NONLINEAR CONNECTIONS AND ISOTOPIC CLIFFORD STUCTURES

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Abstract

The aim of this work is to review the differential geometry of generalized Finsler isospaces and to formulate the isotopic geometry of Clifford and spinor structures and to consider possible applications in particle physics and gravity.

We emphasize that the geometric constructions on isospaces are very similar to those from noncommutative and quantum geometry (see, for instance, [**?**, **?**]) consisting different alternatives to number fields, algebras and geometries. All type of such geometric constructions like noncommutative spaces, quantum groups and isospaces can be defined by some specific realisations on matrix spaces and further generalisations. This is our basic reason to revive the isomathematic formalism and to develop the generalized Finsler geometry and spinor theory on isospaces.

The article consists from three sections:

The first section is devoted to Isotopies of Generalized Finsler Spaces. It begins with an introduction to the isotopic generalized Finsler gravity. We outline the basic concepts of Santilli's isomathematics and isogravity. Then we present an introduction into the theory of nonlinear connections in vector isobundles and study the isogeometry of tangent isobundles. The basic equations of isogravity and its locally anisotropic generalizations are formulated.

The sections 2 and 3 are on Isotopies, Anisotropy and Clifford structures and is devoted to the theory of anisotropic spinors in isotopic spaces of arbitrary dimensions. We formulate the theory of distinguished Clifford isoalgebras and define Clifford d–module isostructures and Clifford isotopic fibrations. The differential isogeometry of distinguished isospinors is formulated.

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Contents

1 Isotopies of Generalized Finsler Spaces

1.1 Introduction

This section is devoted to a generalization [?, ?] of the geometry of Santilli's locally anisotropic and inhomogeneous isospaces [?, ?, ?, ?, ?, ?, ?] to the geometry of vector isobundles provided with nonlinear and distinguished isoconnections and isometric structures. It will serve as a geometric background for definition of isotopies of Clifford and spinor structures in the sections 2 and 3 of the work.

The main purpose of the section 1 is to outline a synthesis of the Santilli isotheory and the approach on modeling locally anisotropic geometries and physical models on bundle spaces provided with nonlinear connection and distinguished connection and metric structures [?, ?, ?]. The isotopic variants of generalized Lagrange and Finsler geometry will be analyzed. Basic geometric constructions such as nonlinear isoconnections in vector isobundles, the isotopic curvatures and torsions of distinguished isoconnections and theirs structure equations and invariant values will be defined. A model of locally anisotropic and inhomogeneous gravitational isotheory will be presented.

Our study of Santilli's isospaces and isogeometries over isofields will be treated via the isodifferential calculus according to their latest formulation [?] (we extend this calculus for isospaces provided with nonlinear isoconnection structure). We shall also use Kadeisvili's notion of isocontinuity [?, ?] and the novel Santilli–Tsagas–Sourlas isodifferential topology [?, ?, ?].

After reviewing the basic elements for completeness as well as for notational convenience, we shall extend Santilli's foundations of the isosympletic geometry [?] to isobundles and related aspects (by applying, in an isotopic manner, the methods summarized in Miron and Anastasiei [?, ?] and Yano and Ishihara [?] monographs). We apply our results on isotopies of Lagrange, Finsler and Kaluza–Klein geometries to further studies of the isogravitational theories (for Riemannian isospaces firstly considered by Santilli [?]) on vector isobundles provided with compatible nonlinear and distinguished isoconnections and isometric structures. Such isogeometrical models of isofield interaction isotheories are in general nonlinear, nonlocal and nonhamiltonian and contain a very large class of local anisotropies and inhomogeneities induced by four fundamental isostructures: the partition of unity, nonlinear isoconnection, distinguished isoconnections and isometric.

1.2 Isotopies of the unit and isospaces

A number of physical problems connected with the general interior dynamics of deformable particles while moving within inhomogeneous and anisotropic physical media result in a study of the most general known systems which are nonlinear in coordinates x and their derivatives $\dot{x}, \ddot{x}, ...$, on wave functions and ψ and their derivatives $\partial\psi, \partial\partial\psi, ...$ Such systems are also nonlocal because of possible integral dependencies on all of the proceeding quantities and noncanonical with violation of integrability conditions for the existence of a Lagrangian or a Hamiltonian.

The mathematical methods for a quantitative treatment of the latter nonlinear, nonlocal and nonhamiltonian systems have been identified by Santilli in a series of contributions beginning the late 1970's [?, ?, ?, ?, ?, ?, ?] under the name of isotopies, and include axiom preserving liftings of fields of numbers, vector and metric spaces, differential and integral calculus, algebras and geometries. These studies were then continued by a number of authors (see ref. [?] for a comprehensive literature up to 1985, and monographs [?, ?, ?, ?, ?, ?, ?, ?, ?] for subsequent literature).

In this subsection we shall mainly recall some necessary fundamental notions and refer to works [?, ?, ?] for details and references on Lie–Santilli isotheory.

For simplicity, we consider that maps $I \to \widehat{I}$ are of necessary Kadeisvili Class I (II), the Class III being considered as the union of the first two, i. e. they are sufficiently smooth, bounded, nowhere degenerate, Hermitian and positive (negative) definite, characterizing isotopies (isodualities).

One demands a compatible lifting of all associative products AB of some generic quantities A and B into the isoproduct $A * B$ satisfying the properties:

$$\begin{aligned} AB &\Rightarrow A * B = A\widehat{T}B,\ IA = AI \equiv A \to \widehat{I} * A = A * \widehat{I} \equiv A, \\ A\left(BC\right) &= \left(AB\right)C \to A * \left(B * C\right) = \left(A * B\right) * C, \end{aligned}$$

where the fixed and invertible matrix $\widehat{T}$ is called the isotopic element.

To follow our outline, a conventional field $F\left(a,+,\times\right)$, for instance of real, complex or quaternion numbers, with elements a, conventional sum $+$ and product $a \times b \doteq ab$, must be lifted into the so–called isofield $\widehat{F}\left(\widehat{a},+,*\right)$, satisfying the properties

$$\begin{aligned} F\left(a,+,*\right) &\to \widehat{F}\left(\widehat{a},+,*\right),\ \widehat{a} = a\widehat{I} \\ \widehat{a} * \widehat{b} &= \widehat{a}\widehat{T}\widehat{b} = \left(ab\right)\widehat{I},\ \widehat{I} = \widehat{T}^{-1} \end{aligned}$$

with elements $\widehat{a}$ called isonumbers, $+$ and $*$ are conventional sum and isoproduct preserving the axioms of the former field $F\left(a,+,\times\right)$. All operations in F are generalized for $\widehat{F}$, for instance we have isosquares $\widehat{a^2} = \widehat{a} * \widehat{a} = \widehat{A}\widehat{T}\widehat{a} = a^2\widehat{I}$, isoquotient $\widehat{a}\widehat{/}\widehat{b} = \left(a/b\right)\widehat{I}$, isosquare roots $\widehat{a}^{1/2} = a^{1/2}\widehat{I}, ...;\ \widehat{a} * A \equiv aA$. We note that in the literature one uses two types of denotation for isotopic product $*$ or $\widehat{\times}$ (in our work we shall consider $* \equiv \widehat{\times}$).

Let us consider, for example, the main lines of the isotopies of a n–dimensional Euclidean space $E^n\left(x,g,\mathcal{R}\right)$, where $\mathcal{R}\left(n,+,\times\right)$ is the real number field, provided with a local coordinate chart $x = \{x^k\}, k = 1,2,...,n$, and n–dimensional metric $\rho = \left(\rho_{ij}\right) = diag\left(1,1,...,1\right)$. The scalar product of two vectors $x, y \in E^n$ is defined as

$$\left(x-y\right)^2 = \left(x^i - y^i\right)\rho_{ij}\left(x^j - y^j\right) \in \mathcal{R}\left(n,+,\times\right)$$

were the Einstein summation rule on repeated indices is assumed hereon.

The **Santilli's Euclidean** isospaces $\widehat{E}\left(\widehat{x},\widehat{\rho},\widehat{R}\right)$ of Class III are introduced as n–dimensional metric spaces defined over an isoreal isofield $\widehat{R}\left(\widehat{n},+,\widehat{\times}\right)$ with an $n \times n$–dimensional real–valued and symmetrical isounit $\widehat{I} = \widehat{I}^t$ of the same class, equipped with the "isometric"

$$\widehat{\rho}\,(t,x,v,a,\mu,\tau,...)=(\widehat{\rho}_{ij})=\widehat{T}\,(t,x,v,a,\mu,\tau,...)\times\rho=\widehat{\rho}^{t},$$

where $\widehat{I}=\widehat{T}^{-1}=\widehat{I}^{t}$.

A local coordinate cart on $\widehat{E}\left(\widehat{x},\widehat{\rho},\widehat{R}\right)$ can be defined in contravariant

$$\widehat{x}=\{\widehat{x}^{k}=\widehat{x^{k}}\}=\{x^{k}\times\widehat{I}_{k}^{\widehat{k}}\}$$

or covariant form

$$\widehat{x}_{k}=\widehat{\rho}_{kl}\widehat{x}^{l}=\widehat{T}_{k}^{r}\rho_{ri}x^{i}\times\widehat{I},$$

where $x^{k},x_{k}\in\widehat{E}$. The square of "isoeuclidean distance" between two points $\widehat{x},\widehat{y}\in\widehat{E}$ is defined as

$$\widehat{(\widehat{x}-\widehat{y})^{2}}=\left[\left(\widehat{x}^{i}-\widehat{y}^{i}\right)\times\widehat{\rho}_{ij}\times\left(\widehat{x}^{j}-\widehat{y}^{j}\right)\right]\times\widehat{I}\in\widehat{R}$$

and the isomultiplication is given by

$$\widehat{x^{\widehat{2}}}=\widehat{x}^{k}\widehat{\times}\widehat{x}_{k}=\left(x^{k}\times\widehat{I}\right)\times\widehat{T}\times\left(x_{k}\times\widehat{I}\right)=\left(x^{k}\times x_{k}\right)\times\widehat{I}=n\times\widehat{I}.$$

Whenever confusion does not arise isospaces can be practically treated via the conventional coordinates x^{k} rather than the isotopic ones $\widehat{x}^{k}=x^{k}\times\widehat{I}$. The symbols $x,v,a,...$ will be used for conventional spaces while symbols $\widehat{x},\widehat{v},\widehat{a},...$ will be used for isospaces; the letter $\widehat{\rho}\,(x,v,a,...)$ refers to the projection of the isometric $\widehat{\rho}$ in the original space.

We note that an isofield of Class III, explicitly denoted as $\widehat{F}_{III}\left(\widehat{a},+,\widehat{\times}\right)$ is a union of two disjoint isofields, one of Class I, $\widehat{F}_{I}\left(\widehat{a},+,\widehat{\times}\right)$, in which the isounit is positive definite, and one of Class II, $\widehat{F}_{II}\left(\widehat{a},+,\widehat{\times}\right)$, in which the isounit is negative–definite. The Class II of isofields is usually written as $\widehat{F}^{d}\left(\widehat{a}^{d},+,\widehat{\times}^{d}\right)$ and called isodual fields with isodual unit $\widehat{I}^{d}=-\widehat{I}<0$, isodual isonumbers $\widehat{a}^{d}=a\times\widehat{I}^{d}=-\widehat{a}$, isodual isoproduct $\widehat{\times}^{d}=\times\widehat{T}^{d}\times=-\widehat{\times}$, etc. For simplicity, in our further considerations we shall use the general terms isofields, isonumbers even for isodual fields, isodual numbers and so on if this will not give rise to ambiguities.

1.3 Isocontinuity and isotopology

The isonorm of an isofield of Class III is defined as

$$\uparrow\widehat{a}\uparrow=|a|\times\widehat{I}$$

where $|a|$ is the conventional norm. Having defined a function $\widehat{f}\,(\widehat{x})$ on isospace $\widehat{E}\left(\widehat{x},\widehat{\delta},\widehat{R}\right)$ over isofield $\widehat{R}\left(\widehat{n},+,\widehat{\times}\right)$ one introduces (see details and references in [**?**, **?**]) the isomodulus

$$\left\uparrow\widehat{f}\,(\widehat{x})\right\uparrow=\left|\widehat{f}\,(\widehat{x})\right|\times\widehat{I}$$

where $\left|\widehat{f}(\widehat{x})\right|$ is the conventional modulus.

One says that an infinite sequence of isofunctions of Class I $\widehat{f}_1, \widehat{f}_2, ...$ is "strongly isoconvergent" to the isofunction $\widehat{f}$ of the same class if

$$\lim_{k\to\infty} \uparrow \widehat{f}_k - \widehat{f} \uparrow = \widehat{0}.$$

The Cauchy isocondition is expressed as

$$\uparrow \widehat{f}_m - \widehat{f}_n \uparrow < \widehat{\rho} = \rho \times \widehat{I}$$

where δ is real and m and n are greater than a suitably chosen $N(\rho)$. Now the isotopic variants of continuity, limits, series, etc, can be easily constructed in a traditional manner.

The notion of n–dimensional isomanifold was studied by Tsagas and Sourlas (we refer the reader for details in [?, ?]). Their constructions are based on idea that every isounit of Class III can always be diagonalized into the form

$$\widehat{I} = diag\,(B_1, B_2, ..., B_n)\,, B_k\,(x, ...) \neq 0, k = 1, 2, ..., n.$$

In result of this one defines an isotopology $\widehat{\tau}$ on $\widehat{R}^n$ which coincides everywhere with the conventional topology τ on R^n except at the isounit $\widehat{I}$. In particular, $\widehat{\tau}$ is everywhere local–differential, except at $\widehat{I}$ which can incorporate integral terms. The above structure is called the Tsagas–Sourlas isotopology or an integro–differential topology. Finally, in this subsection, we note that Prof. Tsagas and Sourlas used a conventional topology on isomanifolds. The isotopology was first introduced by Prof. Santilli in ref. [?].

1.4 Isodifferential and isointegral calculus

Now we are able to introduce isotopies of the ordinary differential calculus, i.e. the isodifferential calculus (for short).

The **isodifferentials** of Class I of the contravariant and covariant coordinates $\widehat{x}^k = x^{\widehat{k}}$ and $\widehat{x}_k = x_{\widehat{k}}$ on an isoeuclidean space $\widehat{E}$ of the same class is given by

$$\widehat{d}\widehat{x}^k = \widehat{I}^k_{\ i}\,(x, ...)\,dx^i,\ \widehat{d}\widehat{x}_k = \widehat{T}_k^{\ i}\,(x, ...)\,dx_i \qquad (1)$$

where $\widehat{d}\widehat{x}^k$ and $\widehat{d}\widehat{x}_k$ are defined on $\widehat{E}$ while the $\widehat{I}^k_{\ i}dx^i$ and $\widehat{T}_k^{\ i}dx_i$ are the projections on the conventional Euclidean space.

For a sufficiently smooth isofunction $\widehat{f}(\widehat{x})$ on a closed domain $\widehat{U}\left(\widehat{x}^k\right)$ covered by contravariant isocoordinates $\widehat{x}^k$ we can define the partial isoderivatives $\widehat{\partial}_{\widehat{k}} = \frac{\widehat{\partial}}{\widehat{\partial}\widehat{x}^k}$ at a point $\widehat{x}^k_{(0)} \in \widehat{U}\left(\widehat{x}^k\right)$ by considering the limit

$$\begin{aligned} \widehat{f}'\left(\widehat{x}^k_{(0)}\right) &= \widehat{\partial}_{\widehat{k}}\widehat{f}(\widehat{x})\,|_{\widehat{x}^k_{(0)}} = \frac{\widehat{\partial}\widehat{f}(\widehat{x})}{\widehat{\partial}\widehat{x}^k}\,|_{\widehat{x}^k_{(0)}} = \widehat{T}_k^{\ i}\frac{\partial f(x)}{\partial x^i}\,|_{\widehat{x}^k_{(0)}} \\ &= \lim_{\widehat{d}\widehat{x}^k \to \widehat{0}^k} \frac{\widehat{f}\left(\widehat{x}^k_{(0)} + \widehat{d}\widehat{x}^k\right) - \widehat{f}\left(\widehat{x}^k_{(0)}\right)}{\widehat{d}x^k} \end{aligned} \qquad (2)$$

where $\widehat{\partial}\widehat{f}(\widehat{x})/\widehat{\partial}\widehat{x}^k$ is computed on $\widehat{E}$ and $\widehat{T}_k^{\ i}\partial f(x)/\partial x^i$ is the projection in E.

