - \\
& T^M_{JL}\tilde{P}_{K^I\cdot MD} + P^A_{LD}\tilde{P}_{K^I\cdot JA} - (-)^{|LJ|}P^A_{JD}\tilde{P}_{K^I\cdot LA} - R^A_{JL}\tilde{S}_{K^I\cdot AD} = 0, \\
& P_{C^A_{JD|L}} - (-)^{|LJ|}P_{C^A_{LD|J}} + \tilde{R}^A_{C^A\cdot LJ|D} + C^M_{LD}R_{C^A_{JM}} - (-)^{|LJ|}C^M_{JD}R_{C^A_{LM}} - \\
& T^M_{JL}P_{C^A_{MD}} + P^F_{LD}P_{C^A_{JF}} - (-)^{|LJ|}P^F_{JD}P_{C^A_{LF}} - R^F_{JL}S_{C^A_{FD}} = 0, \\
& \tilde{P}_{K^I\cdot JD\perp C} - (-)^{|CD|}\tilde{P}_{K^I\cdot JC\perp D} + S_{K^I_{DC|J}} + C^M_{JD}\tilde{P}_{K^I\cdot MC} - (-)^{|CD|}C^M_{JC}\tilde{P}_{K^I\cdot MD} + \\
& P^A_{JC}\tilde{S}_{K^I\cdot DA} - (-)^{|CD|}P^A_{JD}\tilde{S}_{K^I\cdot CA} + S^A_{CD}\tilde{P}_{K^I\cdot JA} = 0, \\
& P_{B^A_{JD\perp C}} - (-)^{|CD|}P_{B^A_{JC\perp D}} + S_{B^A_{CD|J}} + C^M_{JD}P_{B^A_{MC}} - (-)^{|CD|}C^M_{JC}P_{B^A_{MD}} + \\
& P^F_{JC}S_{B^A_{DF}} - (-)^{|CD|}P^F_{JD}S_{B^A_{CF}} + S^F_{CD}P_{B^A_{JF}} = 0, \\
& \sum_{SC\{B,C,D\}} [S_{K^I_{BC\perp D}} - S^A_{BC}\tilde{S}_{K^I\cdot DA}] = 0,
\end{aligned}$$

$$\sum_{SC\{B,C,D\}} [S_F^A{}_{BC\perp D} - S^E{}_{BC} S_F^A{}_{DE}] = 0,$$

where, for instance,  $\sum_{SC\{B,C,D\}}$  means the supersymmetric cyclic sum over indices  $B, C, D$ .

Identities (19) can be detailed for even and odd components of d-connection, torsion and curvature and become very simple if  $T^I{}_{JK} = 0$  and  $S^A{}_{BC} = 0$ .

As a consequence of a corresponding arrangement of (14) we obtain the Ricci identities (for simplicity we establish them only for ds-vector fields, although they may be written for every ds-tensor field):

$$D_{[X}^{(h)} D_{Y]}^{(h)} hZ = R(hX, hY)hZ + D_{[hX, hY]}^{(h)} hZ + D_{[hX, hY]}^{(v)} hZ, \quad (20)$$

$$D_{[X}^{(v)} D_{Y]}^{(h)} hZ = R(vX, hY)hZ + D_{[vX, hY]}^{(h)} hZ + D_{[vX, hY]}^{(v)} hZ,$$

$$D_{[X}^{(v)} D_{Y]}^{(v)} hZ = R(vX, vY)hZ + D_{[vX, vY]}^{(v)} hZ$$

and

$$D_{[X}^{(h)} D_{Y]}^{(h)} vZ = R(hX, hY)vZ + D_{[hX, hY]}^{(h)} vZ + D_{[hX, hY]}^{(v)} vZ, \quad (21)$$

$$D_{[X}^{(v)} D_{Y]}^{(h)} vZ = R(vX, hY)vZ + D_{[vX, hY]}^{(v)} vZ + D_{[vX, hY]}^{(h)} vZ,$$

$$D_{[X}^{(v)} D_{Y]}^{(v)} vZ = R(vX, vY)vZ + D_{[vX, vY]}^{(v)} vZ.$$

Considering  $X = X^I(u) \frac{\delta}{\delta x^I} + X^A(u) \frac{\partial}{\partial y^A}$  and taking into account the local form of the h- and v-covariant s-derivatives and (11),(12),(14),(15) we can express respectively identities (20) and (21) in this form:

$$X^A{}_{|K|L} - (-)^{|KL|} X^A{}_{|L|K} = R_B{}^A{}_{KL} X^B - T^H{}_{KL} X^A{}_{|H} - R^B{}_{KL} X^A{}_{\perp B},$$

$$X^I{}_{|K\perp D} - (-)^{|KD|} X^I{}_{\perp D|K} = \tilde{P}_{H\cdot KD}^I X^H - C^H{}_{KD} X^I{}_{|H} - P^A{}_{KD} X^I{}_{\perp A},$$

$$X^I{}_{\perp B\perp C} - (-)^{|BC|} X^I{}_{\perp C\perp B} = \tilde{S}_{H\cdot BC}^I X^H - S^A{}_{BC} X^I{}_{\perp A}$$

and

$$X^A{}_{|K|L} - (-)^{|KL|} X^A{}_{|L|K} = R_B{}^A{}_{KL} X^B - T^H{}_{KL} X^A{}_{|H} - R^B{}_{KL} X^A{}_{\perp B},$$

$$X^A{}_{|K\perp B} - (-)^{|BK|} X^A{}_{\perp B|K} = P_B{}^A{}_{KC} X^C - C^H{}_{KB} X^A{}_{|H} - P^D{}_{KB} X^A{}_{\perp D},$$

$$X^A{}_{\perp B\perp C} - (-)^{|CB|} X^A{}_{\perp C\perp B} = S_D{}^A{}_{BC} X^D - S^D{}_{BC} X^A{}_{\perp D}.$$

#### 4.4 Structure Equations of a d-Connection in a VS-Bundle

Let, for instance, consider ds-tensor field:

$$t = t_A^I \delta_I \otimes \delta^A.$$

We introduce the so-called d-connection 1-forms  $\omega_J^I$  and  $\tilde{\omega}_B^A$  as

$$Dt = (Dt_A^I) \delta_I \otimes \delta^A$$

with

$$Dt_A^I = dt_A^I + \omega_J^I t_A^J - \tilde{\omega}_A^B t_B^I = t_{A|J}^I dx^J + t_{A\perp B}^I \delta y^B.$$