In a similar manner we can define the **partial isoderivatives** $\widehat{\partial^k} = \frac{\widehat{\partial}}{\widehat{\partial x_k}}$ with respect to a covariant variable $\widehat{x}_k$:

$$\begin{aligned}\widehat{f}'\left(\widehat{x}_{k(0)}\right) &= \widehat{\partial^k}\widehat{f}(\widehat{x})\mid_{\widehat{x}_{k(0)}} = \frac{\widehat{\partial}\widehat{f}(\widehat{x})}{\widehat{\partial}\widehat{x}_k}\mid_{\widehat{x}_{k(0)}} = \widehat{T}_k^{\ i}\frac{\partial f(x)}{\partial x^i}\mid_{\widehat{x}_{k(0)}} \\ &= \lim_{\widehat{dx}_{\widehat{k}}\rightarrow\widehat{0}_k}\frac{\widehat{f}\left(\widehat{x}_{k(0)}+\widehat{d}\widehat{x}_k\right)-\widehat{f}\left(\widehat{x}_{k(0)}\right)}{\widehat{d}x_k}. \end{aligned} \tag{3}$$

The isodifferentials of an isofunction of contravariant or covariant coordinates, $\widehat{x}^k$ or $\widehat{x}_k$, are defined according the formulas

$$\widehat{d}\widehat{f}(\widehat{x})\mid_{contrav} = \widehat{\partial}_{\widehat{k}}\widehat{f}\widehat{d}\widehat{x}^k = \widehat{T}_k^{\ i}\frac{\partial f(x)}{\partial x^i}\widehat{I}^k_{\ j}dx^j = \frac{\partial f(x)}{\partial x^k}dx^k = \frac{\partial f(x)}{\partial x^i}\widehat{T}_j^{\ i}dx^j$$

and

$$\widehat{d}\widehat{f}(\widehat{x})\mid_{covar} = \widehat{\partial^k}\widehat{f}\widehat{d}\widehat{x}_k = \widehat{I}^k_{\ i}\frac{\partial f(x)}{\partial x_i}\widehat{T}_k^{\ j}dx_j = \frac{\partial f(x)}{\partial x_k}dx_k = \frac{\partial f}{\partial x_j}\widehat{I}^i_{\ j}dx_i.$$

The second order isoderivatives there are introduced by iteration of the notion of isoderivative:

$$\begin{aligned}\widehat{\partial}^2_{\widehat{ij}}\widehat{f}(\widehat{x}) &= \frac{\widehat{\partial}^2\widehat{f}(\widehat{x})}{\widehat{\partial}\widehat{x}^i\widehat{\partial}\widehat{x}^j} = \widehat{T}_{\widehat{i}}^{\ i}\widehat{T}_{\widehat{j}}^{\ j}\frac{\partial^2 f(x)}{\partial x^i\partial x^j}, \\ \widehat{\partial}^{2\widehat{ij}}\widehat{f}(\widehat{x}) &= \frac{\widehat{\partial}^2\widehat{f}(\widehat{x})}{\widehat{\partial}\widehat{x}_i\widehat{\partial}\widehat{x}_j} = \widehat{I}^{\widehat{i}}_{\ i}\widehat{I}^{\widehat{j}}_{\ j}\frac{\partial^2 f(x)}{\partial x_i\partial x_j}, \\ \widehat{\partial}^2_{\widehat{i}}{}^{\widehat{j}}\widehat{f}(\widehat{x}) &= \frac{\widehat{\partial}^2\widehat{f}(\widehat{x})}{\widehat{\partial}\widehat{x}^i\widehat{\partial}\widehat{x}_{\widehat{j}}} = \widehat{T}_{\widehat{i}}^{\ i}\widehat{I}^{\widehat{j}}_{\ j}\frac{\partial^2 f(x)}{\partial x^i\partial x_j}.\end{aligned}$$

The Laplace isooperator on Euclidean space $\widehat{E}\left(\widehat{x},\widehat{\delta},\widehat{R}\right)$ is given by

$$\widehat{\Delta} = \widehat{\partial}_k\widehat{\partial}^k = \widehat{\partial}^i\rho_{ij}\widehat{\partial}^j = \widehat{\widehat{I}^i}_{\ k}\partial^k\rho_{ij}\partial^j \tag{4}$$

where there are also used usual partial derivatives $\partial^j = \partial/\partial x_j$ and $\partial_k = \partial/\partial x^k$.

The isodual isodifferential calculus is characterized by the following isodual differentials and isodual isoderivatives

$$\widehat{d}^{(d)}\widehat{x}^{(d)k} = \widehat{I}^{(d)k}_{\ \ i}d\widehat{x}^{(d)i} \equiv \widehat{d}x^k, \quad \widehat{\partial}^{(d)}/\widehat{\partial}\widehat{x}^{(d)i}\widehat{T}_k^{\ i(d)}\partial/\partial\widehat{x}^{i(d)} \equiv \widehat{T}_k^{\ i}\partial/\partial\widehat{x}^i.$$

The formula (4) is different from the expression for the Laplace operator

$$\Delta = \widehat{\rho}^{-1/2}\partial_i\widehat{\rho}^{1/2}\widehat{\rho}^{ij}\partial_j$$

even though the Euclidean isometric $\widehat{\rho}(x, v, a, ...)$ is more general than the Riemannian metric $g(x)$. For partial isoderivations one follows the next properties:

$$\frac{\widehat{\partial}\widehat{x}^i}{\widehat{\partial}\widehat{x}^j} = \delta^i{}_j, \ \frac{\widehat{\partial}\widehat{x}_i}{\widehat{\partial}\widehat{x}_j} = \delta_i{}^j, \ \frac{\widehat{\partial}\widehat{x}_i}{\widehat{\partial}\widehat{x}^j} = \widehat{T}_i{}^j, \ \frac{\widehat{\partial}\widehat{x}^i}{\widehat{\partial}\widehat{x}_j} = \widehat{I}^i{}_j.$$

Here we remark that isointegration (the inverse to isodifferential) is defined [**?**] as to satisfy conditions

$$\int^{\hat{}} \widehat{d}\widehat{x} = \int \widehat{T}\widehat{I}dx = \int dx = x,$$

where $\int^{\hat{}} = \int \widehat{T}$.

1.5 Santilli's Riemannian isospaces

Let consider $\mathcal{R} = \mathcal{R}(x, g, R)$ a (pseudo) Riemannian space over the reals $R(n, +, \times)$ with local coordinates $x = \{x^\mu\}$ and nonwere singular, symmetrical and real–valued metric $g(x) = (g_{\mu\nu}) = g^t$ and the tangent flat space $M(x, \eta, R)$ provided with flat real metric η (for a corresponding signature and dimension we can consider M as the well known Minkowski space). The metric properties of the Riemannian spaces are defined by scalar square of a tangent vector x,

$$x^2 = x^\mu g_{\mu\nu}(x) x^v \in R$$

or, in infinitesimal form by the line element

$$ds^2 = dx^\mu g_{\mu\nu}(x) dx^\nu$$

and related formalism of covariant derivation (see for instance [**?**]).

The isotopies of the Riemannian spaces and geometry, were first studied and applied by [**?**] and are called Santilli's Riemannian isospaces and isogeometry. In this subsection we consider Riemannian isospaces equipped with the Santilli–Tsagas–Sourlas isotopology[**?**, **?**, **?**] in a similar manner as we have done in the previous subsection for isoeuclidean spaces but with respect to a general, non flat, isometric. A Riemannian isospace $\widehat{\mathcal{R}}{=}\widehat{\mathcal{R}}\left(\widehat{x}, \widehat{g}, \widehat{R}\right)$, over the isoreals $\widehat{R} = \widehat{R}(\widehat{n}, +, \widehat{x})$ with common isounits $\widehat{I} = \left(\widehat{I}^\mu{}_\nu\right) = \widehat{T}^{-1}$, is provided with local isocoordinates $\widehat{x} = \{\widehat{x}^\mu\} = \{x^\mu\}$ and isometric $\widehat{g}(x, v, a, \mu, \tau, ...) = \widehat{T}(x, v, \mu, \tau, ...) g(x)$, where $\widehat{T} = \left(\widehat{T}_\mu{}^\nu\right)$ is nowhere singular, real valued and symmetrical matrix of Class I with C^∞ elements. The corresponding isoline and infinitesimal elements are written as

$$\widehat{x^{\widehat{2}}} = [\widehat{x}^\mu \widehat{g}_{\mu\nu}(x, v, a, \mu, \tau, ...) \widehat{x}^\nu] \times \widehat{I} \in \widehat{R}$$

with infinitesimal version

$$d\widehat{s}^2 = (\widehat{d}\widehat{x}^\mu \widehat{g}_{\mu\nu}(x) \widehat{d}\widehat{x}^\nu) \times \widehat{I} \in \widehat{R}.$$

The **covariant isodifferential calculus** has been introduced in ref. [**?**] via the expression

$$\widehat{D}\widehat{X}^{\beta} = \widehat{d}\widehat{X}^{\beta} + \widehat{\Gamma}^{\beta}_{\alpha\gamma}\widehat{X}^{\alpha}\widehat{d}\widehat{x}^{\gamma}$$

with corresponding covariant isoderivative

$$\widehat{X}^{\beta}_{\uparrow\mu} = \widehat{\partial}_{\mu}\widehat{X}^{\beta} + \widehat{\Gamma}^{\beta}_{\alpha\mu}\widehat{X}^{\alpha}$$

with the **isocristoffel symbols** written as

$$\{\widehat{\alpha\beta\gamma}\} = \frac{1}{2}\left(\widehat{\partial}_{\alpha}\widehat{g}_{\beta\gamma} + \widehat{\partial}_{\gamma}\widehat{g}_{\alpha\beta} - \widehat{\partial}_{\beta}\widehat{g}_{\alpha\gamma}\right) = \{\widehat{\gamma\beta\alpha}\}, \tag{5}$$

$$\widehat{\Gamma}^{\beta}_{\alpha\gamma} = \widehat{g}^{\beta\rho}\{\widehat{\alpha\rho\gamma}\} = \widehat{\Gamma}^{\beta}_{\alpha\gamma},$$

where $\widehat{g}^{\beta\rho}$ is inverse to $\widehat{g}_{\alpha\beta}$.

The crucial difference between Riemannian spaces and isospaces is obvious if the corresponding auto–parallel equations

$$\frac{Dx_{\beta}}{Ds} = \frac{dv_{\beta}}{ds} + \{\alpha\beta\gamma\}\,(x)\,\frac{dx^{\alpha}}{ds}\frac{dx^{\gamma}}{ds} = 0 \tag{6}$$

and auto–isoparallel equations

$$\frac{\widehat{D}\widehat{x}_{\beta}}{\widehat{D}\widehat{s}} = \frac{\widehat{d}v_{\beta}}{\widehat{d}\widehat{s}} + \{\widehat{\alpha\beta\gamma}\}\,(\widehat{x},\widehat{v},\widehat{a},...)\,\frac{\widehat{d}\widehat{x}^{\alpha}}{\widehat{d}\widehat{s}}\frac{\widehat{d}\widehat{x}^{\gamma}}{\widehat{d}\widehat{s}} = 0 \tag{7}$$

where $\widehat{v} = \widehat{d}\widehat{x}/\widehat{d}\widehat{s} = \widehat{I}_S \times dx/ds, \widehat{s}$ is the proper isotime and $\widehat{I}_S$ is the related one–dimensional isounit, can be identified by observing that equations (6) are at most quadratic in the velocities while the isotopic equations (7) are arbitrary nonlinear in the velocities and another possible variables and parameters $(\widehat{a},...)$.

By using coefficients $\widehat{\Gamma}^{\beta}_{\alpha\gamma}$ we introduce the next isotopic values [**?**]:
the **curvature isotensor**

$$\widehat{R}_{\alpha\ \gamma\delta}^{\ \beta} = \widehat{\partial}_{\delta}\widehat{\Gamma}^{\beta}_{\alpha\gamma} - \widehat{\partial}_{\gamma}\widehat{\Gamma}^{\beta}_{\alpha\delta} + \widehat{\Gamma}^{\beta}_{\varepsilon\delta}\widehat{\Gamma}^{\varepsilon}_{\alpha\gamma} - \widehat{\Gamma}^{\beta}_{\varepsilon\gamma}\widehat{\Gamma}^{\varepsilon}_{\alpha\delta}; \tag{8}$$

the **Ricci isotensor** $\widehat{R}_{\alpha\gamma} = \widehat{R}_{\alpha\ \gamma\beta}^{\ \beta}$;
the **curvature isoscalar** $\widehat{R} = \widehat{g}^{\alpha\gamma}\widehat{R}_{\alpha\gamma}$;
the **Einstein isotensor**

$$\widehat{G}_{\mu\nu} = \widehat{R}_{\mu\nu} - \frac{1}{2}\widehat{g}_{\mu\nu}\widehat{R} \tag{9}$$

and the **isoscalar**

$$\widehat{\Theta} = \widehat{g}^{\alpha\beta}\widehat{g}^{\gamma\delta}\left(\{\widehat{\rho\alpha\delta}\}\widehat{\Gamma}^{\rho}_{\gamma\beta} - \{\widehat{\rho\alpha\beta}\}\widehat{\Gamma}^{\rho}_{\gamma\delta}\right) \tag{10}$$

(the later is a new object for the Riemannian isometry).

The isotopic lifting of the Einstein equations (see the history, details and references in [**?**]) is written as

$$\widehat{R}^{\alpha\beta} - \frac{1}{2}\widehat{g}^{\alpha\beta}(\widehat{R} + \widehat{\Theta}) = \widehat{t}^{\alpha\beta} - \widehat{\tau}^{\alpha\beta}, \qquad (11)$$

where $\widehat{t}^{\alpha\beta}$ is a **source isotensor** and $\widehat{\tau}^{\alpha\beta}$ is the **stress–energy isotensor** and there is satisfied the Freud isoidentity [**?**]

$$\begin{aligned}
\widehat{G}^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}\widehat{\Theta} &= \widehat{U}^{\alpha}{}_{\beta} + \widehat{\partial}_{\rho}\widehat{V}^{\alpha\rho}{}_{\beta}, \\
\widehat{U}^{\alpha}{}_{\beta} &= -\frac{1}{2}\frac{\widehat{\partial}\widehat{\Theta}}{\widehat{\partial}\widehat{g}^{\gamma\delta}{}_{\uparrow\alpha}}\widehat{g}^{\gamma\delta}{}_{\uparrow\beta}, \\
\widehat{V}^{\alpha\rho}{}_{\beta} &= \frac{1}{2}[\widehat{g}^{\gamma\delta}\left(\delta^{\alpha}{}_{\beta}\widehat{\Gamma}^{\rho}_{\alpha\delta} - \delta^{\alpha}{}_{\delta}\widehat{\Gamma}^{\rho}_{\alpha\beta}\right) + \widehat{g}^{\rho\gamma}\widehat{\Gamma}^{\alpha}_{\beta\gamma} - \widehat{g}^{\alpha\gamma}\widehat{\Gamma}^{\rho}_{\beta\gamma} + \\
&\quad \left(\delta^{\rho}{}_{\beta}\widehat{g}^{\alpha\gamma} - \delta^{\alpha}{}_{\beta}\widehat{g}^{\rho\gamma}\right)\widehat{\Gamma}^{\rho}_{\gamma\rho}].
\end{aligned}$$

Finally, we remark that for antiautomorphic maps of isoduality we have to modify correspondingly the above presented formulas holding true for Riemannian isodual spaces $\widehat{\mathcal{R}}^{(d)} = \widehat{\mathcal{R}}^{(d)}\left(\widehat{x}^{(d)}, \widehat{g}^{(d)}, \widehat{R}^{(d)}\right)$, over the isodual reals $\widehat{R}^{(d)} = \widehat{R}^{(d)}\left(\widehat{n}^{(d)}, +, \widehat{\times}^{(d)}\right)$ with curvature, Ricci, Einstein and so on isodual tensors. For simplicity we omit such details in this work.

1.6 Lie–Santilli isoalgebras and isogroups

The Lie–Santilli isotheory is based on a generalization of the very notion of numbers and fields. If the Lie's theory is centrally dependent on the basic n–dimensional unit $I = diag\,(1,1,...,1)$ in, for instance, enveloping algebras, Lie algebras, Lie groups, representation theory, and so on, the Santilli's main idea is the reformulation of the entire conventional theory with respect to the most general possible, integro–differential isounit. In this subsection we introduce some necessary definitions and formulas on Lie–Santilli isoalgebra and isogroups following [**?**, **?**] where details, developments and basic references on Santilli original result are contained. A Lie–Santilli algebra is defined as a finite–dimensional isospaces $\widehat{L}$ over the isofield $\widehat{F}$ of isoreal or isocomplex numbers with isotopic element T and isounit $\widehat{I} = T^{-1}$. In brief one uses the term isoalgebra (when there is not confusion with isotopies of non–Lie algebras) which is defined by isolinear isocommutators of type $[A, \hat{}B] \in \widehat{L}$ satisfying the conditions:

$$\begin{gathered}
[A, \hat{}B] = -[B, \hat{}A], \\
[A, \hat{}[B, \hat{}C]] + [B, \hat{}[C, \hat{}A]] + [C, \hat{}[A, \hat{}B]] = 0, \\
[A * B, \hat{}C] = A * [B, \hat{}C] + [A, \hat{}C] * B
\end{gathered}$$

for all $A, B, C \in \widehat{L}$. The structure functions $\widehat{C}$ of the Lie–Santilli algebras are introduced according the relations

$$\begin{aligned}
[X_i, \hat{}X_j] &= X_i * X_j - X_j * X_i \\
&= X_i T\,(x, ...)\, X_j - X_j T\,(x, ...)\, X_i = \widehat{C}_{ij}{}^{k}\,(x, \dot{x}, \ddot{x}, ...) * X_k.
\end{aligned}$$

It should be noted that, in fact, the basis $e_k, (k = 1, 2, ..., N)$ of a Lie algebra L is not changed under isotopy except the renormalization factors $\widehat{e}_k$: the isocommutation rules of the isotopies $\widehat{L}$ are

$$[\widehat{e}_i, \widehat{e}_j] = \widehat{e}_i T \widehat{e}_j - \widehat{e}_j T \widehat{e}_i = \widehat{C}_{ij}^{\ k} (x, \dot{x}, \ddot{x}, ...) \widehat{e}_k$$

where $\widehat{C} = CT$.

An isomatrix $\hat{M}$ is and ordinary matrix whose elements are isoscalars. All operations among isomatrices are therefore isotopic.

The isotrace of a isomatrix A is introduced by using the unity $\widehat{I}$:

$$\widehat{Tr}A = (TrA)\,\widehat{I} \in \widehat{F}$$

where TrA is the usual trace. One holds properties

$$\widehat{Tr}(A * B) = \left(\widehat{Tr}A\right) * \left(\widehat{Tr}B\right)$$

and

$$\widehat{Tr}A = \widehat{Tr}\left(BAB^{-1}\right).$$

The Killing isoform is determined by the isoscalar product

$$(A, \hat{}B) = \widehat{Tr}\left[\left(\widehat{Ad}X\right) * \left(\widehat{Ad}B\right)\right]$$

where the isolinear maps are introduced as $\widehat{ad}A\,(B) = [A, \hat{}B], \forall A, B \in \widehat{L}$. Let $e_k, k = 1, 2, ..., N$ be the basis of a Lie algebra with an isomorphic map $e_k \rightarrow \widehat{e}_k$ to the basis $\widehat{e}_k$ of a Lie–Santilli isoalgebra $\widehat{L}$. We can write the elements in $\widehat{L}$ in local coordinate form. For instance, considering $A = x^i \widehat{e}_i, B = y^j \widehat{e}_j$ and $C = z^k \widehat{e}_k = [A, \hat{}B]$ we have

$$C = z^k \widehat{e}_k = [A, \hat{}B] = x^i y^j [\widehat{e}_i, \hat{}\widehat{e}_j] = x^i x^j \widehat{C}_{ij}^{\ k} \widehat{e}_k$$

and

$$\left[\widehat{Ad}A\,(B)\right]^k = [A, \hat{}B]^k = x^i x^j \widehat{C}_{ij}^{\ k}.$$

In standard manner there is introduced the **isocartan tensor**

$$\widehat{q}_{ij}\,(x, \dot{x}, \ddot{x}, ...) = \widehat{C}_{ip}^{\ k} \widehat{C}_{ik}^{\ p} \in \widehat{L}$$

via the definition

$$(A, \hat{}B) = \widehat{q}_{ij} x^i y^j.$$

Considering that $\widehat{L}$ is an isoalgebra with generators X_k and isounit $\widehat{I} = T^{-1} > 0$ the isodual Lie–Santilli algebras $\widehat{L}^d$ of $\widehat{L}$ (we note that $\widehat{L}$ and $\widehat{L}^d$ are (anti) isomorphic).

The conventional structure of the Lie theory admits a conventional isotopic lifting. Let give some examples. The general isolinear and isocomplex Lie–Santilli algebras $\widehat{gl}\left(n, \widehat{C}\right)$ are introduced as the vector isospaces of all $n \times n$ isocomplex matrices over $\widehat{C}$. For the isoreal numbers $\widehat{R}$ we shall write $\widehat{gl}\left(n,\widehat{R}\right)$. By using "hats" we denote respectively the special, isocomplex, isolinear isoalgebra $\widehat{sl}\left(n,\widehat{C}\right)$ and the isoorthogonal algebra $\widehat{o}\,(n)$.

A right Lie–Santilli isogroup $\widehat{Gr}$ on an isospace $\widehat{S}\left(x,\widehat{F}\right)$ over an isofield $\widehat{F},\widehat{I}=T^{-1}$ (in brief isotransformation group or isogroup) is introduced in standard form but with respect to isonumbers and isofields as a group which maps each element $x\in\widehat{S}\left(x,\widehat{F}\right)$ into a new element $x'\in\widehat{S}\left(x,\widehat{F}\right)$ via the isotransformations $x'=\widehat{U}*x=\widehat{U}Tx$, where T is fixed such that

1. The map $(U,x)\to\widehat{U}*x$ of $\widehat{Gr}\times\widehat{S}\left(x,\widehat{F}\right)$ onto $\widehat{S}\left(x,\widehat{F}\right)$ is isodifferentiable;
2. $\widehat{I}*\widehat{U}=\widehat{U}*\widehat{I}=\widehat{U},\ \forall\widehat{U}\in\widehat{Gr}$;
3. $\widehat{U}_1*(\widehat{U}_2*x)=\left(\widehat{U}_1*\widehat{U}_2\right)*x,\ \forall x\in\widehat{S}\left(x,\widehat{F}\right)$ and $\widehat{U}_1,\widehat{U}_2\in\widehat{Gr}$.

We can define accordingly a left isotransformation group.

1.7 Fiber isobundles

In this subsection we present the isotopies of fibre bundles and related topics.

The notion of locally trivial fiber isobundle naturally generalizes that of the isomanifold. The fiber isobundles will be used to get some results in isogeometry as well as to build geometrical models for physical isotheories. In general the proofs, being corresponding reformulation in isotopic manner of standard results, will be omitted. The reader is referred to some well–known books containing the theory of fibre bundles and the mathematical foundations of the Lie–Santilli isotheory.

Let $\widehat{Gr}$ be a Lie–Santilli isogroup which acts isodifferentiably and effectively on a isomanifold $\widehat{V}$, i.e. every element $\widehat{q}\in\widehat{Gr}$ defines an isotopic diffeomorphism $L_{\widehat{Gr}}:\widehat{V}\to\widehat{V}$.

As a rule, all isomanifolds are assumed to be isocontinuous, finite dimensional and having the isotopic variants of the conditions to be Hausdorff, paracompact and isoconnected; all isomaps are isocontinous.

A locally trivial **fibre isobundle** is defined by the data $\left(\widehat{E},\widehat{p},\widehat{M},\widehat{V},\widehat{Gr}\right)$, where $\widehat{M}$ (the base isospace) and $\widehat{E}$ (the total isospace) are isomanifolds, $\widehat{E},\widehat{p}:\widehat{E}\to\widehat{M}$ is a surjective isomap and the following conditions are satisfied:

1/ the isomanifold $\widehat{M}$ can be covered by a set $\mathcal{E}$ of open isotopic sets $\widehat{U},\widehat{W},...$ such that for every open set $\widehat{U}$ there exist a bijective isomap $\widehat{\varphi}_{\widehat{U}}:\widehat{p}^{-1}\left(\widehat{U}\right)\to\widehat{U}\times\widehat{V}$ so that $\widehat{p}\left(\varphi_{\widehat{U}}^{-1}\left(\widehat{x},\widehat{y}\right)\right)=\widehat{x},\ \forall\widehat{x}\in\widehat{U},\forall\widehat{y}\in\widehat{V}$;

2/ if $\widehat{x}\in\widehat{U}\cap\widehat{W}\neq\oslash$, than $\widehat{\varphi}_{\widehat{W},x}\circ\widehat{\varphi}_{\widehat{U},x}^{-1}:\widehat{V}\to\widehat{V}$ is an isotopic diffeomorphism $L_{\widehat{gr}}$ with $\widehat{gr}\in\widehat{Gr}$ where $\widehat{\varphi}_{\widehat{U},x}$ denotes the restriction of $\widehat{\varphi}_{\widehat{U}}$ to $p^{-1}\left(\widehat{x}\right)$ and $\widehat{U},\widehat{W}\in\mathcal{E}$;

3/ the isomap $q_{\widehat{U}\widehat{V}}:\widehat{U}\cap\widehat{V}\to\widehat{Gr}$ defined by structural isofunctions $q_{\widehat{U}\widehat{V}}\left(\widehat{x}\right)=$

$\widehat{\varphi}_{\widehat{W},x} \circ \widehat{\varphi}^{-1}_{\widehat{U},x}$ is isocontinous.

Let In_U and In_V be sets of indices and denote by $\left(\widehat{U}_\alpha, \widehat{\varphi}_\alpha\right)_{\alpha \in In_U}$ and $\left(\widehat{V}_\beta, \widehat{\psi}_\beta\right)_{\beta \in In_V}$ be correspondingly isocontinous atlases on $\widehat{U}_\alpha$ and $\widehat{V}_\beta$. One obtains an isotopic topology on $\widehat{E}$ for which the bijections $\widehat{\varphi}_{\widehat{U}}, \widehat{\varphi}_{\widehat{W},x}$... become isotopic homeomorphisms. Denoting respectively by n and m the dimensions of the isomanifolds $\widehat{M}$ and $\widehat{V}$ we can define the isotopic maps

$$\phi_{\alpha\beta} : \varpi_{\alpha\beta} \to \widehat{R}^{n+m}, \phi_{\alpha\beta} = \left(\widehat{\varphi}_\alpha \times \widehat{\psi}_\beta\right) \circ \varphi^{\alpha\beta}_{\widehat{U}}$$

where $\varphi^{\alpha\beta}_{\widehat{U}}$ is the restriction to the isomap $\widehat{\varphi}_{\widehat{U}}$ to $\varpi_{\alpha\beta}$. Than the set $(\varpi_{\alpha\beta}, \phi_{\alpha\beta})_{(\alpha,\beta) \in In_U \times In_V}$ is a isocontinous atlas on $\widehat{E}$.

A locally trivial **principal isobundle** $\left(\widehat{P}, \widehat{\pi}, \widehat{M}, \widehat{G}r\right)$ is a fibre isobundle $\left(\widehat{E}, \widehat{p}, \widehat{M}, \widehat{V}, \widehat{G}r\right)$ for which the type fibre coincides with the structural group, $\widehat{V} = \widehat{G}r$ and the action of $\widehat{G}r$ on $\widehat{G}r$ is given by the left isotransform $\widehat{L}_q(a) = qa, \forall q, a \in \widehat{G}r$.

The structural functions of the principal isobundle $\left(\widehat{P}, \widehat{\pi}, \widehat{M}, \widehat{G}r\right)$ are

$$q_{\widehat{U}\widehat{W}} : \widehat{U}, \widehat{W} \to \widehat{G}r,\ q_{\widehat{U}\widehat{W}}(\widehat{\pi}(u)) = \widehat{\varphi}_{\widehat{W}}(u) \circ \widehat{\varphi}^{-1}_{\widehat{U}}(u), u \in \widehat{\pi}^{-1}\left(\widehat{U} \cap \widehat{W}\right).$$

A **morphism of principal isobundles** $\left(\widehat{P}, \widehat{\pi}, \widehat{M}, \widehat{G}r\right)$ and $\left(\widehat{P}', \widehat{\pi}', \widehat{M}', \widehat{G}r'\right)$ is a pair $\left(\widehat{f}, \widehat{f}'\right)$ of isomaps for which the following conditions hold:

1/ $\widehat{f} : \widehat{P} \to \widehat{P}'$ is a isocontinous isomap,

2/ $\widehat{f} : \widehat{G}r \to \widehat{G}r'$ is an isotopic morphism of Lie–Santilli isogroups.

3/ $\widehat{f}(\widehat{u}\widehat{q}) = f(\widehat{u}) f'(\widehat{q}), \widehat{u} \in \widehat{P}, \widehat{q} \in \widehat{G}r.$

We can define isotopic isomorphisms, automorphisms and subbundles in a usual manner but with respect to isonumbers, isofields, isogroups and isomanifold when corresponding isotopic transforms and maps provide the isotopic properties.

A **isotopic subbundle** $\left(\widehat{P}, \widehat{\pi}, \widehat{M}, \widehat{G}r\right)$ of the principal isobundle $\left(\widehat{P}', \widehat{\pi}', \widehat{M}', \widehat{G}r'\right)$ is called a reduction of the structural isogroup $\widehat{G}r'$ to $\widehat{G}r$.

An isotopic frame (isoframe) in a point $\widehat{x} \in \widehat{M}$ is a set of n linearly independent isovectors tangent to $\widehat{M}$ in $\widehat{x}$. The set $\widehat{L}\left(\widehat{M}\right)$ of all isoframes in all points of $\widehat{M}$ can be naturally provided (as in the non isotopic case, see, for instance, [**?**]) with an isomanifold structure. The principal isobundle $\left(\widehat{L}\left(\widehat{M}\right), \widehat{\pi}, \widehat{M}, \widehat{Gl}\left(n, \widehat{R}\right)\right)$of isoframes on $\widehat{M}$, denoted in brief also by $\widehat{L}\left(\widehat{M}\right)$, has $\widehat{L}\left(\widehat{M}\right)$ as the total space and the general linear isogroup
$\widehat{Gl}\left(n, \widehat{R}\right)$ as the structural isogroup.

Having introduced the isobundle $\widehat{L}\left(\widehat{M}\right)$ we can give define an **isotopic** G–**structure** on a isomanifold $\widehat{M}$ is a subbundle $\left(\widehat{P},\widehat{\pi},\widehat{M},\widehat{Gr}\right)$ of the principal isobundle $\widehat{L}\left(\widehat{M}\right)$ being an isotopic reduction of the structural isogroup $\widehat{Gl}\left(n,\widehat{R}\right)$ to a isotopic subgroup $\widehat{Gr}$ of it.

A very important class of bundle spaces used for modeling of locally anisotropic interactions is that of vector bundles. We present here the necessary isotopic generalizations.

An locally trivial **isovector bundle** (equivalently, **vector isobundle, v–isobundle**) $\left(\widehat{E},\widehat{p},\widehat{M},\widehat{V},\widehat{Gr}\right)$ is defined as a corresponding fibre isobundle if $\widehat{V}$ is a linear isospace and $\widehat{Gr}$ is the Lie–Santilli isogroup of isotopic authomorphisms of $\widehat{V}$.

For $\widehat{V}=\widehat{R}^m$ and $\widehat{Gr}=\widehat{Gl}\left(m,\widehat{R}\right)$ the v–isobundle $\left(\widehat{E},\widehat{p},\widehat{M},\widehat{R}^m,\widehat{Gl}\left(m,\widehat{R}\right)\right)$ is denoted shortly as $\widehat{\xi}=\left(\widehat{E},\widehat{p},\widehat{M}\right)$. Here we also note that the transformations of the isocoordinates $\left(\widehat{x}^k,\widehat{y}^a\right)\rightarrow\left(\widehat{x}^{k'},\widehat{y}^{a'}\right)$ on $\widehat{\xi}$ are of the form

$$\begin{aligned}\widehat{x}^{k'} &= \widehat{x}^{k'}\left(\widehat{x}^1,...,\widehat{x}^n\right), \quad rank\left(\frac{\widehat{\partial}\widehat{x}^{k'}}{\widehat{\partial}\widehat{x}^k}\right)=n;\\ \widehat{y}^{a'} &= \widehat{Y}_a^{a'}(x)\,y^a, \quad \widehat{Y}_a^{a'}(x)\in\widehat{Gl}\left(m,\widehat{R}\right).\end{aligned}$$

A local isocoordinate parametrization of $\widehat{\xi}$ naturally defines an isocoordinate basis

$$\frac{\partial}{\partial\widehat{u}^\alpha}=\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^a}\right), \tag{12}$$

in brief we shall write $\widehat{\partial}_\alpha=(\widehat{\partial}_i,\widehat{\partial}_a)$, and the reciprocal to (12) coordinate basis

$$d\widehat{u}^\alpha=(d\widehat{x}^i,d\widehat{y}^a), \tag{13}$$

or, in brief, $\widehat{d}^\alpha=(\widehat{d}^i,\widehat{d}^a)$, which is uniquely defined from the equations

$$\widehat{d}^\alpha\circ\widehat{\partial}_\beta=\delta_\beta^\alpha,$$

where δ_β^α is the Kronecher symbol and by "$\circ$" we denote the inner (scalar) product in the isotangent isobundle $\widehat{T\xi}$ (see the definition of isodifferentials and partial isoderivations in (1)–(3)). Here we note that the tangent isobundle (in brief t–isobundle) of a isomanifold $\widehat{M}$, denoted as $\widehat{TM}=\cup_{x\in\widehat{M}}\widehat{T_xM}$, where $\widehat{T_xM}$ is tangent isospaces of tangent isovectors in the point $\widehat{x}\in\widehat{M}$, is defined as a v–isobundle $\widehat{E}=\widehat{TM}$. By $\widehat{TM}^*$ we define the dual (not confusing with isotopic dual) of the t–isobundle $\widehat{TM}$. We note that for $\widehat{TM}$ and $\widehat{TM}^*$ isobundles the fibre and the base have both the same dimension and it is not necessary to distinguish always the fiber and base indices by different letters.

1.8 Nonlinear and Distinguished Isoconnections

The concept of **nonlinear connection,** in brief, N–connection, is fundamental in the geometry of locally anisotropic spaces (see a detailed study and basic references in [?, ?]. In our works [?, ?] the locally anisotropic spaces and superspaces are respectively modelled on vector bundles and vector superbundles enabled with compatible nonlinear and distinguished connections and metric structures (as particular cases, on tangent bundles and superbundles, for corresponding classes of metrics and nonlinear connections, one constructs generalized Lagrange and Finsler spaces and superspaces). The geometrical objects on a locally anisotropic space are called **distinguished** (see detailes in [?, ?]) if they are compatible with the N–connection structure (one considers, for instance, distinguished connections and distinguished tensors, in brief, d–connections and d–tensors).

In this subsection we study, apparently for the first time, the isogeometry of N–connection in vector isobundle. We note that a type of generic nonlinearity is contained by definition in the structure of isospace (it can be associated to a corresponding class of nonlinear isoconnections which can be turned into linear ones under corresponding isotopic transforms). As to a general N–connection introduced as a global decompositions of a vector isobundle into horizontal and vertical isotopic subbundles (see below) it can not be isolinearized if its isocurvature is nonzero.

Let consider a v–isobundle $\widehat{\xi} = \left(\widehat{E}, \widehat{p}, \widehat{M}\right)$ whose type fibre is $\widehat{R}^m$ and $\widehat{p}^T : \widehat{TE} \to \widehat{TM}$ is the isodifferential of the isomap $\widehat{p}$. The kernel of the isomap $\widehat{p}^T$ (which is a fibre–preserving isotopic morphism of the t–isobundle $\left(\widehat{TE}, \widehat{\tau}_E, \widehat{E}\right)$ to $\widehat{E}$ and of t–isobundle $\left(\widehat{TM}, \widehat{\tau}_M, \widehat{M}\right)$ to $\widehat{M}$) defines the vertical isotopic subbundle $\left(\widehat{VE}, \widehat{\tau}_V, \widehat{E}\right)$ over $\widehat{E}$ being an isovector subbundle of the v–isobundle $\left(\widehat{TE}, \widehat{\tau}_E, \widehat{E}\right)$.