For the d-connection 1-forms of a d-connection  $D$  on  $\mathcal{E}$  defined by  $\omega_J^I$  and  $\tilde{\omega}_B^A$  one holds the following structure equations:

$$\begin{aligned} d(d^I) - d^H \wedge \omega_H^I &= -\Omega, \\ d(\delta^A) - \delta^B \wedge \tilde{\omega}_B^A &= -\tilde{\Omega}^A, \\ d\omega_J^I - \omega_J^H \wedge \omega_H^I &= -\Omega_J^I, \\ d\tilde{\omega}_B^A - \tilde{\omega}_B^C \wedge \tilde{\omega}_C^A &= -\tilde{\Omega}_B^A, \end{aligned}$$

in which the torsion 2-forms  $\Omega^I$  and  $\tilde{\Omega}^A$  are given respectively by formulas:

$$\Omega^I = \frac{1}{2} T^I_{JK} d^J \wedge d^K + \frac{1}{2} C^I_{JK} d^J \wedge \delta^C,$$

$$\tilde{\Omega}^A = \frac{1}{2} R^A_{JK} d^J \wedge d^K + \frac{1}{2} P^A_{JC} d^J \wedge \delta^C + \frac{1}{2} S^A_{BC} \delta^B \wedge \delta^C,$$

and

$$\Omega_J^I = \frac{1}{2} R_J^I{}_{KH} d^K \wedge d^H + \frac{1}{2} \tilde{P}_{J\cdot KC}^I d^K \wedge \delta^C + \frac{1}{2} \tilde{S}_{J\cdot KC}^I \delta^B \wedge \delta^C,$$

$$\tilde{\Omega}_B^A = \frac{1}{2} \tilde{R}_{B\cdot KH}^A d^K \wedge d^H + \frac{1}{2} P_B^A{}_{KC} d^K \wedge \delta^C + \frac{1}{2} S_B^A{}_{CD} \delta^C \wedge \delta^D.$$

We have defined the exterior product on s-space to satisfy the property  $\delta^\alpha \wedge \delta^\beta = -(-)^{|\alpha\beta|} \delta^\beta \wedge \delta^\alpha$ .

#### 4.5 Metric Structure of the Total Spase of a SV-Bundle

We consider the base  $M$  of a vs-bundle  $\mathcal{E} = (E, \pi_E, M)$  to be a connected and paracompact s-manifold.

**Definition 3** A *metric structure* on the total space  $E$  of a vs-bundle  $\mathcal{E}$  is a supersymmetric, second order, covariant s-tensor field  $G$  which in every point  $u \in \mathcal{E}$  is given by nondegenerate s-matrix  $G_{\alpha\beta} = G(\partial_\alpha, \partial_\alpha)$  (with nonvanishing superdeterminant,  $sdetG \neq 0$ ).



Similarly as for usual vector bundles [26,27] we establish this concordance between metric and N-connection structures on  $\mathcal{E}$ :

$$G(\delta_I, \partial_A) = 0,$$

or, in consequence,

$$G_{IA} - N_I^B h_{AB} = 0, \tag{22}$$

where

$$G_{IA} = G(\partial_I, \partial_A),$$

which gives

$$N_I^B = h^{BA} G_{IA},$$

where matrix  $h^{AB}$  is inverse to matrix  $h_{AB} = G(\partial_A, \partial_B)$ . Thus, in this case, the coefficients of N-connection  $N_B^A(u)$  are uniquely determined by the components of the metric on  $\mathcal{E}$ .

If the equality (22) holds, the metric on  $\mathcal{E}$  decomposes as

$$G(X, Y) = G(hX, hY) + G(vX, vY), \quad X, Y \in \Xi(E),$$

and looks locally as

$$G = g_{\alpha\beta}(u) \delta^\alpha \otimes \delta^\beta = g_{IJ} d^I \otimes d^J + h_{AB} \delta^A \otimes \delta^B. \tag{23}$$

**Definition 4** A d-connection  $D$  on  $\mathcal{E}$  is *metric*, or *compatible with metric  $G$* , if conditions

$$D_\alpha G_{\beta\gamma} = 0$$

are satisfied.

We can prove that a d-connection  $D$  on  $\mathcal{E}$  provided with a metric  $G$  is a metric d-connection if and only if

$$D_X^{(h)}(hG) = 0, D_X^{(h)}(vG) = 0, D_X^{(v)}(hG) = 0, D_X^{(v)}(vG) = 0, \tag{24}$$

for every  $X \in \Xi(E)$ . Conditions (24) are written in locally adapted form as

$$g_{IJ|K} = 0, g_{IJ\perp A} = 0, h_{AB|K} = 0, h_{AB\perp C} = 0.$$

In the total space  $E$  of sv-bundle  $\mathcal{E}$  endowed with a metric  $G$  given by (23) one exists a metric d-connection depending only on components of G-metric and N-connection called the canonical d-connection associated to  $G$ . Its local coefficients  $C\Gamma = (\dot{L}_{JK}^I, \dot{L}_{BK}^A, \dot{C}_{JC}^I, \dot{C}_{BC}^A)$  are as follows:

$$\begin{aligned} \dot{L}_{JK}^I &= \frac{1}{2} g^{IH} (\delta_K g_{HJ} + \delta_J g_{HK} - \delta_H g_{JK}), \\ \dot{L}_{BK}^A &= \partial_B N_K^A + \frac{1}{2} h^{AC} [\delta_K h_{BC} - (\partial_B N_K^D) h_{DC} - (\partial_C N_K^D) h_{DB}], \end{aligned} \tag{25}$$

$$\begin{aligned}\dot{C}_{JC}^I &= \frac{1}{2}g^{IK}\partial_C g_{JK}, \\ \dot{C}_{BC}^A &= \frac{1}{2}h^{AD}(\partial_C h_{DB} + \partial_B h_{DC} - \partial_D h_{BC}).\end{aligned}$$

We point out that, in general, the torsion of  $C\Gamma$ -connection (25) does not vanish (see formulas (12)).

It is very important to note that on sv-bundles provided with N-connection and d-connection and metric structures really it is defined a multiconnection d-structure, i.e. we can use in an equivalent geometric manner different variants of d-connections with various properties. For example, for modeling of some physical processes we can use the Berwald type d-connection (see (5))

$$B\Gamma = (L^I{}_{JK}, \partial_B N_K^A, 0, C^A{}_{BC}), \quad (26)$$

where  $L^I{}_{JK} = \dot{L}^I{}_{JK}$  and  $C^A{}_{BC} = \dot{C}^A{}_{BC}$ , which is hv-metric, i.e. satisfies conditions:

$$D_X^{(h)} hG = 0$$

and

$$D_X^{(v)} vG = 0,$$

for every  $X \in \Xi(E)$ , or in locally adapted coordinates,

$$g_{IJ|K} = 0$$

and

$$h_{AB\perp C} = 0.$$

As well we can introduce the Levi-Civita connection

$$\left\{ \frac{\alpha}{\beta\gamma} \right\} = \frac{1}{2}G^{\alpha\beta}(\partial_\beta G_{\tau\gamma} + \partial_\gamma G_{\tau\beta} - \partial_\tau G_{\beta\gamma}),$$

constructed as in the Riemann geometry from components of metric  $G_{\alpha\beta}$  by using partial derivations  $\partial_\alpha = \frac{\partial}{\partial u^\alpha} = (\frac{\partial}{\partial x^I}, \frac{\partial}{\partial y^A})$  which is metric but not a d-connection.

Another metric d-connection can be defined as

$$\tilde{\Gamma}_{\beta\gamma}^\alpha = \frac{1}{2}G^{\alpha\tau}(\delta_\beta G_{\tau\gamma} + \delta_\gamma G_{\tau\beta} - \delta_\tau G_{\beta\gamma}), \quad (27)$$

with components  $C\tilde{\Gamma} = (L^I{}_{JK}, 0, 0, C^A{}_{BC})$ , where coefficients  $L^I{}_{JK}$  and  $C^A{}_{BC}$  are computed as in formulas (26). We call the coefficients (27) the generalized Christoffel symbols on vs-bundle  $\mathcal{E}$ .