An isovector $\widehat{X}_u$ tangent to $\widehat{E}$ in a point $\widehat{u} \in \widehat{E}$ locally defined by the decomposition $\widehat{X}^i \widehat{\partial}_i + \widehat{Y}^a \widehat{\partial}_a$ is locally represented by the isocoordinates

$$\widehat{X}_u = \left(\widehat{x}, \widehat{y}, \widehat{X}, \widehat{Y}\right) = \left(\widehat{x}^i, \widehat{y}^a, \widehat{X}^i, \widehat{Y}^a\right).$$

Since $\widehat{p}^T \left(\widehat{\partial}_a\right) = 0$ it results that $\widehat{p}^T \left(\widehat{x}, \widehat{y}, \widehat{X}, \widehat{Y}\right) = \left(\widehat{x}, \widehat{X}\right)$. We also consider the isotopic imbedding map $\widehat{i} : \widehat{VE} \to \widehat{TE}$ and the isobundle of inverse image $\widehat{p}^* \widehat{TM}$ of the $\widehat{p} : \widehat{E} \to \widehat{M}$ and define in result the isomap $\widehat{p}! : \widehat{TE} \to \widehat{p}^* \widehat{TM}, \widehat{p}! \left(\widehat{X}_u\right) = \left(\widehat{u}, \widehat{p}^T \left(\widehat{X}_u\right)\right)$ for which one holds $Ker\ \widehat{p}! = Ker\ \widehat{p}^T = \widehat{VE}.$

A **nonlinear isoconnection,** (in brief, **N–isoconnection)** in the isovector bundle $\widehat{\xi} = \left(\widehat{E}, \widehat{p}, \widehat{M}\right)$ is defined as a splitting on the left of the exact sequence of isotopic maps

$$0 \longrightarrow \widehat{VE} \stackrel{\widehat{i}}{\longrightarrow} \widehat{TE} \stackrel{\widehat{p}!}{\longrightarrow} \widehat{TE}/\widehat{VE} \longrightarrow 0$$

that is an isotopic morphism of vector isobundles $\widehat{C} : \widehat{TE} \to \widehat{VE}$ such that $\widehat{C} \circ \widehat{i}$ is the identity on $\widehat{VE}$.

The kernel of the isotopic morphism $\widehat{C}$ is a isovector subbundle of $\left(\widehat{TE}, \widehat{\tau}_E, \widehat{E}\right)$ and will be called the horizontal isotopic subbundle $\left(\widehat{HE}, \widehat{\tau}_H, \widehat{E}\right)$.

As a consequence of the above presented definition we can consider that a N–isoconnection in v–isobundle $\widehat{E}$ is a isotopic distribution $\{\widehat{N} : \widehat{E}_u \to H_u\widehat{E}, T_u\widehat{E} = H_u\widehat{E} \oplus V_u\widehat{E}\}$ on $\widehat{E}$ such that it is defined a global decomposition, as a Whitney sum, into horizontal,$\widehat{HE}$, and vertical, $\widehat{VE}$, subbundles of the tangent isobundle $\widehat{TE}$:

$$\widehat{TE} = \widehat{HE} \oplus \widehat{VE}. \tag{14}$$

Locally a N–isoconnection in $\widehat{\xi}$ is given by its components $\widehat{N}_i^a(\widehat{u}) = \widehat{N}_i^a(\widehat{x}, \widehat{y}) = N_{\widehat{i}}^{\widehat{a}}(\widehat{x}, \widehat{y})$ with respect to local isocoordinate bases (12) and (13)):

$$\widehat{\mathbf{N}} = \widehat{N}_i^a(\widehat{u})\widehat{d}^i \otimes \widehat{\partial}_a.$$

We note that a linear isoconnection in a v–isobundle $\widehat{\xi}$ can be considered as a particular case of a N–isoconnection when $\widehat{N}_i^a(\widehat{x}, \widehat{y}) = \widehat{K}_{bi}^a(\widehat{x})\,\widehat{y}^b$, where isofunctions $\widehat{K}_{ai}^b(\widehat{x})$ on the base $\widehat{M}$ are called the Christoffel isocoefficients.

To coordinate locally geometric constructions with the global splitting of isobundle defined by a N–isoconnection structure, we have to introduce a locally adapted isobasis (isoframe):

$$\frac{\widehat{\delta}}{\delta\widehat{u}^\alpha} = \left(\frac{\widehat{\delta}}{\delta\widehat{x}^i} = \widehat{\partial}_i - \widehat{N}_i^a(\widehat{u})\,\widehat{\partial}_a, \frac{\widehat{\partial}}{\partial\widehat{y}^a}\right), \tag{15}$$

or, in brief, $\widehat{\delta}_\alpha = \delta_{\widehat{\alpha}} = \left(\widehat{\delta}_i, \widehat{\partial}_a\right)$, and its dual locally isotropic isobasis

$$\widehat{\delta}\widehat{u}^\alpha = \left(\widehat{\delta}\widehat{x}^i = \widehat{d}\widehat{x}^i, \widehat{\delta}\widehat{y}^a + \widehat{N}_i^a(\widehat{u})\,\widehat{d}\widehat{x}^i\right), \tag{16}$$

or, in brief, $\widehat{\delta}^{\ \alpha} = \left(\widehat{d}^i, \widehat{\delta}^a\right)$. We note that isooperators (15) and (16) generalize correspondingly the partial isoderivations and isodifferentials (2)–(3) and (12) for the case when a N—isoconnection is defined.

The **nonholonomic isocoefficients** $\widehat{\mathbf{w}} = \{\widehat{w}_{\beta\gamma}^\alpha(\widehat{u})\}$ of locally isotropic isoframes are defined as

$$\left[\widehat{\delta}_\alpha, \widehat{\delta}_\beta\right] = \widehat{\delta}_\alpha\widehat{\delta}_\beta - \widehat{\delta}_\beta\widehat{\delta}_\alpha = \widehat{w}_{\beta\gamma}^\alpha(\widehat{u})\,\widehat{\delta}_\alpha.$$

The **algebra of tensorial distinguished isofields** $D\widehat{T}\left(\widehat{\xi}\right)$ (d–isofields, d–isotensors, d–tensor isofield, d–isobjects) on $\widehat{\xi}$ is introduced as the tensor algebra $\mathcal{T} = \{\widehat{\mathcal{T}}_{qs}^{pr}\}$ of the v–isobundle $\widehat{\xi}_{(d)}$, $\widehat{p}_d : \widehat{HE} \oplus \widehat{VE} \to \widehat{E}$. An element $\widehat{\mathbf{t}} \in \widehat{\mathcal{T}}_{qs}^{pr}$, d–tensor isofield of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, can be written in local form as

$$\widehat{\mathbf{t}} = \widehat{t}_{j_1\ldots j_q b_1\ldots b_r}^{i_1\ldots i_p a_1\ldots a_r}(u)\,\widehat{\delta}_{i_1} \otimes \ldots \otimes \widehat{\delta}_{i_p} \otimes \widehat{\partial}_{a_1} \otimes \ldots \otimes \widehat{\partial}_{a_r} \otimes$$

$$\widehat{d}^{j_1} \otimes ... \otimes \widehat{d}^{j_q} \otimes \widehat{\delta}^{b_1} ... \otimes \widehat{\delta}^{b_r}.$$

We shall respectively use denotations $\mathcal{X}\left(\widehat{\xi}\right)$ (or $\mathcal{X}\left(\widehat{M}\right)$), $\Lambda^p\left(\widehat{\xi}\right)$ (or $\Lambda^p\left(\widehat{M}\right)$) and $\mathcal{F}\left(\widehat{\xi}\right)$ (or $\mathcal{F}\left(\widehat{M}\right)$) for the isotopic module of d-vector isofields on $\widehat{\xi}$ (or $\widehat{M}$), the exterior algebra of p-forms on $\widehat{\xi}$ (or $\widehat{M}$) and the set of real functions on $\widehat{\xi}$ (or $\widehat{M}$).

In general, d–objects on $\widehat{\xi}$ are introduced as geometric objects with various isogroup and isocoordinate transforms coordinated with the N–connection isostructure on $\widehat{\xi}$. For example, a d–connection $\widehat{D}$ on $\widehat{\xi}$ is defined as a isolinear connection $\widehat{D}$ on $\widehat{E}$ conserving under a parallelism the global decomposition (14) into horizontal and vertical subbundles of $\widehat{TE}$.

A N–connection in $\widehat{\xi}$ induces a corresponding decomposition of d–isotensors into sums of horizontal and vertical parts, for example, for every d–isovector $\widehat{X} \in \mathcal{X}\left(\widehat{\xi}\right)$ and 1–form $\widetilde{X} \in \Lambda^1\left(\widehat{\xi}\right)$ we have respectively

$$\widehat{X} = \widehat{hX} + \widehat{vX} \quad \text{and} \quad \widetilde{X} = h\widetilde{X} + v\widetilde{X} \ .$$

In consequence, we can associate to every d–covariant isoderivation $\widehat{D_X} = \widehat{X} \circ \widehat{D}$ two new operators of h- and v-covariant isoderivations defined respectively as

$$\widehat{D}_X^{(h)}\widehat{Y} = \widehat{D}_{hX}\widehat{Y} \quad \text{and} \quad \widehat{D}_X^{(v)}\widehat{Y} = \widehat{D}_{vX}\widehat{Y}, \quad \forall \widehat{Y} \in \mathcal{X}\left(\widehat{\xi}\right)$$

for which the following conditions hold:

$$\begin{aligned} \widehat{D}_X\widehat{Y} &= \widehat{D}_X^{(h)}\widehat{Y} + \widehat{D}_X^{(v)}\widehat{Y}, \\ \widehat{D}_X^{(h)} f &= (h\widehat{X})f \quad \text{and} \quad \widehat{D}_X^{(v)} f = (v\widehat{X})f, \quad \widehat{X}, \widehat{Y} \in \mathcal{X}\left(\widehat{\xi}\right), f \in \mathcal{F}\left(\widehat{M}\right). \end{aligned} \tag{17}$$

An **isometric structure** $\widehat{\mathbf{G}}$ in the total space $\widehat{E}$ of v–isobundle $\widehat{\xi} = \left(\widehat{E}, \widehat{p}, \widehat{M}\right)$ over a connected and paracompact base $\widehat{M}$ is introduced as a symmetrical covariant tensor isofield of type $(0,2)$, $\widehat{G}_{\alpha\beta}$, being nondegenerate and of constant signature on $\widehat{E}$.

Nonlinear isoconnection $\widehat{\mathbf{N}}$ and isometric $\widehat{\mathbf{G}}$ structures on $\widehat{\xi}$ are mutually compatible it there are satisfied the conditions:

$$\widehat{\mathbf{G}}\left(\widehat{\delta}_i, \widehat{\partial}_a\right) = 0$$

which in component form are written as

$$\widehat{G}_{ia}\left(\widehat{u}\right) - \widehat{N}_i^b\left(\widehat{u}\right)\widehat{h}_{ab}\left(\widehat{u}\right) = 0, \tag{18}$$

where $\widehat{h}_{ab} = \widehat{\mathbf{G}}\left(\widehat{\partial}_a, \widehat{\partial}_b\right)$ and $\widehat{G}_{ia} = \widehat{\mathbf{G}}\left(\widehat{\partial}_i, \widehat{\partial}_a\right)$ (the matrix $\widehat{h}^{ab}$ is inverse to $\widehat{h}_{ab}$).

In consequence one obtains the following decomposition of isotopic metric :

$$\widehat{\mathbf{G}}(\widehat{X},\widehat{Y})=\widehat{\mathbf{hG}}(\widehat{X},\widehat{Y})+\widehat{\mathbf{vG}}(\widehat{X},\widehat{Y})$$

where the d-tensor $\widehat{\mathbf{hG}}(\widehat{X},\widehat{Y})=\widehat{\mathbf{G}}(\widehat{hX},\widehat{hY})$ is of type $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and the d–isotensor $\widehat{\mathbf{vG}}(\widehat{X},\widehat{Y})=\widehat{\mathbf{G}}(v\widehat{X},v\widehat{Y})$ is of type $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. With respect to locally anisotropic isobasis (16) the d–isometric is written as

$$\widehat{\mathbf{G}} = \widehat{g}_{\alpha\beta}\left(\widehat{u}\right)\widehat{\delta}^{\alpha}\otimes\widehat{\delta}^{\beta} = \widehat{g}_{ij}\left(\widehat{u}\right)\widehat{d}^{i}\otimes\widehat{d}^{j} + \widehat{h}_{ab}\left(\widehat{u}\right)\widehat{\delta}^{a}\otimes\widehat{\delta}^{b}, \tag{19}$$

where $\widehat{g}_{ij} = \widehat{\mathbf{G}}\left(\widehat{\delta}_i,\widehat{\delta}_j\right).$

A metric isostructure of type (19) on $\widehat{E}$ with components satisfying constraints (16)) defines an adapted to the given N–isoconnection inner (d–scalar) isoproduct on the tangent isobundle $\widehat{TE}$.

A d–isoconnection $\widehat{D}_X$ is **compatible** with an isometric $\widehat{\mathbf{G}}$ on $\widehat{\xi}$ if

$$\widehat{D}_X\widehat{\mathbf{G}} = 0, \forall\widehat{X}\in\widehat{\mathcal{X}}\left(\widehat{\xi}\right).$$

Locally adapted components $\widehat{\Gamma}^{\alpha}_{\beta\gamma}$ of a d–isoconnection $\widehat{D}_\alpha = (\widehat{\delta}_\alpha\circ\widehat{D})$ are defined by the equations

$$\widehat{D}_\alpha\widehat{\delta}_\beta = \widehat{\Gamma}^{\gamma}_{\alpha\beta}\widehat{\delta}_\gamma,$$

from which one immediately follows

$$\widehat{\Gamma}^{\gamma}_{\alpha\beta}\left(\widehat{u}\right) = \left(\widehat{D}_\alpha\widehat{\delta}_\beta\right)\circ\widehat{\delta}^{\gamma}. \tag{20}$$

The operations of h- and v–covariant isoderivations,

$$\widehat{D}^{(h)}_k = \{\widehat{L}^i_{jk},\widehat{L}^a_{bk}\} \text{ and } \widehat{D}^{(v)}_c = \{\widehat{C}^i_{jk},\widehat{C}^a_{bc}\},$$

are introduced as corresponding horizontal and vertical parametrizations:

$$\widehat{L}^i_{jk} = \left(\widehat{D}_k\widehat{\delta}_j\right)\circ\widehat{d}^i, \quad \widehat{L}^a_{bk} = \left(\widehat{D}_k\widehat{\partial}_b\right)\circ\widehat{\delta}^a \tag{21}$$

and

$$\widehat{C}^i_{jc} = \left(\widehat{D}_c\widehat{\delta}_j\right)\circ\widehat{d}^i, \quad \widehat{C}^a_{bc} = \left(\widehat{D}_c\widehat{\partial}_b\right)\circ\widehat{\delta}^a. \tag{22}$$

Components (21) and (22), $\widehat{D}\widehat{\Gamma} = \left(\widehat{L}^i_{jk},\widehat{L}^a_{bk},\widehat{C}^i_{jc},\widehat{C}^a_{bc}\right)$, completely defines the local action of a d–isoconnection $\widehat{D}$ in $\widehat{\xi}$. For instance, taken a d–tensor isofield of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$,

$$\widehat{\mathbf{t}} = \widehat{t}^{ia}_{jb}\widehat{\delta}_i\otimes\widehat{\partial}_a\otimes\widehat{\partial}^j\otimes\widehat{\delta}^b,$$

and a d-vector $\widehat{\mathbf{X}} = \widehat{X}^i \widehat{\delta}_i + \widehat{X}^a \widehat{\partial}_a$ we have

$$\widehat{D}_X \widehat{\mathbf{t}} = \widehat{D}_X^{(h)} \widehat{\mathbf{t}} + \widehat{D}_X^{(v)} \widehat{\mathbf{t}} = \left(\widehat{X}^k \widehat{t}^{ia}_{jb|k} + \widehat{X}^c \widehat{t}^{ia}_{jb\perp c} \right) \widehat{\delta}_i \otimes \widehat{\partial}_a \otimes \widehat{d}^j \otimes \widehat{\delta}^b,$$

where the **h–covariant and v–covariant isoderivatives** are written respectively as

$$\widehat{t}^{ia}_{jb|k} = \frac{\widehat{\delta} \widehat{t}^{ia}_{jb}}{\delta \widehat{x}^k} + \widehat{L}^i_{hk} \widehat{t}^{ha}_{jb} + \widehat{L}^a_{ck} \widehat{t}^{ic}_{jb} - \widehat{L}^h_{jk} \widehat{t}^{ia}_{hb} - \widehat{L}^c_{bk} \widehat{t}^{ia}_{jc}$$

and

$$\widehat{t}^{ia}_{jb\perp c} = \frac{\widehat{\partial} \widehat{t}^{ia}_{jb}}{\partial \widehat{y}^c} + \widehat{C}^i_{hc} \widehat{t}^{ha}_{jb} + \widehat{C}^a_{dc} \widehat{t}^{id}_{jb} - \widehat{C}^h_{jc} \widehat{t}^{ia}_{hb} - \widehat{C}^d_{bc} \widehat{t}^{ia}_{jd}.$$

For a scalar isofunction $f \in \mathcal{F}(\widehat{\xi})$ we have

$$\widehat{D}_k^{(h)} = \frac{\widehat{\delta} f}{\delta \widehat{x}^k} = \frac{\widehat{\partial} f}{\partial \widehat{x}^k} - \widehat{N}^a_k \frac{\widehat{\partial} f}{\partial \widehat{y}^a} \text{ and } \widehat{D}_c^{(v)} f = \frac{\widehat{\partial} f}{\partial \widehat{y}^c}.$$

We emphasize that the geometry of connections in a v–isobundle $\widehat{\xi}$ is very reach. For instance, if a triple of fundamental isogeometric objects $(\widehat{N}^a_i(\widehat{u}), \widehat{\Gamma}^\alpha_{\beta\gamma}(\widehat{u}), \widehat{G}_{\alpha\beta}(\widehat{u}))$ is fixed on $\widehat{\xi}$, a multi–isoconnection structure (with corresponding rules of covariant isoderivation, which are, or not, mutually compatible and with the same, or not, induced d–scalar products in $\widehat{TE}$) is defined.

Let enumerate some of isoconnections and covariant isoderivations which can present interest in investigation of locally anisotropic and homogeneous gravitational and matter field isotopic interactions:

1. Every N–isoconnection in $\widehat{\xi}$ with coefficients $\widehat{N}^a_i(\widehat{x}, \widehat{y})$ being isodifferentiable on y-variables induces a structure of isolinear isoconnection $\widetilde{N}^\alpha_{\beta\gamma}$, where $\widetilde{N}^a_{bi} = \frac{\widehat{\partial} \widehat{N}^a_i}{\partial \widehat{y}^b}$ and $\widetilde{N}^a_{bc}(\widehat{x}, \widehat{y}) = 0$. For some $\widehat{Y}(\widehat{u}) = \widehat{Y}^i(\widehat{u}) \widehat{\partial}_i + \widehat{Y}^a(\widehat{u}) \widehat{\partial}_a$ and $\widehat{B}(\widehat{u}) = \widehat{B}^a(\widehat{u}) \widehat{\partial}_a$ one writes

$$\widehat{D}_Y^{(\widetilde{N})} \widehat{B} = \left[\widehat{Y}^i \left(\frac{\widehat{\partial} \widehat{B}^a}{\partial \widehat{x}^i} + \widetilde{N}^a_{bi} \widehat{B}^b \right) + \widehat{Y}^b \frac{\widehat{\partial} \widehat{B}^a}{\partial \widehat{y}^b} \right] \frac{\widehat{\partial}}{\partial \widehat{y}^a}.$$

2. The d–isoconnection of Berwald type [**?**]

$$\widehat{\Gamma}^{(B)\alpha}_{\beta\gamma} = \left(\widehat{L}^i_{jk}, \frac{\widehat{\partial} \widehat{N}^a_k}{\partial \widehat{y}^b}, 0, \widehat{C}^a_{bc} \right),$$

where

$$\begin{aligned} \widehat{L}^i_{.jk}(\widehat{x}, \widehat{y}) &= \frac{1}{2} \widehat{g}^{ir} \left(\frac{\widehat{\delta} \widehat{g}_{jk}}{\delta \widehat{x}^k} + \frac{\widehat{\delta} \widehat{g}_{kr}}{\delta \widehat{x}^j} - \frac{\widehat{\delta} \widehat{g}_{jk}}{\delta \widehat{x}^r} \right), \qquad (23) \\ \widehat{C}^a_{.bc}(\widehat{x}, \widehat{y}) &= \frac{1}{2} \widehat{h}^{ad} \left(\frac{\widehat{\partial} \widehat{h}_{bd}}{\partial \widehat{y}^c} + \frac{\widehat{\partial} \widehat{h}_{cd}}{\partial \widehat{y}^b} - \frac{\widehat{\partial} \widehat{h}_{bc}}{\partial \widehat{y}^d} \right), \end{aligned}$$

is **hv-isometric,** i.e. $\widehat{D}_k^{(B)} \widehat{g}_{ij} = 0$ and $\widehat{D}_c^{(B)} \widehat{h}_{ab} = 0$.

3. The isocanonical d–isoconnection $\widehat{\mathbf{\Gamma}}^{(\mathbf{c})}$ is associated to a isometric $\widehat{\mathbf{G}}$ of type (18) $\widehat{\Gamma}_{\beta\gamma}^{(c)\alpha} = (\widehat{L}_{jk}^{(c)i}, \widehat{L}_{bk}^{(c)a}, \widehat{C}_{jc}^{(c)i}, \widehat{C}_{bc}^{(c)a})$, with coefficients (see (23))

$$\begin{aligned} \widehat{L}_{jk}^{(c)i} &= \widehat{L}_{.jk}^{i}, \widehat{C}_{bc}^{(c)a} = \widehat{C}_{.bc}^{a}) \\ \widehat{L}_{bi}^{(c)a} &= \widetilde{N}_{bi}^{a} + \frac{1}{2}\widehat{h}^{ac}\left(\frac{\widetilde{\delta h}_{bc}}{\delta \widehat{x}^{i}} - \widetilde{N}_{bi}^{d}\widehat{h}_{dc} - \widetilde{N}_{ci}^{d}\widehat{h}_{db}\right), \\ \widehat{C}_{jc}^{(c)i} &= \frac{1}{2}\widehat{g}^{ik}\frac{\widehat{\partial g}_{jk}}{\partial \widehat{y}^{c}}. \end{aligned} \tag{24}$$

This is a isometric d–isoconnection which satisfies compatibility conditions

$$\widehat{D}_{k}^{(c)}\widehat{g}_{ij} = 0, \widehat{D}_{c}^{(c)}\widehat{g}_{ij} = 0, \widehat{D}_{k}^{(c)}\widehat{h}_{ab} = 0, \widehat{D}_{c}^{(c)}\widehat{h}_{ab} = 0.$$

4. We can consider N–adapted **isotopic Christoffel distinguished symbols** (as in (5))

$$\widetilde{\Gamma}_{\beta\gamma}^{\alpha} = \frac{1}{2}\widehat{G}^{\alpha\tau}\left(\widehat{\delta}_{\gamma}\widehat{G}_{\tau\beta} + \widehat{\delta}_{\beta}\widehat{G}_{\tau\gamma} - \widehat{\delta}_{\tau}\widehat{G}_{\beta\gamma}\right),$$

which have the components of d–connection $\widetilde{\Gamma}_{\beta\gamma}^{\alpha} = \left(\widehat{L}_{jk}^{i}, 0, 0, \widehat{C}_{bc}^{a}\right)$, with $\widehat{L}_{jk}^{i}$ and $\widehat{C}_{bc}^{a}$ as in (23) if $\widehat{G}_{\alpha\beta}$ is taken in the form (19).

Arbitrary isolinear isoconnections on a v–isobundle $\widehat{\xi}$ can be also characterized by theirs deformation isotensors with respect, for instance, to d–isoconnection (4):

$$\widehat{\Gamma}_{\beta\gamma}^{(B)\alpha} = \widetilde{\Gamma}_{\beta\gamma}^{\alpha} + \widehat{P}_{\beta\gamma}^{(B)\alpha}, \widehat{\Gamma}_{\beta\gamma}^{(c)\alpha} = \widetilde{\Gamma}_{\beta\gamma}^{\alpha} + \widehat{P}_{\beta\gamma}^{(c)\alpha}$$

or, in general,

$$\widehat{\Gamma}_{\beta\gamma}^{\alpha} = \widetilde{\Gamma}_{\beta\gamma}^{\alpha} + \widehat{P}_{\beta\gamma}^{\alpha}, \tag{25}$$

where $\widehat{P}_{\beta\gamma}^{(B)\alpha}, \widehat{P}_{\beta\gamma}^{(c)\alpha}$ and $\widehat{P}_{\beta\gamma}^{\alpha}$ are corresponding deformation d–isotensors of d–isoconnections.

1.9 Isotorsions and Isocurvatures

The notions of isotorsion and isocurvature were introduced in the ref. [**?**] for an Riemannian isospaces. In this subsection we reformulate these notions on isobundles provided with N–isoconnection and d–isoconnection structures.

The **isocurvature** $\widehat{\mathbf{\Omega}}$ of a **nonlinear isoconnection** $\widehat{\mathbf{N}}$ in a v–isobundle $\widehat{\xi}$ can be defined as the Nijenhuis tensor isofield $\widehat{N}_{v}\left(\widehat{X}, \widehat{Y}\right)$ associated to $\widehat{\mathbf{N}}$ (this is an isotopic transform for N–curvature considered, for instance, in [**?**, **?**]:

$$\widehat{\mathbf{\Omega}} = \widehat{N}_{v} = \left[\mathbf{v}\widehat{\mathbf{X}}, \mathbf{v}\widehat{\mathbf{Y}}\right] + \mathbf{v}\left[\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right] - \mathbf{v}\left[\mathbf{v}\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right] - \mathbf{v}\left[\widehat{\mathbf{X}}, \mathbf{v}\widehat{\mathbf{Y}}\right], \widehat{\mathbf{X}}, \widehat{\mathbf{Y}} \in \mathcal{X}\left(\widehat{\xi}\right)$$

having this local representation

$$\widehat{\mathbf{\Omega}} = \frac{1}{2}\widehat{\Omega}^a_{ij}\widehat{d}^i \bigwedge \widehat{d}^j \otimes \widehat{\partial}_a,$$

where

$$\widehat{\Omega}^a_{ij} = \frac{\partial \widehat{N}^a_i}{\partial \widehat{x}^j} - \frac{\partial \widehat{N}^a_j}{\partial \widehat{x}^i} + \widehat{N}^b_i \widetilde{N}^a_{bj} - \widehat{N}^b_j \widetilde{N}^a_{bi}. \tag{26}$$

The **isotorsion** $\widehat{\mathbf{T}}$ of a **d–isoconnection** $\widehat{\mathbf{D}}$ in $\widehat{\xi}$ is defined by the equation

$$\widehat{\mathbf{T}}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right) = \widehat{D}_X \widehat{\mathbf{Y}} - \widehat{D}_Y \widehat{\mathbf{X}} - \left[\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right]. \tag{27}$$

One holds the following h- and v–decompositions

$$\widehat{\mathbf{T}}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right) = \widehat{\mathbf{T}}\left(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}}\right) + \widehat{\mathbf{T}}\left(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{v}\widehat{\mathbf{Y}}\right) + \widehat{\mathbf{T}}\left(\mathbf{v}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}}\right) + \widehat{\mathbf{T}}\left(\mathbf{v}\widehat{\mathbf{X}}, \mathbf{v}\widehat{\mathbf{Y}}\right). \tag{28}$$

We consider the projections:

$$\mathbf{h}\widehat{\mathbf{T}}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right), \mathbf{v}\widehat{\mathbf{T}}\left(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}}\right), \mathbf{h}\widehat{\mathbf{T}}\left(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}}\right), ...$$

and say that, for instance, $\mathbf{h}\widehat{\mathbf{T}}\left(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}}\right)$ is the h(hh)–isotorsion of $\widehat{\mathbf{D}}$, $\mathbf{v}\widehat{\mathbf{T}}\left(\mathbf{h}\widehat{\mathbf{X}}, \mathbf{h}\widehat{\mathbf{Y}}\right)$ is the v(hh)–isotorsion of $\widehat{\mathbf{D}}$ and so on.

The isotorsion (27) is locally determined by five d–tensor isofields, isotorsions, defined as

$$\begin{aligned} \widehat{T}^i_{jk} &= \mathbf{h}\widehat{\mathbf{T}}\left(\widehat{\delta}_k, \widehat{\delta}_j\right) \cdot \widehat{d}^i, \quad \widehat{T}^a_{jk} = \mathbf{v}\widehat{\mathbf{T}}\left(\widehat{\delta}_k, \widehat{\delta}_j\right) \cdot \widehat{\delta}^a, \\ \widehat{P}^i_{jb} &= \mathbf{h}\widehat{\mathbf{T}}\left(\widehat{\partial}_b, \widehat{\delta}_j\right) \cdot \widehat{d}^i, \quad \widehat{P}^a_{jb} = \mathbf{v}\widehat{\mathbf{T}}\left(\widehat{\partial}_b, \widehat{\delta}_j\right) \cdot \widehat{\delta}^a, \widehat{S}^a_{bc} = \mathbf{v}\widehat{\mathbf{T}}\left(\widehat{\partial}_c, \widehat{\partial}_b\right) \cdot \widehat{\delta}^a. \end{aligned}$$

Using formulas (15), (16), (25) and (27) we compute in explicit form the components of isotorsions (28) for a d–isoconnection of type (21) and (22):

$$\begin{aligned} \widehat{T}^i_{.jk} &= \widehat{T}^i_{jk} = \widehat{L}^i_{jk} - \widehat{L}^i_{kj}, \quad \widehat{T}^i_{ja} = \widehat{C}^i_{.ja}, \widehat{T}^i_{aj} = -\widehat{C}^i_{ja}, \\ \widehat{T}^i_{.ja} &= 0, \widehat{T}^a_{.bc} = \widehat{S}^a_{.bc} = \widehat{C}^a_{bc} - \widehat{C}^a_{cb}, \\ \widehat{T}^a_{.ij} &= \frac{\widehat{\delta} N^a_i}{\delta \widehat{x}^j} - \frac{\widehat{\delta} \widehat{N}^a_j}{\delta \widehat{x}^i}, \quad \widehat{T}^a_{.bi} = \widehat{P}^a_{.bi} = \frac{\partial \widehat{N}^a_i}{\partial \widehat{y}^b} - \widehat{L}^a_{.bj}, \quad \widehat{T}^a_{.ib} = -\widehat{P}^a_{.bi}. \end{aligned} \tag{29}$$

The **isocurvature** $\widehat{\mathbf{R}}$ of a **d–isoconnection** in $\widehat{\xi}$ is defined by the equation

$$\widehat{\mathbf{R}}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right)\widehat{\mathbf{Z}} = \widehat{D}_X \widehat{D}_Y \widehat{\mathbf{Z}} - \widehat{D}_Y \widehat{D}_X \widehat{\mathbf{Z}} - \widehat{D}_{[X,Y]}\widehat{\mathbf{Z}}. \tag{30}$$

One holds the next properties for the h- and v–decompositions of isocurvature:

$$\begin{aligned} \mathbf{v}\widehat{\mathbf{R}}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right)\mathbf{h}\widehat{\mathbf{Z}} &= \mathbf{0}, \ \mathbf{h}\widehat{\mathbf{R}}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right)\mathbf{v}\widehat{\mathbf{Z}} = \mathbf{0}, \\ \widehat{\mathbf{R}}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right)\widehat{\mathbf{Z}} &= \mathbf{h}\widehat{\mathbf{R}}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right)\mathbf{h}\widehat{\mathbf{Z}} + \mathbf{v}\widehat{\mathbf{R}}\left(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}\right)\mathbf{v}\widehat{\mathbf{Z}}. \end{aligned}$$

¿From (30) and the equation $\widehat{\mathbf{R}}\left(\widehat{\mathbf{X}},\widehat{\mathbf{Y}}\right)=-\widehat{\mathbf{R}}\left(\widehat{\mathbf{Y}},\widehat{\mathbf{X}}\right)$ we conclude that the curvature of a d-connection $\widehat{\mathbf{D}}$ in $\widehat{\xi}$ is completely determined by the following six d–tensor isofields:

$$\begin{aligned}
\widehat{R}_{h.jk}^{.i} &= \widehat{d}^i\cdot\widehat{\mathbf{R}}\left(\widehat{\delta}_k,\widehat{\delta}_j\right)\widehat{\delta}_h,\ \widehat{R}_{b.jk}^{.a}=\widehat{\delta}^a\cdot\widehat{\mathbf{R}}\left(\widehat{\delta}_k,\widehat{\delta}_j\right)\widehat{\partial}_b, \\
\widehat{P}_{j.kc}^{.i} &= \widehat{d}^i\cdot\widehat{\mathbf{R}}\left(\widehat{\partial}_c,\widehat{\partial}_k\right)\widehat{\delta}_j,\ \widehat{P}_{b.kc}^{.a}=\widehat{\delta}^a\cdot\widehat{\mathbf{R}}\left(\widehat{\partial}_c,\widehat{\partial}_k\right)\widehat{\partial}_b, \\
\widehat{S}_{j.bc}^{.i} &= \widehat{d}^i\cdot\widehat{\mathbf{R}}\left(\widehat{\partial}_c,\widehat{\partial}_b\right)\widehat{\delta}_j,\ \widehat{S}_{b.cd}^{.a}=\widehat{\delta}^a\cdot\widehat{\mathbf{R}}\left(\widehat{\partial}_d,\widehat{\partial}_c\right)\widehat{\partial}_b.
\end{aligned} \tag{31}$$