For our further considerations it is useful to express arbitrary d-connection as a deformation of the background d-connection (26):

$$\Gamma^\alpha{}_{\beta\gamma} = \tilde{\Gamma}^\alpha{}_{\beta\gamma} + P^\alpha{}_{\beta\gamma}, \quad (28)$$

where  $P^\alpha_{\beta\gamma}$  is called the deformation ds-tensor. Putting splitting (29) into (12) and (16) we can express torsion  $T^\alpha_{\beta\gamma}$  and curvature  $R^\alpha_{\beta\gamma\delta}$  of a d-connection  $\Gamma^\alpha_{\beta\gamma}$  as respective deformations of torsion  $\tilde{T}^\alpha_{\beta\gamma}$  and torsion  $\tilde{R}^\alpha_{\beta\gamma\delta}$  for connection  $\tilde{\Gamma}^\alpha_{\beta\gamma}$  :

$$T^\alpha_{\beta\gamma} = \tilde{T}^\alpha_{\beta\gamma} + \ddot{T}^\alpha_{\beta\gamma} \tag{29}$$

and

$$R^\alpha_{\beta\gamma\delta} = \tilde{R}^\alpha_{\beta\gamma\delta} + \ddot{R}^\alpha_{\beta\gamma\delta}, \tag{30}$$

where

$$\tilde{T}^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma} - (-)^{|\beta\gamma|} \tilde{\Gamma}^\alpha_{\gamma\beta} + w^\alpha_{\gamma\delta}, \quad \ddot{T}^\alpha_{\beta\gamma} = \ddot{\Gamma}^\alpha_{\beta\gamma} - (-)^{|\beta\gamma|} \ddot{\Gamma}^\alpha_{\gamma\beta},$$

and

$$\tilde{R}^\alpha_{\beta\gamma\delta} = \delta_\delta \tilde{\Gamma}^\alpha_{\beta\gamma} - (-)^{|\gamma\delta|} \delta_\gamma \tilde{\Gamma}^\alpha_{\beta\delta} + \tilde{\Gamma}^\varphi_{\beta\gamma} \tilde{\Gamma}^\alpha_{\varphi\delta} - (-)^{|\gamma\delta|} \tilde{\Gamma}^\varphi_{\beta\delta} \tilde{\Gamma}^\alpha_{\varphi\gamma} + \tilde{\Gamma}^\alpha_{\beta\varphi} w^\varphi_{\gamma\delta},$$

$$\ddot{R}^\alpha_{\beta\gamma\delta} = \ddot{D}_\delta P^\alpha_{\beta\gamma} - (-)^{|\gamma\delta|} \ddot{D}_\gamma P^\alpha_{\beta\delta} + P^\varphi_{\beta\gamma} P^\alpha_{\varphi\delta} - (-)^{|\gamma\delta|} P^\varphi_{\beta\delta} P^\alpha_{\varphi\gamma} + P^\alpha_{\beta\varphi} w^\varphi_{\gamma\delta},$$

the nonholonomy coefficients  $w^\alpha_{\beta\gamma}$  are defined as

$$[\delta_\alpha, \delta_\beta] = \delta_\alpha \delta_\beta - (-)^{\alpha\beta} \delta_\beta \delta_\alpha = w^\tau_{\alpha\beta} \delta_\tau.$$

Finally, in this section we remark that if from geometric point of view all considered d-connections are "equal in rights", the construction of physical models on la-spaces requires an explicit fixing of the type of d-connection and metric structures.

## 5 Supersymmetric Generalized Lagrange Spaces

Let us fix our attention to the st-bundle  $TM$ . The aim of this section is to formulate some results in the supergeometry of  $TM$  and to use them in order to develop the geometry of Finsler and Lagrange superspaces (classical and new approaches to Finsler geometry, its generalizations and applications in physics are contained, for example, in [20-30]).

All presented in the previous section basic results on sv-bundles provided with N-connection, d-connection and metric structures hold good for  $TM$ . In this case the dimension of the base space and typical fibre coincides and we can write locally, for instance, s-vectors as

$$X = X^I \delta_I + Y^J \partial_I = X^I \delta_I + Y^{(I)} \partial_{(I)},$$

where  $u^\alpha = (x^I, y^J) = (x^I, y^{(J)})$ .

On st-bundles we can define a global map

$$J : \Xi(TM) \rightarrow \Xi(TM) \tag{31}$$

which does not depend on N-connection structure:

$$J\left(\frac{\delta}{\delta x^I}\right) = \frac{\partial}{\partial y^I}$$

and

$$J\left(\frac{\partial}{\partial y^I}\right) = 0.$$

This endomorphism is called the natural (or canonical) almost tangent structure on  $TM$ ; it has the properties:

$$1) J^2 = 0, \quad 2) \text{Im} J = \text{Ker} J = VTM$$

and 3) the Nigenhuis s-tensor,

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] \\ (X, Y \in \Xi(TN))$$

identically vanishes, i.e. the natural almost tangent structure  $J$  on  $TM$  is integrable.

### 5.1 Notions of Generalized Lagrange, Lagrange and Finsler Superspaces

Let  $M$  be a supersmooth  $(n+m)$ -dimensional s-manifold and  $(TM, \tau, M)$  its st-bundle. The metric of type  $g_{ij}(x, y)$  was introduced by P.Finsler as a generalization of that for Riemannian spaces. Variables  $y = (y^i)$  can be interpreted as parameters of local anisotropy or of fluctuations in nonhomogeneous and turbulent media. The most general form of metrics with local anisotropy have been recently studied in the frame of the so-called generalized Lagrange geometry (GL-geometry, the geometry of GL-spaces) [26,27]. For s-spaces we introduce this

**Definition 5** A *generalized Lagrange superspace*, *GLS-space*, is a pair  $GL^{n,m} = (M, g_{IJ}(x, y))$ , where  $g_{IJ}(x, y)$  is a ds-tensor field on  $\tilde{TM} = TM - \{0\}$ , supersymmetric of superrank  $(n, m)$ .

We call  $g_{IJ}$  as the fundamental ds-tensor, or metric ds-tensor, of GLS-space. In this work we shall not introduce a supersymmetric notion of signature in order to be able to consider physical models with variable signature on the even part of the s-spaces.

It is well known that if  $M$  is a paracompact s-manifold there exists at least a nonlinear connection in the its tangent bundle. Thus it is quite natural to fix a nonlinear connection  $N$  in  $TM$  and to relate it to  $g_{IJ}(x, y)$ , by using equations (22) written on  $TM$ . For simplicity, we can consider  $N$ -connection with vanishing torsion, when

$$\partial_K N_J^I - (-)^{|JK|} \partial_J N_K^I = 0.$$

Let denote a normal d-connection, defined by using  $N$  and adapted to the almost tangent structure (31) as  $D\Gamma = (L^A_{JK}, C^A_{JK})$ . This d-connection is compatible with metric  $g_{IJ}(x, y)$  if  $g_{IJ|K} = 0$  and  $g_{IJ\perp K} = 0$ .

There exists a unique d-connection  $CT(N)$  which is compatible with  $g_{IJ}(u)$  and has vanishing torsions  $T^I_{JK}$  and  $S^I_{JK}$  (see formulas (12) rewritten for st-bundles).