By a direct computation, using (15),(16),(21),(22) and (31) we get:

$$\begin{aligned}
\widehat{R}_{h.jk}^{.i} &= \frac{\widehat{\delta}\widehat{L}_{.hj}^{i}}{\delta\widehat{x}^h}-\frac{\widehat{\delta}\widehat{L}_{.hk}^{i}}{\delta\widehat{x}^j}+\widehat{L}_{.hj}^{m}\widehat{L}_{mk}^{i}-\widehat{L}_{.hk}^{m}\widehat{L}_{mj}^{i}+\widehat{C}_{.ha}^{i}\widehat{R}_{.jk}^{a}, \\
\widehat{R}_{b.jk}^{.a} &= \frac{\widehat{\delta}\widehat{L}_{.bj}^{a}}{\delta\widehat{x}^k}-\frac{\widehat{\delta}\widehat{L}_{.bk}^{a}}{\delta\widehat{x}^j}+\widehat{L}_{.bj}^{c}\widehat{L}_{.ck}^{a}-\widehat{L}_{.bk}^{c}\widehat{L}_{.cj}^{a}+\widehat{C}_{.bc}^{a}\widehat{R}_{.jk}^{c}, \\
\widehat{P}_{j.ka}^{.i} &= \frac{\widehat{\partial}\widehat{L}_{.jk}^{i}}{\partial\widehat{y}^k}-\left(\frac{\widehat{\delta}\widehat{C}_{.ja}^{i}}{\partial\widehat{x}^k}+\widehat{L}_{.lk}^{i}\widehat{C}_{.ja}^{l}-\widehat{L}_{.jk}^{l}\widehat{C}_{.la}^{i}-\widehat{L}_{.ak}^{c}\widehat{C}_{.jc}^{i}\right)+\widehat{C}_{.jb}^{i}\widehat{P}_{.ka}^{b}, \\
\widehat{P}_{b.ka}^{.c} &= \frac{\widehat{\partial}\widehat{L}_{.bk}^{c}}{\partial\widehat{y}^a}-\left(\frac{\widehat{\delta}\widehat{C}_{.ba}^{c}}{\partial\widehat{x}^k}+\widehat{L}_{.dk}^{c}\widehat{C}_{.ba}^{d}-\widehat{L}_{.bk}^{d}\widehat{C}_{.da}^{c}-\widehat{L}_{.ak}^{d}\widehat{C}_{.bd}^{c}\right)+\widehat{C}_{.bd}^{c}\widehat{P}_{.ka}^{d}, \\
\widehat{S}_{j.bc}^{.i} &= \frac{\widehat{\partial}\widehat{C}_{.jb}^{i}}{\partial\widehat{y}^c}-\frac{\widehat{\partial}\widehat{C}_{.jc}^{i}}{\partial\widehat{y}^b}+\widehat{C}_{.jb}^{h}\widehat{C}_{.hc}^{i}-\widehat{C}_{.jc}^{h}\widehat{C}_{hb}^{i}, \\
\widehat{S}_{b.cd}^{.a} &= \frac{\widehat{\partial}\widehat{C}_{.bc}^{a}}{\partial\widehat{y}^d}-\frac{\widehat{\partial}\widehat{C}_{.bd}^{a}}{\partial\widehat{y}^c}+\widehat{C}_{.bc}^{e}\widehat{C}_{.ed}^{a}-\widehat{C}_{.bd}^{e}\widehat{C}_{.ec}^{a}.
\end{aligned} \tag{32}$$

We note that isotorsions (29) and isocurvatures (32) can be computed by particular cases of d–isoconnections when d–isoconnections
(24), or (4) are used instead of (21) and (22). The above presented formulas are similar to (8),(9) and (10) being distinguished (in the case of locally anisotropic and inhomogeneous isospaces) by N–isoconnection structure.

For some considerations it is useful to use an alternative way of definition isotorsion (27) and isocurvature (30) by using the commutator

$$\widehat{\Delta}_{\alpha\beta}\doteq\widehat{\nabla}_\alpha\widehat{\nabla}_\beta-\widehat{\nabla}_\beta\widehat{\nabla}_\alpha=2\widehat{\nabla}_{[\alpha}\widehat{\nabla}_{\beta]}. \tag{33}$$

For components (33) of d–isotorsion we have

$$\widehat{\Delta}_{\alpha\beta}\widehat{f}=\widehat{T}_{.\alpha\beta}^{\gamma}\widehat{\nabla}_\gamma\widehat{f}$$

for every scalar function $\widehat{f}$ on $\widehat{\xi}$. Curvature can be introduced as an operator acting on arbitrary d–isovector $\widehat{V}^\delta$:

$$(\widehat{\Delta}_{\alpha\beta}-\widehat{T}_{.\alpha\beta}^{\gamma}\widehat{\nabla}_\gamma)\widehat{V}^\delta=\widehat{R}_{\gamma.\alpha\beta}^{\cdot\delta}\widehat{V}^\gamma$$

(in this work we are following conventions similar to Miron and Anastasiei [**?**, **?**] on d–isotensors; we can obtain corresponding Penrose and Rindler abstract index formulas [**?**, **?**, **?**] just for a trivial N–connection structure and by changing denotations for components of isotorsion and isocurvature in this manner: $T^{\gamma}_{.\alpha\beta} \rightarrow T_{\alpha\beta}{}^{\gamma}$ and $R^{\cdot\delta}_{\gamma.\alpha\beta} \rightarrow R_{\alpha\beta\gamma}{}^{\delta}$).

For our further considerations it is useful to compute deformations of isotorsion (27) and isocurvature (30) under deformations of d–connections (25). Putting the splitting (25), $\widehat{\Gamma}^{\alpha}{}_{\beta\gamma} = \tilde{\Gamma}^{\alpha}_{\cdot\beta\gamma} + \widehat{P}^{\alpha}{}_{\beta\gamma}$, into (27) and (30) we can express isotorsion $\widehat{T}^{\alpha}{}_{\beta\gamma}$ and isocurvature $\widehat{R}_{\beta}{}^{\alpha}{}_{\gamma\delta}$ of a d–isoconnection $\widehat{\Gamma}^{\alpha}{}_{\beta\gamma}$ as respective deformations of isotorsion $\tilde{T}^{\alpha}_{\beta\gamma}$ and isotorsion $\tilde{R}^{\cdot\alpha}_{\beta\cdot\gamma\delta}$ for connection $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$:

$$T^{\alpha}{}_{\beta\gamma} = \tilde{T}^{\alpha}_{\cdot\beta\gamma} + \ddot{T}^{\alpha}_{\cdot\beta\gamma}$$

and

$$R_{\beta}{}^{\alpha}{}_{\gamma\delta} = \tilde{R}^{\cdot\alpha}_{\beta\cdot\gamma\delta} + \ddot{R}^{\cdot\alpha}_{\beta\cdot\gamma\delta},$$

where

$$\tilde{T}^{\alpha}_{\beta\gamma} = \tilde{\Gamma}^{\alpha}_{\beta\gamma} - \tilde{\Gamma}^{\alpha}_{\gamma\beta} + w^{\alpha}{}_{\gamma\delta}, \qquad \ddot{T}^{\alpha}_{\beta\gamma} = \ddot{\Gamma}^{\alpha}_{\beta\gamma} - \ddot{\Gamma}^{\alpha}_{\gamma\beta},$$

and

$$\begin{aligned} \tilde{R}^{\cdot\alpha}_{\beta\cdot\gamma\delta} &= \delta_{\delta}\tilde{\Gamma}^{\alpha}_{\beta\gamma} - \delta_{\gamma}\tilde{\Gamma}^{\alpha}_{\beta\delta} + \tilde{\Gamma}^{\varphi}_{\beta\gamma}\tilde{\Gamma}^{\alpha}_{\varphi\delta} - \tilde{\Gamma}^{\varphi}_{\beta\delta}\tilde{\Gamma}^{\alpha}_{\varphi\gamma} + \tilde{\Gamma}^{\alpha}_{\beta\varphi}w^{\varphi}{}_{\gamma\delta}, \\ \ddot{R}^{\cdot\alpha}_{\beta\cdot\gamma\delta} &= \tilde{D}_{\delta}P^{\alpha}{}_{\beta\gamma} - \tilde{D}_{\gamma}P^{\alpha}{}_{\beta\delta} + P^{\varphi}{}_{\beta\gamma}P^{\alpha}{}_{\varphi\delta} - P^{\varphi}{}_{\beta\delta}P^{\alpha}{}_{\varphi\gamma} + P^{\alpha}{}_{\beta\varphi}w^{\varphi}{}_{\gamma\delta}. \end{aligned}$$

Finally, in this subsection we note that the isotopies of Bianchi and Ricci identities were first studied by Santilli [**?**] on an Riemannian isospace. On spaces with N–connection structures the general formulas for Bianchi and Ricci identities (for osculator and vector bundles, generalized Lagrange and Finsler geometry) have been considered by Miron and Anastasiei [**?**, **?**] and Miron and Atanasiu [**?**]. S. Vacaru extended the Miron–Anastasiei–Atanasiu constructions for superspaces with local and higher order anisotropy in refs. [**?**, **?**, **?**, **?**, **?**]. For isotopic generalized Finsler spaces the Bianchi and Ricci identities are proved in [**?**, **?**] (we refer the reader for details in the mentioned works).

2 Distinguished Clifford Isoalgebras and Isospinors

The theory of distinguished Clifford isoalgebras can be defined in a usual manner but by substituting the real (or, correspondingly, complex) numbers into isonumbers. This way the constructions are developed to Clifford module structures and isobundles. In order to state the difference between the usual objects (with real/complex number fields) and those constructed with isonumbers we shall provide "iso–objects" with "hat" symbols or indices.

2.1 Distinguished Clifford Isoalgebras

The typical fiber of vector isobundle (v-isobundle) $\tilde{\xi}_d$, $\pi_d : \ H\tilde{E} \oplus V\tilde{E} \to \tilde{E}$ is a d–vector isospace, $\hat{\mathcal{F}} = h\hat{\mathcal{F}} \oplus v\hat{\mathcal{F}}$, split into horizontal $h\hat{\mathcal{F}}$ and vertical $v\hat{\mathcal{F}}$ subspaces, with metric $\hat{G}(g,h)$ induced by v–isobundle metric (19). Clifford algebras (see, for example, Refs. [**?**, **?**, **?**]) formulated for d–vector isospaces will be called Clifford d-isoalgebras [**?**, **?**, **?**, **?**, **?**]. In this subsection we shall consider the main properties of Clifford d–isoalgebras. The proof of theorems will be based on the technique developed in Ref. [**?**, **?**, **?**, **?**] correspondingly adapted to the distinguished character of isospaces in consideration.

Let $\hat{k}$ be a isonumber field (for our purposes and define $\hat{\mathcal{F}}$, as a d–vector isospace on $\hat{k}$ provided with nondegenerate symmetric quadratic form (metric) $\hat{G}$. Let $\hat{C}$ be an algebra on $\hat{k}$ (not necessarily commutative) and $j : \ \hat{\mathcal{F}} \to \hat{C}$ a homomorphism of underlying vector spaces such that $j(u)^2 = \ \hat{G}(u) \cdot \hat{1}$ (1 is an isounity in isoalgebra $\hat{C}$ and d-isovector $u \in \hat{\mathcal{F}}$). We are interested in definition of the pair $\left(\hat{C}, j\right)$ satisfying the next universitality conditions. For every $\hat{k}$–algebra $\hat{A}$ and arbitrary homomorphism $\hat{\varphi} : \ \hat{\mathcal{F}} \to \hat{A}$ of the underlying d-vector isospaces, such that $\left(\hat{\varphi}(u)\right)^2 \to \hat{G}\left(u\right) \cdot \hat{1}$, there is a unique homomorphism of isoalgebras $\hat{\psi} \ : \ \hat{C} \to \hat{A}$ transforming the diagram 1 into a commutative one. The isoalgebra solving this problem is denoted $\hat{C}\left(\hat{\mathcal{F}}, \hat{A}\right)$

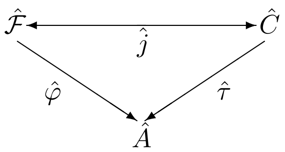

Figure 1: Diagram 1

[equivalently, $\hat{C}\left(\hat{G}\right)$ or $\hat{C}\left(\hat{\mathcal{F}}\right)$] and called as Clifford d-isoalgebra associated with pair $\left(\hat{\mathcal{F}}, \hat{G}\right)$.

Theorem 1 *The above–presented diagram has a unique solution* $\left(\hat{C}, \hat{j}\right)$ *up to isomorphism.*

Proof: (We adapt for d-isoalgebras that of Ref. [**?**], p. 127.) For a universal problem the uniqueness is obvious if we prove the existence of solution $\hat{C}\left(\hat{G}\right)$. To do this we use tensor isoalgebra $\hat{\mathcal{L}}^{(F)} = \oplus \hat{\mathcal{L}}^{pr}_{qs}\left(\hat{\mathcal{F}}\right) = \oplus_{i=0}^{\infty} T^i\left(\hat{\mathcal{F}}\right)$, where $T^0\left(\hat{\mathcal{F}}\right) = \hat{k}$ and $T^i\left(\hat{\mathcal{F}}\right) = \hat{k}$ and $T^i\left(\hat{\mathcal{F}}\right) = \hat{\mathcal{F}} \otimes ... \otimes \hat{\mathcal{F}}$ for $i > 0$. Let $I\left(\hat{G}\right)$ be the bilateral ideal generated by elements of form $\epsilon\left(u\right) = u \otimes u - \hat{G}\left(u\right) \cdot \hat{1}$ where $\hat{u} \in \hat{\mathcal{F}}$ and $\hat{1}$ is the isounity element of algebra $\hat{\mathcal{L}}\left(\hat{\mathcal{F}}\right)$. Every element from $I\left(\hat{G}\right)$ can be written as $\sum_i \lambda_i \epsilon(\hat{u}_i)\mu_i$, where $\lambda_i, \mu_i \in \hat{\mathcal{L}}(\hat{\mathcal{F}})$ and $\hat{u}_i \in \hat{\mathcal{F}}$. Let $\hat{C}\left(\hat{G}\right) = \hat{\mathcal{L}}(\hat{\mathcal{F}})/I\left(\hat{G}\right)$ and define $\hat{j} : \hat{\mathcal{F}} \to \hat{C}\left(\hat{G}\right)$ as the composition of monomorphism $\hat{\imath} : \hat{\mathcal{F}} \to \hat{L}^1(\hat{\mathcal{F}}) \subset \hat{\mathcal{L}}(\hat{\mathcal{F}})$ and projection $\hat{p} : \hat{\mathcal{L}}\left(\hat{\mathcal{F}}\right) \to \hat{C}\left(\hat{G}\right)$. In this case pair $\left(\hat{C}\left(\hat{G}\right), \hat{j}\right)$ is the solution of our problem. From the general properties of tensor isoalgebras the homomorphism $\hat{\varphi} : \hat{\mathcal{F}} \to \hat{A}$ can be extended to $\hat{\mathcal{L}}(\hat{\mathcal{F}})$, i.e., the diagram 2 is commutative, where

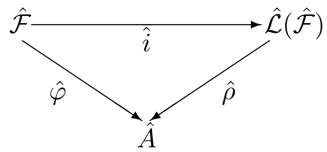

Figure 2: Diagram 2

$\hat{\rho}$ is a monomorphism of isoalgebras. Because $(\hat{\varphi}(\hat{u}))^2 = \hat{G}(\hat{u}) \cdot \hat{1}$, then $\hat{\rho}$ vanishes on isoideal $\hat{I}\left(\hat{G}\right)$ and in this case the necessary homomorphism $\hat{\tau}$ is defined. As a consequence of uniqueness of $\hat{\rho}$, the homomorphism $\hat{\tau}$ is unique.

Tensor d-isoalgebra $\hat{\mathcal{L}}(\hat{\mathcal{F}})$ can be considered as a $\mathcal{Z}/2$ graded isoalgebra. Really, let us introduce $\hat{\mathcal{L}}^{(0)}(\hat{\mathcal{F}}) = \sum_{i=1}^{\infty} T^{2i}\left(\hat{\mathcal{F}}\right)$ and $\hat{\mathcal{L}}^{(1)}(\hat{\mathcal{F}}) = \sum_{i=1}^{\infty} T^{2i+1}\left(\hat{\mathcal{F}}\right)$. Setting $\hat{I}^{(\alpha)}\left(\hat{G}\right) = \hat{I}\left(\hat{G}\right) \cap \hat{\mathcal{L}}^{(\alpha)}(\hat{\mathcal{F}})$. Define $\hat{C}^{(\alpha)}\left(\hat{G}\right)$ as $\hat{p}\left(\hat{\mathcal{L}}^{(\alpha)}(\hat{\mathcal{F}})\right)$, where $\hat{p} : \hat{\mathcal{L}}\left(\hat{\mathcal{F}}\right) \to \hat{C}\left(\hat{G}\right)$ is the canonical projection. Then $\hat{C}\left(\hat{G}\right) = \hat{C}^{(0)}\left(\hat{G}\right) \oplus \hat{C}^{(1)}\left(\hat{G}\right)$ and in consequence we obtain that the Clifford d-isoalgebra is $\mathcal{Z}/2$ graded.

It is obvious that Clifford d-isoalgebra functorially depends on pair $\left(\hat{\mathcal{F}}, \hat{G}\right)$. If $f : \hat{\mathcal{F}} \to \hat{\mathcal{F}}'$ is a homomorphism of $\hat{k}$–vector isospaces, such that $\hat{G}'(f(\hat{u})) = \hat{G}(\hat{u})$, where $\hat{G}$ and $\hat{G}'$ are, respectively, isometrics on $\hat{\mathcal{F}}$ and $\hat{\mathcal{F}}'$, then f induces an homomorphism of d-isoalgebras

$$\hat{C}(f) : \hat{C}\left(\hat{G}\right) \to \hat{C}\left(\hat{G}'\right)$$

with identities $\hat{C}(\hat{\varphi} \cdot f) = \hat{C}(\hat{\varphi})\,\hat{C}(f)$ and $\hat{C}(Id_{\mathcal{F}}) = Id_{C(\mathcal{F})}$.

If $\mathcal{A}^\alpha$ and $\mathcal{B}^\beta$ are $\mathcal{Z}/2$–graded d–isoalgebras, then their graded tensorial product $\mathcal{A}^\alpha \otimes \mathcal{B}^\beta$ is defined as a d–isoalgebra for $\hat{k}$–vector d–isospace $\mathcal{A}^\alpha \otimes \mathcal{B}^\beta$ with the graded product induced as $(a \otimes b)(c \otimes d) = (-1)^{\alpha\beta} ac \otimes bd$, where $b \in \mathcal{B}^\alpha$ and $c \in \mathcal{A}^\alpha \quad (\alpha, \beta = 0, 1)$.

Now we reformulate for d–isoalgebras the Chevalley theorem [**?**]:

Theorem 2 *The Clifford d–isoalgebra* $\hat{C}\left(h\hat{\mathcal{F}} \oplus v\hat{\mathcal{F}}, \hat{g} + \hat{h}\right)$ *is naturally isomorphic to*
$\hat{C}(\hat{g}) \otimes \hat{C}\left(\hat{h}\right)$.

Proof. The proof is similar to that for d–algebras with that difference that we are using isoobjects.

In consequence of theorem 2.2 we conclude that all operations with Clifford d-isoalgebras can be reduced to calculations for $\hat{C}\left(h\hat{\mathcal{F}},\hat{g}\right)$ and $\hat{C}\left(v\hat{\mathcal{F}},\hat{h}\right)$ which are usual Clifford algebras of dimension 2^n and, respectively, 2^m (a result similar to that for usual Clifford algebras [**?**, **?**]).

Considerations presented in the proof of theorem 2.2 show that map $\hat{j}:\hat{\mathcal{F}}\to\hat{C}\left(\hat{\mathcal{F}}\right)$ is monomorphic, so we can identify space $\hat{\mathcal{F}}$ with its image in $\hat{C}\left(\hat{\mathcal{F}},\hat{G}\right)$, denoted as $\hat{u}\to\overline{u}$, if $\hat{u}\in\hat{C}^{(0)}\left(\hat{\mathcal{F}},\hat{G}\right)$ $\left(\hat{u}\in\hat{C}^{(1)}\left(\hat{\mathcal{F}},\hat{G}\right)\right)$; then $\hat{u}=\overline{u}$ (respectively, $\overline{u}=-\hat{u}$).

Definition 1 The set of elements $\hat{u}\in\hat{C}\left(\hat{G}\right)^*$, where $\hat{C}\left(\hat{G}\right)^*$ denotes the multiplicative isogroup of invertible elements of $\hat{C}\left(\hat{\mathcal{F}},\hat{G}\right)$ satisfying $\overline{u}\hat{\mathcal{F}}\hat{u}^{-1}\in\hat{\mathcal{F}}$, is called the twisted Clifford d-isogroup, denoted $\widetilde{\Gamma}\left(\hat{\mathcal{F}}\right)$.

Let $\widetilde{\rho}:\ \widetilde{\Gamma}\left(\hat{\mathcal{F}}\right)\to GL\left(\hat{\mathcal{F}}\right)$ ($GL\left(\hat{\mathcal{F}}\right)$ is the isogroup of linear isotransforms) be the homorphism given by $\hat{u}\to\hat{\rho}\widetilde{u}$, where $\widetilde{\rho}_u\left(\hat{w}\right)=\overline{u}\ \hat{w}\ \hat{u}^{-1}$. We can verify that $\ker\widetilde{\rho}$ is a subisogroup in $\widetilde{\Gamma}\left(\hat{\mathcal{F}}\right)$.

Canonical map $\hat{j}:\hat{\mathcal{F}}\to\hat{C}\left(\hat{\mathcal{F}}\right)$ can be interpreted as the linear isomap $\hat{\mathcal{F}}\to\hat{C}\left(\hat{\mathcal{F}}\right)^0$ satisfying the universal property of Clifford d–isoalgebras. This leads to a homomorphism of isoalgebras, $\hat{C}\left(\hat{\mathcal{F}}\right)\to\hat{C}\left(\hat{\mathcal{F}}\right)^t$, considered by an anti-involution of $\hat{C}\left(\hat{\mathcal{F}}\right)$ and denoted as $\hat{u}\to\ {}^t\hat{u}$. More exactly, if $\hat{u}_1...\hat{u}_n\in\hat{\mathcal{F}}$, then $\hat{t}_u=\hat{u}_n...\hat{u}_1$ and ${}^t\overline{u}=\overline{{}^tu}=(-1)^n\,\hat{u}_n...\hat{u}_1$.

Definition 2 The isospinor norm of arbitrary $\hat{u}\in\hat{C}\left(\hat{\mathcal{F}}\right)$ is defined as
$S\left(\hat{u}\right)=\ {}^t\overline{u}\cdot\hat{u}\in\hat{C}\left(\hat{\mathcal{F}}\right).$

It is obvious that if $\hat{u},u',u''\in\widetilde{\Gamma}\left(\hat{\mathcal{F}}\right)$, then $S(\hat{u},u')=S\left(\hat{u}\right)S\left(u'\right)$ and
$S\left(\hat{u}\ u'u''\right)=S\left(\hat{u}\right)S\left(u'\right)S\left(u''\right)$. For $\hat{u},u'\in\mathcal{F}S\left(\hat{u}\right)=-G\left(\hat{u}\right)$ and $S\left(\hat{u},u'\right)=$
$S\left(\hat{u}\right)S\left(u'\right)=S\left(\hat{u}\ u'\right)$.

Let us introduce the orthogonal group $O\left(\hat{G}\right)\subset GL\left(\hat{G}\right)$ defined by metric $\hat{G}$ on $\hat{\mathcal{F}}$ and denote sets $SO\left(\hat{G}\right)=\{\hat{u}\in O\left(\hat{G}\right),\det|\hat{u}|=1\}$, $Pin\left(\hat{G}\right)=\{\hat{u}\in\widetilde{\Gamma}\left(\hat{\mathcal{F}}\right),S\left(\hat{u}\right)=1\}$ and $\hat{S}\ pin\left(\hat{G}\right)=Pin\left(\hat{G}\right)\cap C^0\left(\hat{\mathcal{F}}\right)$. For $\hat{\mathcal{F}}\cong\hat{\mathcal{R}}^{n+m}$, were $\hat{\mathcal{R}}$ is the isonumber field, we write $\hat{S}\ pin\left(n+m\right)$. By straightforward calculations we can verify the exactness of these sequences:

$$1 \to \mathcal{Z}/2 \to Pin\left(\hat{G}\right) \to O\left(\hat{G}\right) \to 1,$$

$$1 \to \mathcal{Z}/2 \to \hat{S}\ pin\left(\hat{G}\right) \to SO\left(\hat{G}\right) \to 0,$$

$$1 \to \mathcal{Z}/2 \to \hat{S}\ pin\,(n+m) \to SO\,(n+m) \to 1.$$

We conclude this subsection by emphasizing that the isospinor norm was defined with respect to a quadratic form induced by a metric in v-isobundle $\hat{\xi}_d$.

2.2 Clifford Isoundles and D-Isopinor Structures

There are two possibilities for generalizing our isospinor constructions defined for d-vector isospaces to the case of vector bundle isospaces enabled with the structure of N-connection. The firs is to use the extension to the category of vector isobundles. The second is to define the Clifford fibration associated with compatible linear d-connection and isometric $\hat{G}$ on a vector isobundle. Let us consider both variants.

2.2.1 Clifford d-module structure in v-isobundles

Because functor $\hat{\mathcal{F}} \to \hat{C}(\hat{\mathcal{F}})$ is smooth we can extend it to the category of vector isobundles of type $\hat{\xi}_d = \{\pi_d : H\ \hat{E} \oplus V\ \hat{E} \to \hat{E}\}$. Recall that by $\hat{\mathcal{F}}$ we denote the typical fiber of such isobundles. For $\hat{\xi}_d$ we obtain a bundle of isoalgebras, denoted as $\hat{C}\left(\hat{\xi}_d\right)$, such that $\hat{C}\left(\hat{\xi}_d\right)_u = \hat{C}\left(\hat{\mathcal{F}}_u\right)$. Multiplication in every fibre defines a continuous map $\hat{C}\left(\hat{\xi}_d\right) \times \hat{C}\left(\hat{\xi}_d\right) \to \hat{C}\left(\hat{\xi}_d\right)$. If $\hat{\xi}_d$ is a vector isobundle on isonumber field $\hat{k}$, the structure of the $\hat{C}\left(\hat{\xi}_d\right)$-isomodule, the d–isomodule, on $\hat{\xi}_d$ is given by the continuous map $\hat{C}\left(\hat{\xi}_d\right) \times_E \hat{\xi}_d \to \hat{\xi}_d$ with every fiber $\hat{\mathcal{F}}_u$ provided with the structure of the $\hat{C}\left(\hat{\mathcal{F}}_u\right)$ $-$module, correlated with its $\hat{k}$-module structure, Because $\hat{\mathcal{F}} \subset \hat{C}\left(\hat{\mathcal{F}}\right)$, we have a fiber to fiber map $\hat{\mathcal{F}} \times_E \hat{\xi}_d \to \hat{\xi}_d$, inducing on every fiber the map $\hat{\mathcal{F}}_u \times_E \hat{\xi}_{d(u)} \to \hat{\xi}_{d(u)}$ ($\hat{\mathcal{R}}$-linear on the first factor and $\hat{k}$-linear on the second one). Inversely, every such bilinear map defines on $\hat{\xi}_d$ the structure of the $\hat{C}\left(\hat{\xi}_d\right)$-module by virtue of universal properties of Clifford d-isoalgebras. Equivalently, the above-mentioned bilinear map defines a morphism of v-isobundles $m : \hat{\xi}_d \to HOM\left(\hat{\xi}_d, \hat{\xi}_d\right)$, where $HOM\left(\hat{\xi}_d, \hat{\xi}_d\right)$ denotes the bundles of homomorphisms when $\left(m\left(\hat{u}\right)\right)^2 = \hat{G}\left(u\right)$ on every point.

Vector isobundles $\hat{\xi}_d$ provided with $\hat{C}\left(\hat{\xi}_d\right)$-structures are objects of the category with morphisms being morphisms of v-isobundles, which induce on every point $\hat{u} \in \hat{\xi}$ morphisms of $\hat{C}\left(\hat{\mathcal{F}}_u\right)$ $-$modules. This is a Banach category contained in the category of finite-dimensional d-vector isospaces on filed $\hat{k}$.

Let us denote by $H^s\left(\hat{\xi}, GL_{n+m}\left(\hat{\mathcal{R}}\right)\right)$ the s-dimensional cohomology isogroup of the isoalgebraic sheaf of germs of continuous maps of v-isobundle $\hat{\xi}$ with group $GL_{n+m}\left(\hat{\mathcal{R}}\right)$ the group of automorphisms of $\hat{\mathcal{R}}^{n+m}$ (for the language of algebraic topology see, for example, Refs. [?] and [?]). We shall also use the isogroup $SL_{n+m}\left(\hat{\mathcal{R}}\right) = \{\hat{A} \subset GL_{n+m}\left(\hat{\mathcal{R}}\right), \det \hat{A} = \hat{1}\}$. Here we point out that cohomologies $H^s(\hat{M}, iGr)$ characterize the class of a principal isobundle $\pi : \hat{P} \rightarrow \hat{M}$ on $\hat{M}$ with structural group iGr. Taking into account that we deal with bundles distinguished by an N-connection we introduce into consideration cohomologies $H^s\left(\hat{\xi}, GL_{n+m}\left(\hat{\mathcal{R}}\right)\right)$ as distinguished classes (d-classes) of isobundles $\hat{\xi}$ provided with a global N-connection structure.

For a real vector isobundle $\hat{\xi}_d$ on compact base $\hat{\xi}$ we can define the orientation on $\hat{\xi}_d$ as an element $\alpha_d \in H^1\left(\xi, GL_{n+m}\left(\hat{\mathcal{R}}\right)\right)$ whose image on map

$$H^1\left(\hat{\xi}, SL_{n+m}\left(\hat{\mathcal{R}}\right)\right) \rightarrow H^1\left(\xi, GL_{n+m}\left(\hat{\mathcal{R}}\right)\right)$$

is the d-class of isobundle $\hat{\xi}$.

Definition 3 The isospinor structure on $\hat{\xi}_d$ is defined as an element $\hat{\beta}_d \in H^1\left(\hat{\xi}, \hat{S}\ pin\left(n+m\right)\right)$ whose image in the composition

$$H^1\left(\hat{\xi}, \hat{S}\ pin\left(n+m\right)\right) \rightarrow H^1\left(\hat{\xi}, SO\left(n+m\right)\right) \rightarrow H^1\left(\hat{\xi}, GL_{n+m}\left(\hat{\mathcal{R}}\right)\right)$$

is the d-class of $\hat{\xi}$.

The above definition of isospinor structures can be reformulated in terms of principal isobundles. Let $\hat{\xi}_d$ be a real vector isobundle of rank n+m on a compact isobase $\hat{\xi}$. If there is a principal isobundle $\hat{P}_d$ with structural isogroup $\hat{S}\ O\left(n+m\right)$ [or $\hat{S}\ pin\left(n+m\right)$], this isobundle $\hat{\xi}_d$ can be provided with orientation (or isospinor) structure. The isobundle $\hat{P}_d$ is associated with element $\alpha_d \in H^1\left(\hat{\xi}, \hat{S}\ O(n+m)\right)$ [or $\beta_d \in H^1\left(\hat{\xi}, \hat{S}\ pin\left(n+m\right)\right)$].

We remark that a real bundle is oriented if and only if its first Stiefel-Whitney d-class vanishes,

$$w_1\left(\hat{\xi}_d\right) \in H^1\left(\hat{\xi}, \mathcal{Z}/2\right) = 0,$$

where $H^1\left(\hat{\xi}, \mathcal{Z}/2\right)$ is the first group of Chech cohomology with coefficients in $\mathcal{Z}/2$, Considering the second Stiefel-Whitney class $w_2\left(\hat{\xi}_d\right) \in H^{21}\left(\hat{\xi}, \mathcal{Z}/2\right)$ it is well known that vector isobundle $\hat{\xi}_d$ admits the isospinor structure if and only if $w_2\left(\hat{\xi}_d\right) = 0$. Finally, in this subsection, we emphasize that taking into account that base isospace $\hat{\xi}$ is also a v-isobundle, $\hat{p} :\ \hat{E} \rightarrow \hat{M}$, we have to make explicit calculations in order to express cohomologies $H^s\left(\hat{\xi}, GL_{n+m}\right)$ and $H^s\left(\hat{\xi}, \hat{S}\ O\left(n+m\right)\right)$ through cohomologies

$H^s\left(\hat{M}, GL_n\right)$, $H^s\left(\hat{M}, \hat{S}\ O\left(m\right)\right)$, which depends on global topological structures of spaces $\hat{M}$ and $\hat{\xi}$. For general bundle and base spaces this requires a cumbersome cohomological calculus.

2.2.2 Clifford fibration

Another way of defining the spinor structure is to use Clifford fibrations. Consider the principal isobundle with the structural group Gr being a subgroup of orthogonal group $O\left(\hat{G}\right)$, where $\hat{G}$ is a quadratic nondegenerate form (see(19)) defined on the base (also being a bundle space) space $\hat{\xi}$. The fibration associated to principal fibration $P\left(\hat{\xi}, iGr\right)$ with a typical fiber having Clifford isoalgebra $\hat{C}\left(\hat{G}\right)$ is, by definition, the Clifford fibration $P\ \hat{C}\left(\hat{\xi}, iGr\right)$. We can always define a metric on the Clifford fibration if every fiber is isometric to $P\ \hat{C}\left(\hat{\xi}, \hat{G}\right)$ (this result was proved for arbitrary quadratic forms $\hat{G}$ on pseudo-Riemannian bases [**?**]). If, additionally, $iGr \subset \hat{S}\ O\left(\hat{G}\right)$ a global subsection can be defined on $P\ \hat{C}\left(\hat{G}\right)$.

Let $\mathcal{P}\left(\hat{\xi}, iGr\right)$ be the set of principal isobundles with differentiable base $\hat{\xi}$ and structural group iGr. If $ig : iGr \to iGr'$ is an homomorphism of Lie–Santilli isogroups and $\hat{P}\left(\hat{\xi}, iGr\right) \subset \mathcal{P}\left(\hat{\xi}, iGr\right)$ (for simplicity in this subsection we shall denote mentioned bundles and sets of bundles as $\hat{P}, \hat{P}'$ and respectively, $\mathcal{P}, \mathcal{P}'$), we can always construct a principal bundle with the property that there is as homomorphism $f : \hat{P}' \to \hat{P}$ of principal bundles which can be projected to the identity map of $\hat{\xi}$ and corresponds to isomorphism $g : iGr \to iGr'$. If the inverse statement also holds, the bundle $\hat{P}'$ is called as the extension of $\hat{P}$ associated to g and f is called the extension homomorphism denoted as $\widetilde{g}$.

Now we can define distinguished isospinor structures on bundle isospaces (compare with definition 2.3).

Definition 4 Let $\hat{P} \in \mathcal{P}\left(\hat{\xi}, O\left(\hat{G}\right)\right)$ be a principal isobundle. A distinguished isospinor structure of $\hat{P}$, equivalently a ds–isostructure of $\hat{\xi}$ is an extension $\widetilde{P}$ of $\hat{P}$ associated to homomorphism $h : Pin\ \hat{G} \to O\left(\hat{G}\right)$ where $O\left(\hat{G}\right)$ is the group of orthogonal rotations, generated by metric $\hat{G}$, in bundle $\hat{\xi}$.