This connection, depending only on  $g_{IJ}(u)$  and  $N_J^I(u)$  is called the canonical metric d-connection of GLS-space. It has coefficients

$$L^I{}_{JK} = \frac{1}{2}g^{IH}(\delta_J g_{HK} + \delta_H g_{JK} - \delta_H g_{JK}), \tag{32}$$

$$C^I{}_{JK} = \frac{1}{2}g^{IH}(\partial_J g_{HK} + \partial_H g_{JK} - \partial_H g_{JK}).$$

Of course, metric d-connections different from  $CT(N)$  may be found. For instance, there is a unique normal d-connection  $D\Gamma(N) = (\bar{L}^I{}_{JK}, \bar{C}^I{}_{JK})$  which is metric and has a priori given torsions  $T^I{}_{JK}$  and  $S^I{}_{JK}$ . The coefficients of  $D\Gamma(N)$  are the following ones:

$$\bar{L}^I{}_{JK} = L^I{}_{JK} - \frac{1}{2}g^{IH}(g_{JR}T^R{}_{HK} + g_{KR}T^R{}_{HJ} - g_{HR}T^R{}_{KJ}),$$

$$\bar{C}^I{}_{JK} = C^I{}_{JK} - \frac{1}{2}g^{IH}(g_{JR}S^R{}_{HK} + g_{KR}S^R{}_{HJ} - g_{HR}S^R{}_{KJ}),$$

where  $L^I{}_{JK}$  and  $C^I{}_{JK}$  are the same as for the  $CT(N)$ -connection (32).

The Lagrange spaces were introduced [46] in order to geometrize the concept of Lagrangian in mechanics. The Lagrange geometry is studied in details in [26,27]. For s-spaces we present this generalization:

**Definition 6** A Lagrange s-space, LS-space,  $L^{n,m} = (M, g_{IJ})$ , is defined as a particular case of GLS-space when the ds-metric on M can be expressed as

$$g_{IJ}(u) = \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial y^I \partial y^J}, \tag{33}$$

where  $\mathcal{L} : TM \rightarrow \Lambda$ , is a s-differentiable function called a s-Lagrangian on M.

Now we consider the supersymmetric extension of the Finsler space:

A Finsler s-metric on M is a function  $F_S : TM \rightarrow \Lambda$  having the properties:

1. The restriction of  $F_S$  to  $\tilde{TM} = TM \setminus \{0\}$  is of the class  $G^\infty$  and F is only supersmooth on the image of the null cross-section in the st-bundle to M.
2. The restriction of F to  $\tilde{TM}$  is positively homogeneous of degree 1 with respect to  $(y^I)$ , i.e.  $F(x, \lambda y) = \lambda F(x, y)$ , where  $\lambda$  is a real positive number.
3. The restriction of F to the even subspace of  $\tilde{TM}$  is a positive function.
4. The quadratic form on  $\Lambda^{n,m}$  with the coefficients

$$g_{IJ}(u) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^I \partial y^J} \tag{34}$$

defined on  $\tilde{TM}$  is nondegenerate.

**Definition 7** A pair  $F^{n,m} = (M, F)$  which consists from a supersmooth s-manifold M and a Finsler s-metric is called a Finsler superspace, FS-space.

It's obvious that FS-spaces form a particular class of LS-spaces with s-Lagrangian  $\mathcal{L} = F^2$  and a particular class of GLS-spaces with metrics of type (34).

For a FS-space we can introduce the supersymmetric variant of nonlinear Cartan connection [24,25]:

$$N_J^I(x, y) = \frac{\partial}{\partial y^J} G^{*I},$$

where

$$G^{*I} = \frac{1}{4} g^{*IJ} \left( \frac{\partial^2 \varepsilon}{\partial y^I \partial x^K} y^K - \frac{\partial \varepsilon}{\partial x^J} \right), \quad \varepsilon(u) = g_{IJ}(u) y^I y^J,$$

and  $g^{*IJ}$  is inverse to  $g_{IJ}^*(u) = \frac{1}{2} \frac{\partial^2 \varepsilon}{\partial y^I \partial y^J}$ . In this case the coefficients of canonical metric d-connection (32) gives the supersymmetric variants of coefficients of the Cartan connection of Finsler spaces. A similar remark applies to the Lagrange superspaces.

## 5.2 The Supersymmetric Almost Hermitian Model of the GLS–Space

Consider a GLS–space endowed with the canonical metric d-connection  $CT(N)$ . Let  $\delta_\alpha = (\delta_\alpha, \dot{\partial}_I)$  be a usual adapted frame (6) on TM and  $\delta^\alpha = (\partial^I, \dot{\delta}^I)$  its dual. The linear operator

$$F : \Xi(T\dot{M}) \rightarrow \Xi(T\dot{M}),$$

acting on  $\delta_\alpha$  by  $F(\delta_I = -\dot{\partial}_I, F(\dot{\partial}_I) = \delta_I$ , defines an almost complex structure on  $T\dot{M}$ . We shall obtain a complex structure if and only if the even component of the horizontal distribution N is integrable. For s-spaces, in general with even and odd components, we write the supersymmetric almost Hermitian property (almost Hermitian s-structure) as

$$F_\beta^\alpha F_\delta^\beta = -(-)^{|\alpha\delta|} \delta_\beta^\alpha.$$

The s-metric  $g_{IJ}(x, y)$  on GLS-spaces induces on  $T\dot{M}$  the following metric:

$$G = g_{IJ}(u) dx^I \otimes dx^J + g_{IJ}(u) \delta y^I \otimes \delta y^J. \quad (35)$$

We can verify that pair  $(G, F)$  is an almost Hermitian s-structure on  $T\dot{M}$  with the associated supersymmetric 2-form

$$\theta = g_{IJ}(x, y) \delta y^I \wedge dx^J.$$

The almost Hermitian s-space  $H_S^{2n, 2m} = (TM, G, F)$ , provided with a metric of type (35) is called the lift on TM, or the almost Hermitian s-model, of GLS-space  $GL^{n, m}$ . We say that a linear connection  $D$  on  $T\dot{M}$  is almost Hermitian supersymmetric of Lagrange type if it preserves by parallelism the vertical distribution V and is compatible with the almost Hermitian s-structure  $(G, F)$ , i.e.

$$D_X G = 0, \quad D_X F = 0, \quad (36)$$

for every  $X \in \Xi(TM)$ .

There exists an unique almost Hermitian connection of Lagrange type  $D^{(c)}$  having  $h(hh)$ - and  $v(vv)$ -torsions equal to zero. We can prove (similarly as in [26,27]) that coefficients  $(L^I{}_{JK}, C^I{}_{JK})$  of  $D^{(c)}$  in the adapted basis  $(\delta_I, \delta_J)$  are just the coefficients (32) of the canonical metric d-connection  $CT(N)$  of the GLS-space  $GL^{(n,m)}$ . Inversely, we can say that  $CT(N)$ -connection determines on  $T\tilde{N}$  and supersymmetric almost Hermitian connection of Lagrange type with vanishing  $h(hh)$ - and  $v(vv)$ -torsions. If instead of GLS-space metric  $g_{IJ}$  in (34) the Lagrange (or Finsler) s-metric (32) (or (33)) is taken, we obtain the almost Hermitian s-model of Lagrange (or Finsler) s-spaces  $L^{n,m}$  (or  $F^{n,m}$ ).

We note that the natural compatibility conditions (36) for the metric (35) and  $CT(N)$ -connections on  $H^{2n,2m}$ -spaces plays an important role for developing physical models on la-superspaces. In the case of usual locally anisotropic spaces geometric constructions and d-covariant calculus are very similar to those for the Riemann and Einstein-Cartan spaces. This is exploited for formulation in a selfconsistent manner the theory of spinors on la-spaces [35], for introducing a geometric background for locally anisotropic Yang-Mills and gauge like gravitational interactions [31,32] and for extending the theory of stochastic processes and diffusion to the case of locally anisotropic spaces and interactions on such spaces [47]. In a similar manner we shall use in this work N-lifts to sv- and st-bundles in order to investigate supergravitational la-models.

## 6 Supergravity on Locally Anisotropic Superspaces

In this section we shall introduce a set of Einstein and (equivalent in our case) gauge like gravitational equations, i.e. we shall formulate two variants of la-supergravity, on the total space  $E$  of a sv-bundle  $\mathcal{E}$  over a supersmooth manifold  $M$ . The first model will be a variant of locally anisotropic supergravity theory generalizing the Miron and Anastasiei model [26,27] on vector bundles (they considered prescribed components of N-connection and  $h(hh)$ - and  $v(vv)$ -torsions, in our approach we shall introduce algebraic equations for torsion and its source). The second model will be a la-supersymmetric extension of constructions for gauge la-gravity [31,32] and affine-gauge interpretation of the Einstein gravity [55,56]. There are two ways in developing supergravitational models. We can try to maintain similarity to Einstein's general relativity (see in [48,49] an example of locally isotropic supergravity) and to formulate a variant of Einstein-Cartan theory on sv-bundles, this will be the aim of the subsection A, or to introduce into consideration supervielbein variables and to formulate a supersymmetric gauge like model of la-supergravity (this approach is more accepted in the usual locally isotropic supergravity, see as a review [45]). The last variant will be analysed in subsection B by using the s-bundle of supersymmetric affine adapted frames on la-superspaces. For both models of la-supergravity we shall consider the matter field contributions as giving rise to corresponding sources in la-supergravitational field equations. A detailed study of supersymmetric of la-gravitational and matter fields is a matter of our further investigations [58].

### 6.1 Einstein–Cartan Equations on SV–Bundles

Let consider a sv–bundle  $\mathcal{E} = (E, \pi, M)$  provided with some compatible nonlinear connection  $N$ , d–connection  $D$  and metric  $G$  structures. For a locally  $N$ -adapted frame we write

$$D_{(\frac{\delta}{\delta u^\tau})} \frac{\delta}{\delta u^\beta} = \Gamma_{\beta\gamma}^\alpha \frac{\delta}{\delta u^\alpha},$$

where the d-connection  $D$  has the following coefficients:

$$\begin{aligned} \Gamma^I_{JK} = L^I_{JK}, \Gamma^I_{JA} = C^I_{JA}, \Gamma^I_{AJ} = 0, \Gamma^I_{AB} = 0, \Gamma^A_{JK} = 0, \\ \Gamma^A_{JB} = 0, \Gamma^A_{BK} = L^A_{BK}, \Gamma^A_{BC} = C^A_{BC}. \end{aligned} \quad (37)$$

The nonholonomy coefficients  $w^\gamma_{\alpha\beta}$ , defined as  $[\delta_\alpha, \delta_\beta] = w^\gamma_{\alpha\beta} \delta_\gamma$ , are as follows:

$$\begin{aligned} w^K_{IJ} = 0, w^K_{AJ} = 0, w^K_{IA} = 0, w^K_{AB} = 0, w^A_{IJ} = R^A_{IJ}, \\ w^B_{AI} = -(-)^{|IA|} \frac{\partial N^B_A}{\partial y^A}, w^B_{IA} = \frac{\partial N^B_A}{\partial y^A}, w^C_{AB} = 0. \end{aligned}$$

By straightforward calculations we can obtain respectively these components of torsion,

$\mathcal{T}(\delta_\gamma, \delta_\beta) = \mathcal{T}_{\beta\gamma}^\alpha \delta_\alpha$ , and curvature,  $\mathcal{R}(\delta_\beta, \delta_\gamma) \delta_\tau = \mathcal{R}_{\beta\cdot\gamma\tau}^\alpha \delta_\alpha$ , ds-tensors:

$$\mathcal{T}^I_{JK} = T^I_{JK}, \mathcal{T}^I_{JA} = C^I_{JA}, \mathcal{T}^I_{AJ} = -C^I_{JA}, \mathcal{T}^I_{AB} = 0, \quad (38)$$

$$\mathcal{T}^A_{IJ} = R^A_{IJ}, \mathcal{T}^A_{IB} = -P^A_{BI}, \mathcal{T}^A_{BI} = P^A_{BI}, \mathcal{T}^A_{BC} = S^A_{BC}$$

and

$$\begin{aligned} \mathcal{R}^J_{I\cdot KL} = R^J_{I\cdot KL}, \mathcal{R}^J_{B\cdot KL} = 0, \mathcal{R}^A_{J\cdot KL} = 0, \mathcal{R}^A_{B\cdot KL} = \tilde{R}^A_{B\cdot KL}, \\ \mathcal{R}^I_{J\cdot KD} = P^I_{J\cdot KD}, \mathcal{R}^A_{B\cdot KD} = 0, \mathcal{R}^A_{J\cdot KD} = 0, \mathcal{R}^A_{B\cdot KD} = P^A_{B\cdot KD}, \\ \mathcal{R}^I_{J\cdot DK} = -P^I_{J\cdot DK}, \mathcal{R}^I_{B\cdot DK} = 0, \mathcal{R}^A_{J\cdot DK} = 0, \mathcal{R}^H_{B\cdot DK} = -P^A_{B\cdot KD}, \\ \mathcal{R}^I_{J\cdot CD} = S^I_{J\cdot CD}, \mathcal{R}^I_{B\cdot CD} = 0, \mathcal{R}^A_{J\cdot CD} = 0, \mathcal{R}^A_{B\cdot CD} = S^A_{B\cdot CD} \end{aligned} \quad (39)$$

(for explicit dependencies of components of torsions and curvatures on components of d-connection see formulas (12) and (16)).

The locally adapted components  $\mathcal{R}_{\alpha\beta} = Ric(D)(\delta_\alpha, \delta_\beta)$  (we point that in general on st-bundles  $\mathcal{R}_{\alpha\beta} \neq (-)^{|\alpha\beta|} \mathcal{R}_{\beta\alpha}$ ) of the Ricci tensor are as follows:

$$\mathcal{R}_{IJ} = R_I^K{}_{JK}, \mathcal{R}_{IA} = -^{(2)}P_{IA} = -\tilde{P}^K_{I\cdot KA} \quad (40)$$

$$\mathcal{R}_{AI} = {}^{(1)}P_{AI} = \tilde{P}^B_{A\cdot IB}, \mathcal{R}_{AB} = S^C_{A\cdot BC} = S_{AB}.$$

For scalar curvature,  $\tilde{\mathcal{R}} = Sc(D) = G^{\alpha\beta} \mathcal{R}_{\alpha\beta}$ , we have

$$Sc(D) = R + S, \quad (41)$$

where  $R = g^{IJ} R_{IJ}$  and  $S = h^{AB} S_{AB}$ .



The Einstein–Cartan equations on sv-bundles are written as

$$\mathcal{R}_{\alpha\beta} - \frac{1}{2}G_{\alpha\beta}\tilde{\mathcal{R}} + \lambda G_{\alpha\beta} = \kappa_1 \mathcal{J}_{\alpha\beta}, \quad (42)$$

and

$$T_{\cdot\beta\gamma}^\alpha + G_\beta^\alpha T^\tau_{\cdot\gamma\tau} - (-)^{|\beta\gamma|} G_\gamma^\alpha T^\tau_{\cdot\beta\tau} = \kappa_2 Q_{\cdot\beta\gamma}^\alpha, \quad (43)$$

where  $\mathcal{J}_{\alpha\beta}$  and  $Q_{\cdot\beta\gamma}^\alpha$  are respectively components of energy-momentum and spin-density of matter ds-tensors on la-space,  $\kappa_1$  and  $\kappa_2$  are the corresponding interaction constants and  $\lambda$  is the cosmological constant. To write in a explicit form the mentioned matter sources of la-supergravity in (42) and (43) there are necessary more detailed studies of models of interaction of superfields on la-superspaces (see first results for Yang–Mills and spinor fields on la-spaces in [31,32,35] and, from different points of view, [28,29,38]). We omit such considerations in this paper.

Equations (42), specified in (x,y)-components,

$$R_{IJ} - \frac{1}{2}(R + S - \lambda)g_{IJ} = \kappa_1 \mathcal{J}_{IJ}, \quad (1)P_{AI} = \kappa_1 (1)\mathcal{J}_{AI}, \quad (44)$$

$$S_{AB} - \frac{1}{2}(S + R - \lambda)h_{AB} = \kappa_2 \tilde{\mathcal{J}}_{AB}, \quad (2)P_{IA} = -\kappa_2 (2)\mathcal{J}_{IA},$$

are a supersymmetric, with cosmological term, generalization of the similar ones presented in [26,27], with prescribed N-connection and h(hh)- and v(vv)-torsions. We have added algebraic equations (43) in order to close the system of s-gravitational field equations (really we have also to take into account the system of constraints (22) if locally anisotropic s-gravitational field is associated to a ds-metric (23)).

We point out that on la-superspaces the divergence  $D_\alpha \mathcal{J}^\alpha$  does not vanish (this is a consequence of generalized Bianchi and Ricci identities (17),(19) and (20),(21)). The d-covariant derivations of the left and right parts of (42), equivalently of (44), are as follows:

$$D_\alpha [\mathcal{R}_{\cdot\beta}^\alpha - \frac{1}{2}(\tilde{\mathcal{R}} - 2\lambda)\delta_{\cdot\beta}^\alpha] = \begin{cases} [R_J^I - \frac{1}{2}(R + S - 2\lambda)\delta_J^I]_{|I} + (1)P^A_{I\perp A} = 0, \\ [S_B^A - \frac{1}{2}(R + S - 2\lambda)\delta_B^A]_{\perp A} - (2)P^I_{B|I} = 0, \end{cases}$$

where

$$(1)P^A_{\cdot J} = (1)P_{BJ}h^{AB}, \quad (2)P^I_{\cdot B} = (2)P_{JB}g^{IJ}, \quad R^I_{\cdot J} = R_{KJ}g^{IK}, \quad S^A_{\cdot B} = S_{CB}h^{AC},$$

and

$$D_\alpha \mathcal{J}_{\cdot\beta}^\alpha = \mathcal{U}_\alpha, \quad (45)$$

where

$$D_\alpha \mathcal{J}_{\cdot\beta}^\alpha = \begin{cases} \mathcal{J}_{\cdot J|I}^I + (1)\mathcal{J}_{\cdot J\perp A}^A = \frac{1}{\kappa_1}\mathcal{U}_I, \\ (2)\mathcal{J}_{\cdot A|I}^I + \mathcal{J}_{\cdot A\perp B}^B = \frac{1}{\kappa_1}\mathcal{U}_A, \end{cases}$$

and

$$\mathcal{U}_\alpha = \frac{1}{2}(G^{\beta\delta}\mathcal{R}_{\delta\cdot\varphi\beta}^\gamma \mathcal{T}_{\cdot\alpha\gamma}^\varphi - (-)^{|\alpha\beta|} G^{\beta\delta}\mathcal{R}_{\delta\cdot\varphi\alpha}^\gamma \mathcal{T}_{\cdot\beta\gamma}^\varphi + \mathcal{R}_{\cdot\varphi}^\beta \mathcal{T}_{\cdot\beta\alpha}^\varphi). \quad (46)$$

From the last formula it follows that ds-vector  $\mathcal{U}_\alpha$  vanishes if d-connection (37) is torsionless.

No wonder that conservation laws for values of energy–momentum type, being a consequence of global automorphisms of spaces and s–spaces, or, respectively, of their tangent spaces and s–spaces (for models on curved spaces and s–spaces), on la–superspaces are more sophisticated because, in general, such automorphisms do not exist for a generic local anisotropy. We can construct a la–model of supergravity, in a way similar to that for the Einstein theory if instead an arbitrary metric d–connection the generalized Christoffel symbols  $\tilde{\Gamma}^\alpha_{\beta\gamma}$  (see (27)) are used. This will be a locally anisotropic supersymmetric model on the base s-manifold  $M$  which looks like locally isotropic on the total space of a sv-bundle. More general supergravitational models which are locally anisotropic on the both base and total spaces can be generated by using deformations of d-connections of type (28). In this case the vector  $\mathcal{U}_\alpha$  from (46) can be interpreted as a corresponding source of generic local anisotropy satisfying generalized conservation laws of type (45).

More completely the problem of formulation of conservation laws for both locally isotropic and anisotropic supergravity can be solved in the frame of the theory of nearly autoparallel maps of sv-bundles (with specific deformation of d-connections (28), torsion (29) and curvature (30)), which have to generalize our constructions from [33,34,51]. This is a matter of our further investigations.

We end this subsection with the remark that field equations of type (42), equivalently (44), and (43) for la-supergravity can be similarly introduced for the particular cases of GLS–spaces with metric (35) on  $T\tilde{M}$  with coefficients parametrized as for the Lagrange, (33), or Finsler, (34), spaces.

## 6.2 Gauge Like Locally Anisotropic Supergravity

The great part of theories of locally isotropic s-gravity are formulated as gauge supersymmetric models based on supervielbein formalism (see [45,51–53]). Similar approaches to la-supergravity on vs-bundles can be developed by considering an arbitrary adapted to N-connection frame  $B_{\underline{\alpha}}(u) = (B_{\underline{I}}(u), B_{\underline{C}}(u))$  on  $\mathcal{E}$  and supervielbein, s-vielbein, matrix

$$A_{\alpha}^{\alpha}(u) = \begin{pmatrix} A_I^I(u) & 0 \\ 0 & A_C^C(u) \end{pmatrix} \subset GL_{n,k}^{m,l}(\Lambda) = \\ GL(n, k, \Lambda) \oplus GL(m, l, \Lambda)$$

for which

$$\frac{\delta}{\delta u^\alpha} = A_{\alpha}^{\alpha}(u) B_{\underline{\alpha}}(u),$$

or, equivalently,  $\frac{\delta}{\delta x^I} = A_I^I(x, y) B_{\underline{I}}(x, y)$  and  $\frac{\partial}{\partial y^C} = A_C^C B_{\underline{C}}(x, y)$ , and

$$G_{\alpha\beta}(u) = A_{\alpha}^{\alpha}(u) A_{\beta}^{\beta}(u) \eta_{\underline{\alpha}\underline{\beta}},$$

where, for simplicity,  $\eta_{\underline{\alpha}\underline{\beta}}$  is a constant metric on vs-space  $V^{n,k} \oplus V^{l,m}$ .

We denote by  $LN(\mathcal{E})$  the set of all adapted frames in all points of sv-bundle  $\mathcal{E}$ . Considering the surjective s-map  $\pi_L$  from  $LN(\mathcal{E})$  to  $\mathcal{E}$  and treating  $GL_{n,k}^{m,l}(\Lambda)$  as the structural s-group we define a principal s-bundle,

$$LN(\mathcal{E}) = (LN(\mathcal{E}), \pi_L : LN(\mathcal{E}) \rightarrow \mathcal{E}, GL_{n,k}^{m,l}(\Lambda)),$$

called as the s-bundle of linear adapted frames on  $\mathcal{E}$ .

Let denote the canonical basis of the sl-algebra  $\mathcal{G}_{n,k}^{m,l}$  for a s-group  $GL_{n,k}^{m,l}(\Lambda)$  as  $I_{\hat{\alpha}}$ , where index  $\hat{\alpha} = (\hat{I}, \hat{J})$  enumerates the  $Z_2$ -graded components. The structural coefficients  $f_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}}$  of  $\mathcal{G}_{n,k}^{m,l}$  satisfy s-commutation rules

$$[I_{\hat{\alpha}}, I_{\hat{\beta}}] = f_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} I_{\hat{\gamma}}.$$

On  $\mathcal{E}$  we consider the connection 1-form

$$\Gamma = \Gamma_{\underline{\beta}\gamma}^{\alpha} I_{\underline{\alpha}}^{\beta} du^{\gamma}, \tag{47}$$

where

$$\Gamma_{\underline{\beta}\gamma}^{\alpha} = A^{\alpha}_{\underline{\alpha}} A^{\beta}_{\underline{\beta}} \Gamma^{\alpha}_{\beta\gamma} + A^{\alpha}_{\underline{\alpha}} \frac{\delta}{\delta u^{\gamma}} A^{\beta}_{\underline{\beta}}(u),$$

$\Gamma^{\alpha}_{\beta\gamma}$  are the components of the metric d-connection (37), s-matrix  $A^{\beta}_{\underline{\beta}}$  is inverse to the s-vielbein matrix  $A^{\underline{\beta}}_{\beta}$ , and  $I_{\underline{\beta}}^{\alpha} = \begin{pmatrix} I_{\underline{J}}^I & 0 \\ 0 & I_{\underline{B}}^A \end{pmatrix}$  is the standard distinguished basis in SL-algebra  $\mathcal{G}_{n,k}^{m,l}$ .

The curvature  $\mathcal{B}$  of the connection (47),

$$\mathcal{B} = d\Gamma + \Gamma \wedge \Gamma = \mathcal{R}_{\underline{\alpha}\gamma\delta}^{\cdot\beta} I_{\underline{\beta}}^{\alpha} \delta u^{\gamma} \wedge \delta u^{\delta} \tag{48}$$

has coefficients

$$\mathcal{R}_{\underline{\alpha}\gamma\delta}^{\cdot\beta} = A^{\alpha}_{\underline{\alpha}}(u) A^{\beta}_{\underline{\beta}}(u) \mathcal{R}_{\alpha\gamma\delta}^{\cdot\beta},$$

where  $\mathcal{R}_{\alpha\gamma\delta}^{\cdot\beta}$  are the components of the ds-tensor (39).

Aside from  $LN(\mathcal{E})$  with vs-bundle  $\mathcal{E}$  it is naturally related another s-bundle, the bundle of adapted affine frames  $\mathcal{AN}(\mathcal{E}) = (\mathcal{AN}(\mathcal{E}), \pi_A : \mathcal{AN}(\mathcal{E}) \rightarrow \mathcal{E}, \mathcal{AF}_{n,k}^{m,l}(\Lambda))$  with the structural s-group  $\mathcal{AN}_{n,k}^{m,l}(\Lambda) = GL_{n,k}^{m,l}(\Lambda) \odot \Lambda^{n,k} \oplus \Lambda^{m,l}$  being a semidirect product (denoted by  $\odot$ ) of  $GL_{n,k}^{m,l}(\Lambda)$  and  $\Lambda^{n,k} \oplus \Lambda^{m,l}$ . Because as a linear s-space the LS-algebra  $\mathcal{AF}_{n,k}^{m,l}$  of s-group  $\mathcal{AF}_{n,k}^{m,l}(\Lambda)$ , is a direct sum of  $\mathcal{G}_{n,k}^{m,l}$  and  $\Lambda^{n,k} \oplus \Lambda^{m,l}$  we can write forms on  $\mathcal{AN}(\mathcal{E})$  as  $\Theta = (\Theta_1, \Theta_2)$ , where  $\Theta_1$  is the  $\mathcal{G}_{n,k}^{m,l}$ -component and  $\Theta_2$  is the  $(\Lambda^{n,k} \oplus \Lambda^{m,l})$ -component of the form  $\Theta$ . The connection (47) in  $LN(\mathcal{E})$  induces a Cartan connection  $\bar{\Gamma}$  in  $\mathcal{AN}(\mathcal{E})$  (see, for instance, in [55] the case of usual affine frame bundles). This is the unique connection on s-bundle  $\mathcal{AN}(\mathcal{E})$  represented as  $i^*\bar{\Gamma} = (\Gamma, \chi)$ , where  $\chi$  is the shifting form and  $i : \mathcal{AN}(\mathcal{E}) \rightarrow LN(\mathcal{E})$  is the trivial

reduction of s-bundles. If  $B = (B_\alpha)$  is a local adapted frame in  $\mathcal{LN}(\mathcal{E})$  then  $\bar{B} = i \circ B$  is a local section in  $\mathcal{AN}(\mathcal{E})$  and

$$\begin{aligned}\bar{\Gamma} &= B\Gamma = (\Gamma, \chi), \\ \bar{\mathcal{B}} &= \bar{B}\mathcal{B} = (\mathcal{B}, \mathcal{T}),\end{aligned}\tag{49}$$

where  $\chi = e_\alpha \otimes A^\alpha_\alpha du^\alpha$ ,  $e_\alpha$  is the standard basis in  $\Lambda^{n,k} \oplus \Lambda^{m,l}$  and torsion  $\mathcal{T}$  is introduced as

$$\mathcal{T} = d\chi + [\Gamma \wedge \chi] = \mathcal{T}^\alpha_{\beta\gamma} e_\alpha du^\beta \wedge du^\gamma,$$

$\mathcal{T}^\alpha_{\beta\gamma} = A^\alpha_\alpha T^{\alpha\cdot\beta\gamma}$  are defined by the components of the torsion ds-tensor (38).

By using metric G (35) on sv-bundle  $\mathcal{E}$  we can define the dual (Hodge) operator  $*_G : \bar{\Lambda}^{q,s}(\mathcal{E}) \rightarrow \bar{\Lambda}^{n-q,k-s}(\mathcal{E})$  for forms with values in LS-algebras on  $\mathcal{E}$  (see details, for instance, in [52]), where  $\bar{\Lambda}^{q,s}(\mathcal{E})$  denotes the s-algebra of exterior (q,s)-forms on  $\mathcal{E}$ .

Let operator  $*_G^{-1}$  be the inverse to operator  $*$  and  $\hat{\delta}_G$  be the adjoint to the absolute derivation d (associated to the scalar product for s-forms) specified for (r,s)-forms as

$$\delta_G = (-1)^{r+s} *_G^{-1} \circ d \circ *_G.$$

Both introduced operators act in the space of LS-algebra-valued forms as

$$*_G(I_{\hat{\alpha}} \otimes \phi^{\hat{\alpha}}) = I_{\hat{\alpha}} \otimes (*_G \phi^{\hat{\alpha}})$$

and

$$\delta_G(I_{\hat{\alpha}} \otimes \phi^{\hat{\alpha}}) = I_{\hat{\alpha}} \otimes \delta_G \phi^{\hat{\alpha}}.$$

If the supersymmetric variant of the Killing form for the structural s-group of a s-bundle into consideration is degenerate as a s-matrix (for instance, this holds for s-bundle  $\mathcal{AN}(\mathcal{E})$ ) we use an auxiliary nondegenerate bilinear s-form in order to define formally a metric structure  $G_A$  in the total space of the s-bundle. In this case we can introduce operator  $\delta_{\mathcal{E}}$  acting in the total space and define operator  $\Delta \doteq \hat{H} \circ \delta_A$ , where  $\hat{H}$  is the operator of horizontal projection. After  $\hat{H}$ -projection we shall not have dependence on components of auxiliary bilinear forms.

Methods of abstract geometric calculus, by using operators  $*_G, *_A, \delta_G, \delta_A$  and  $\Delta$ , are illustrated, for instance, in [54-57] for locally isotropic, and in [32] for locally anisotropic, spaces. Because on superspaces these operators act in a similar manner we omit tedious intermediate calculations and present the final necessary results. For  $\Delta \bar{B}$  one computes

$$\Delta \bar{B} = (\Delta \mathcal{B}, \mathcal{R}\tau + \mathcal{R}i),$$

where  $\mathcal{R}\tau = \delta_G \mathcal{J} + *_G^{-1}[\Gamma, *\mathcal{J}]$  and

$$\mathcal{R}i = *_G^{-1}[\chi, *_G \mathcal{R}] = (-1)^{n+k+l+m} \mathcal{R}_{\alpha\mu} G^{\alpha\hat{\alpha}} e_{\hat{\alpha}} \delta u^\mu.\tag{50}$$

Form  $\mathcal{R}i$  from (50) is locally constructed by using the components of the Ricci ds-tensor (40) as it follows from the decomposition with respect to a locally adapted basis  $\delta u^\alpha$  (7).

Equations

$$\Delta \bar{\mathcal{B}} = 0 \tag{51}$$

are equivalent to the geometric form of Yang–Mills equations for the connection  $\bar{\Gamma}$  (see (49)). In [55–57] it is proved that such gauge equations coincide with the vacuum Einstein equations if as components of connection form (47) are used the usual Christoffel symbols. For spaces with local anisotropy the torsion of a metric d–connection in general is not vanishing and we have to introduce the source 1–form in the right part of (51) even gravitational interactions with matter fields are not considered [32].

Let us consider the locally anisotropic supersymmetric matter source  $\bar{\mathcal{J}}$  constructed by using the same formulas as for  $\Delta \bar{\mathcal{B}}$  when instead of  $\mathcal{R}_{\alpha\beta}$  from (50) is taken  $\kappa_1(\mathcal{J}_{\alpha\beta} - \frac{1}{2}G_{\alpha\beta}\mathcal{J}) - \lambda(G_{\alpha\beta} - \frac{1}{2}G_{\alpha\beta}G_{\tau}^{\tau})$ . By straightforward calculations we can verify that Yang–Mills equations

$$\Delta \bar{\mathcal{B}} = \bar{\mathcal{J}} \tag{52}$$

for torsionless connection  $\bar{\Gamma} = (\Gamma, \chi)$  in s–bundle  $\mathcal{AN}(\mathcal{E})$  are equivalent to Einstein equations (42) on sv–bundle  $\mathcal{E}$ . But such types of gauge like la–supergravitational equations, completed with algebraic equations for torsion and s–spin source, are not variational in the total space of the s–bundle  $\mathcal{AL}(\mathcal{E})$ . This is a consequence of the mentioned degeneration of the Killing form for the affine structural group [55,56] which also holds for our la–supersymmetric generalization. We point out that we have introduced equations (52) in a "pure" geometric manner by using operators  $*$ ,  $\delta$  and horizontal projection  $\hat{H}$ .

We end this section by emphasizing that to construct a variational gauge like supersymmetric la–gravitational model is possible, for instance, by considering a minimal extension of the gauge s–group  $AF_{n,k}^{m,l}(\Lambda)$  to the de Sitter s–group  $S_{n,k}^{m,l}(\Lambda) = SO_{n,k}^{m,l}(\Lambda)$ , acting on space  $\Lambda_{n,k}^{m,l} \oplus \mathcal{R}$ , and formulating a nonlinear version of de Sitter gauge s–gravity (see [57] for locally isotropic gauge gravity and [32] for a locally anisotropic variant). Such s–gravitational models will be analyzed in details in [58].

## 7 DISCUSSION AND CONCLUSIONS

In this paper we have formulated the theory of nonlinear and distinguished connections in sv–bundles which is a framework for developing supersymmetric models of fundamental physical interactions on la–superspaces. Our approach has the advantage of making manifest the relevant structures of supersymmetric theories with local anisotropy and putting great emphasis on the analogy with both usual locally isotropic supersymmetric gravitational models and locally anisotropic gravitational theory on vector bundles provided with compatible nonlinear and distinguished linear connections and metric structures.

The proposed supersymmetric differential geometric techniques allows us a rigorous mathematical study and analysis of physical consequences of various variants of supergravitational theories (developed in a manner similar to the Einstein theory, or in a gauge like form). As two examples we have considered in details two models

of locally anisotropic supergravity which have been chosen to be equivalent in order to illustrate the efficiency and particularities of applications of our formalism in supersymmetric theories of la-gravity.

We emphasize that there are a number of arguments for taking into account effects of possible local anisotropy of both the space-time and fundamental interactions. For example, it's well known the result that a selfconsistent description of radiational processes in classical field theories requires adding of higher derivation terms (in classical electrodynamics radiation is modeled by introducing a corresponding term proportional to the third derivation on time of coordinates). A very important argument for developing quantum field models on the tangent bundle is the unclosed character of quantum electrodynamics. The renormalized amplitudes in the framework of this theory tend to  $\infty$  with values of momenta  $p \rightarrow \infty$ . To avoid this problem one introduces additional suppositions, modifications of fundamental principles and extensions of the theory, which are less motivated from physical point of view. Similar constructions, but more sophisticated, are in order for modelling of radiational dissipation in all variants of classical and quantum (super)gravity and (supersymmetric) quantum field theories with higher derivations. It is quite possible that the Early Universe was in a state with local anisotropy caused by fluctuations of quantum space-time "foam".

The above mentioned points to the necessity to extend the geometric background of some models of classical and quantum field interactions if a careful analysis of physical processes with non-negligible back reaction, quantum and statistical fluctuations, turbulence, random dislocations and disclinations in continuous media and so on.

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