So, if $\widetilde{P}$ is a isospinor structure of the isospace $\hat{\xi}$, then $\widetilde{P} \in \mathcal{P}\left(\hat{\xi}, Pin\ \hat{G}\right)$.

The definition of spinor structures on varieties was given in Ref.[**?**]. In Refs. [**?**] and [**?**] it is proved that a necessary and sufficient condition for a space time to be orientable is to admit a global field of orthonormalized frames. We mention that spinor structures can be also defined on varieties modeled on Banach spaces [**?**]. As we have shown in this subsection, similar constructions are possible for the cases when space time has the structure of a v-isobundle with an N-connection.

Definition 5 A special distinguished isospinor structure, ds–isostructure, of principal isobundle $\hat{P} = \hat{P}\left(\hat{\xi}, SO\left(\hat{G}\right)\right)$ is a principal bundle $\widetilde{P} = \widetilde{P}\left(\hat{\xi}, \hat{S}\ pin\ \hat{G}\right)$ for which a homomorphism of principal isobundles $\widetilde{p} : \widetilde{P} \to \hat{P}$, projected on the identity map of $\hat{\xi}$ and corresponding to representation

$$R : \hat{S}\ pin\ \hat{G} \to SO\left(\hat{G}\right),$$

is defined.

In the case when the base isospace variety is oriented, there is a natural bijection between tangent spinor structures with a common base. For special ds–isostructures we can define, as for any isospinor structure, the concepts of isospin tensors, isospinor connections, and isospinor covariant derivations (see Refs. [**?**, **?**, **?**]).

2.3 Spinor Techniques for D–Vector Isospaces

The problem of a rigorous definition of spinors on locally anisotropic spaces (d–spinors) was posed and solved [**?**, **?**, **?**] in the framework of the formalism of Clifford and spinor structures on v-bundles provided with compatible nonlinear and distinguished connections and metric. We introduced d-spinors as corresponding objects of the Clifford d-algebra $\mathcal{C}(\mathcal{F}, G)$, defined for a d-vector space $\mathcal{F}$ in a standard manner (see, for instance, [**?**]) and proved that operations with $\mathcal{C}(\mathcal{F}, G)$ can be reduced to calculations for $\mathcal{C}(h\mathcal{F}, g)$ and $\mathcal{C}(v\mathcal{F}, h)$, which are usual Clifford algebras of respective dimensions 2^n and 2^m (if it is necessary we can use quadratic forms g and h correspondingly induced on $h\mathcal{F}$ and $v\mathcal{F}$ by a metric $\mathbf{G}$. Considering the orthogonal subgroup $O(\mathbf{G}) \subset GL(\mathbf{G})$ defined by a metric $\mathbf{G}$ we defined the d-spinor norm and parametrized d-spinors by ordered pairs of elements of Clifford algebras $\mathcal{C}(h\mathcal{F}, g)$ and $\mathcal{C}(v\mathcal{F}, h)$.

In this subsection, as a rule, we shall omit proofs which in most cases are mechanical but rather tedious. We can apply the methods developed in [**?**, **?**, **?**, **?**] in a straightforward manner on h- and v-subbundles in order to verify the correctness of affirmations. In isotopic cases we must consider isonumber fields.

2.3.1 Clifford d-isoalgebra and d-isospinors

In order to relate the succeeding constructions with Clifford d-isoalgebras we consider an isoframe decomposition of the metric (19):

$$G_{\alpha\beta}(u) = l_{\alpha}^{\widehat{\alpha}}(u)\, l_{\beta}^{\widehat{\beta}}(u)\, G_{\widehat{\alpha}\widehat{\beta}}, \tag{34}$$

where the isoframe d-isovectors and constant isometric matrices are distinguished as

$$l_{\alpha}^{\widehat{\alpha}}(u) = \begin{pmatrix} l_{j}^{\widehat{j}}(u) & 0 \\ 0 & l_{a}^{\widehat{a}}(u) \end{pmatrix}, G_{\widehat{\alpha}\widehat{\beta}} \begin{pmatrix} g_{\widehat{i}\widehat{j}} & 0 \\ 0 & h_{\widehat{a}\widehat{b}} \end{pmatrix}, \tag{35}$$

$g_{\widehat{ij}}$ and $h_{\widehat{ab}}$ are diagonal matrices with $g_{\widehat{ii}} = h_{\widehat{aa}} = \pm\hat{1}$.

To generate Clifford d-isoalgebras we start with matrix equations

$$\sigma_{\widehat{\alpha}}\sigma_{\widehat{\beta}} + \sigma_{\widehat{\beta}}\sigma_{\widehat{\alpha}} = -\hat{G}_{\widehat{\alpha\beta}}\hat{I}, \tag{36}$$

where $\hat{I}$ is the isoidentity matrix, matrices $\sigma_{\widehat{\alpha}}$ (σ-isoobjects) act on a d-vector isospace $\hat{\mathcal{F}} = h\hat{\mathcal{F}} \oplus v\hat{\mathcal{F}}$ and theirs components are distinguished as

$$\sigma_{\widehat{\alpha}} = \left\{ (\sigma_{\widehat{\alpha}})_{\underline{\beta}}^{\cdot\underline{\gamma}} = \begin{pmatrix} (\sigma_{\widehat{i}})_{\underline{j}}^{\cdot\underline{k}} & 0 \\ 0 & (\sigma_{\widehat{a}})_{\underline{b}}^{\cdot\underline{c}} \end{pmatrix} \right\}, \tag{37}$$

indices $\underline{\beta}$,$\underline{\gamma}$,... refer to isospin spaces of type $\hat{\mathcal{S}} = \hat{S}_{(h)} \oplus \hat{S}_{(v)}$ and underlined Latin indices $\underline{j}$,$\underline{k}$,... and $\underline{b}, \underline{c}$,... refer respectively to a h-isospin space $\hat{\mathcal{S}}_{(h)}$ and a v-isospin space $\hat{\mathcal{S}}_{(v)}$, which are correspondingly associated to a h- and v-decomposition of a v-isobundle $\hat{\mathcal{E}}_{(d)}$. The irreducible algebra of matrices $\sigma_{\widehat{\alpha}}$ of minimal dimension $N \times N$, where $N = N_{(n)} + N_{(m)}$, $\dim \hat{\mathcal{S}}_{(h)} = N_{(n)}$ and $\dim \hat{\mathcal{S}}_{(v)} = N_{(m)}$, has these dimensions

$$N_{(n)} = \left\{ \begin{array}{cc} 2^{(n-1)/2}, & n = 2k+1 \\ 2^{n/2}, & n = 2k; \end{array} \right. \quad \text{and} \quad N_{(m)} = \left\{ \begin{array}{cc} 2^{(m-1)/2}, & m = 2k+1 \\ 2^{m/2}, & m = 2k, \end{array} \right.$$

where $k = 1, 2, \ldots$.

The Clifford d–isoalgebra is generated by sums on $n+1$ elements of form

$$A_1 I + B^{\widehat{i}}\sigma_{\widehat{i}} + C^{\widehat{ij}}\sigma_{\widehat{ij}} + D^{\widehat{ijk}}\sigma_{\widehat{ijk}} + \ldots$$

and sums of $m+1$ elements of form

$$A_2 I + B^{\widehat{a}}\sigma_{\widehat{a}} + C^{\widehat{ab}}\sigma_{\widehat{ab}} + D^{\widehat{abc}}\sigma_{\widehat{abc}} + \ldots$$

with antisymmetric coefficients $C^{\widehat{ij}} = C^{[\widehat{ij}]}, C^{\widehat{ab}} = C^{[\widehat{ab}]}, D^{\widehat{ijk}} = D^{[\widehat{ijk}]}, D^{\widehat{abc}} = D^{[\widehat{abc}]}, \ldots$ and matrices $\sigma_{\widehat{ij}} = \sigma_{[\widehat{i}}\sigma_{\widehat{j}]}, \sigma_{\widehat{ab}} = \sigma_{[\widehat{a}}\sigma_{\widehat{b}]}, \sigma_{\widehat{ijk}} = \sigma_{[\widehat{i}}\sigma_{\widehat{j}}\sigma_{\widehat{k}]}, \ldots$. Really, we have 2^{n+1} coefficients $\left(A_1, C^{\widehat{ij}}, D^{\widehat{ijk}}, \ldots\right)$ and 2^{m+1} coefficients $\left(A_2, C^{\widehat{ab}}, D^{\widehat{abc}}, \ldots\right)$ of the Clifford isoalgebra on $\hat{\mathcal{F}}$.

For simplicity, in this subsection, we shall present the necessary geometric constructions only for h–isospin spaces $\hat{\mathcal{S}}_{(h)}$ of dimension $N_{(n)}$. Considerations for a v-isospin space $\hat{\mathcal{S}}_{(v)}$ are similar but with proper characteristics for a dimension $N_{(m)}$.

In order to define the isoscalar (isospinor) product on $\hat{\mathcal{S}}_{(h)}$ we introduce into consideration this finite sum (because of a finite number of elements $\sigma_{[\widehat{ij}\ldots\widehat{k}]}$):

$$^{(\pm)}E_{\underline{km}}^{\underline{ij}} = \delta_{\underline{k}}^{\underline{i}}\delta_{\underline{m}}^{\underline{j}} + \frac{2}{1!}(\sigma_{\widehat{i}})_{\underline{k}}^{\cdot\underline{i}}(\sigma^{\widehat{i}})_{\underline{m}}^{\cdot\underline{j}} + \frac{2^2}{2!}(\sigma_{\widehat{ij}})_{\underline{k}}^{\cdot\underline{i}}(\sigma^{\widehat{ij}})_{\underline{m}}^{\cdot\underline{j}} + \frac{2^3}{3!}(\sigma_{\widehat{ijk}})_{\underline{k}}^{\cdot\underline{i}}(\sigma^{\widehat{ijk}})_{\underline{m}}^{\cdot\underline{j}} + \ldots \tag{38}$$

which can be factorized as

$$^{(\pm)}E_{\underline{km}}^{\underline{ij}} = N_{(n)} \; ^{(\pm)}\epsilon_{\underline{km}} \; ^{(\pm)}\epsilon^{\underline{ij}} \text{ for } n = 2k \tag{39}$$

and

$$
\begin{aligned}
{}^{(+)}E^{\underline{ij}}_{\underline{km}} &= 2N_{(n)}\epsilon_{\underline{km}}\epsilon^{\underline{ij}},\ {}^{(-)}E^{\underline{ij}}_{\underline{km}} = 0 \text{ for } n = 3(mod4), \\
{}^{(+)}E^{\underline{ij}}_{\underline{km}} &= 0,\ {}^{(-)}E^{\underline{ij}}_{\underline{km}} = 2N_{(n)}\epsilon_{\underline{km}}\epsilon^{\underline{ij}} \text{ for } n = 1(mod4).
\end{aligned} \tag{40}
$$

Antisymmetry of $\sigma_{\widetilde{ijk...}}$ and the construction of the isoobjects (**??**)–(**??**) define the properties of ϵ–objects ${}^{(\pm)}\epsilon_{\underline{km}}$ and $\epsilon_{\underline{km}}$ which have an eight-fold periodicity on n (see details in [**?**] and, with respect to locally isotropic spaces, [**?**]).

For even values of n it is possible the decomposition of every h-isospin space $\hat{\mathcal{S}}_{(v)}$ into irreducible h-isospin spaces $\hat{\mathbf{S}}_{(v)}$ and $\hat{\mathbf{S}}'_{(v)}$ (one considers splitting of h-indices, for instance, $\underline{l}= L \oplus L', \underline{m} = M \oplus M', ...$; for v-indices we shall write $\underline{a} = A \oplus A', \underline{b} = B \oplus B', ...$) and defines new $\hat{\epsilon}$–objects

$$
\hat{\epsilon}^{\underline{lm}} = \frac{1}{2}\left({}^{(+)}\hat{\epsilon}^{\underline{lm}} +{}^{(-)} \hat{\epsilon}^{\underline{lm}}\right) \text{ and } \tilde{\epsilon}^{\underline{lm}} = \frac{1}{2}\left({}^{(+)}\hat{\epsilon}^{\underline{lm}} -{}^{(-)} \hat{\epsilon}^{\underline{lm}}\right) \tag{41}
$$

We shall omit similar formulas for ϵ–objects with lower indices.

We can verify, by using expressions (**??**) and straightforward calculations, these parametrizations on symmetry properties of ϵ-objects (**??**) [for symplicity we omit in these formulas the "hat" symbols]

$$
\begin{aligned}
\epsilon^{\underline{lm}} &= \begin{pmatrix} \epsilon^{LM} = \epsilon^{ML} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \tilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\epsilon}^{LM} = \tilde{\epsilon}^{ML} \end{pmatrix} \quad \text{for} \quad n = 0(mod8); \\
\epsilon^{\underline{lm}} &= -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \epsilon^{\underline{ml}}, {}^{(+)}\epsilon^{\underline{lm}} = 0, \text{ and } \tilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \tilde{\epsilon}^{\underline{ml}} \quad \text{for} \quad n = 1(mod8); \\
\epsilon^{\underline{lm}} &= \begin{pmatrix} 0 & 0 \\ \epsilon^{L'M} & 0 \end{pmatrix} \text{ and } \tilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & \tilde{\epsilon}^{LM'} = -\epsilon^{M'L} \\ 0 & 0 \end{pmatrix} \quad \text{for} \quad n = 2(mod8); \\
\epsilon^{\underline{lm}} &= -\frac{1}{2}{}^{(+)}\epsilon^{\underline{lm}}, {}^{(-)}\epsilon^{\underline{lm}} = 0, \text{ and } \tilde{\epsilon}^{\underline{lm}} = \frac{1}{2}{}^{(+)}\epsilon^{\underline{lm}} = -\tilde{\epsilon}^{\underline{ml}} \quad \text{for} \quad n = 3(mod8); \\
\epsilon^{\underline{lm}} &= \begin{pmatrix} \epsilon^{LM} = -\epsilon^{ML} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \tilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\epsilon}^{LM} = -\tilde{\epsilon}^{ML} \end{pmatrix} \quad \text{for} \quad n = 4(mod8); \\
\epsilon^{\underline{lm}} &= -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}}, {}^{(+)}\epsilon^{\underline{lm}} = 0, \text{ and } \tilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = -\tilde{\epsilon}^{\underline{ml}} \quad \text{for} \quad n = 5(mod8); \\
\epsilon^{\underline{lm}} &= \begin{pmatrix} 0 & 0 \\ \epsilon^{L'M} & 0 \end{pmatrix} \text{ and } \tilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & \tilde{\epsilon}^{LM'} = \epsilon^{M'L} \\ 0 & 0 \end{pmatrix} \quad \text{for} \quad n = 6(mod8); \\
\epsilon^{\underline{lm}} &= \frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \epsilon^{\underline{ml}}, {}^{(+)}\epsilon^{\underline{lm}} = 0, \text{ and } \tilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \tilde{\epsilon}^{\underline{ml}} \quad \text{for} \quad n = 7(mod8).
\end{aligned}
$$

2.3.2 Mutual transforms of d–isotensors and d-isospinors

The spinor algebra for spaces of higher dimensions can not be considered as a real alternative to the tensor algebra as for locally isotropic spaces of dimensions $n = 3, 4$ [**?**, **?**, **?**]. The same holds true for locally isotropic isospaces and we emphasize that it is not quite convenient to perform an isospinor calculus for dimensions $n, m >> 4$.

Nevertheless, the concept of isospinors is important for every type of spaces, we can deeply understand the fundamental properties of geometical objects on locally isotropic isospaces, and we shall consider in this subsection some questions concerning transforms of d–isotensor objects into d–isospinor ones.

Transformation of d–isotensors into d–isospinors

In order to pass from d–isotensors to d–isospinors we must use σ-objects (**??**) written in reduced or irreduced form (in dependence of fixed values of dimensions n and m):

$$(\sigma_{\widehat{\alpha}})_{\underline{\beta}}^{\cdot\gamma},\ (\sigma^{\widehat{\alpha}})^{\underline{\beta\gamma}},\ (\sigma^{\widehat{\alpha}})_{\underline{\beta\gamma}},...,(\sigma_{\widehat{a}})^{\underline{bc}},...,(\sigma_{\widehat{i}})_{\underline{jk}},...,(\sigma_{\widehat{a}})^{AA'},...,(\sigma^{\widehat{i}})_{II'},.... \tag{42}$$

It is obvious that contracting with corresponding σ-objects (**??**) we can introduce instead of d-isotensors indices the d-isospinor ones, for instance,

$$\omega^{\underline{\beta\gamma}} = (\sigma^{\widehat{\alpha}})^{\underline{\beta\gamma}}\omega_{\widehat{\alpha}},\quad \omega_{AB'} = (\sigma^{\widehat{a}})_{AB'}\omega_{\widehat{a}},\quad ...,\zeta_{\cdot\underline{j}}^{\underline{i}} = (\sigma^{\widehat{k}})_{\cdot\underline{j}}^{\underline{i}}\zeta_{\widehat{k}},....$$

For d-isotensors containing groups of antisymmetric indices there is a more simple procedure of theirs transforming into d-isospinors because the objects

$$(\sigma_{\widehat{\alpha}\widehat{\beta}...\widehat{\gamma}})^{\underline{\delta\nu}},\quad (\sigma^{\widehat{a}\widehat{b}...\widehat{c}})^{\underline{de}},\quad ...,(\sigma^{\widehat{i}\widehat{j}...\widehat{k}})_{II'},\quad ... \tag{43}$$

can be used for sets of such indices into pairs of d-isospinor indices. Let us enumerate some properties of σ-objects of type (**??**) (for simplicity we consider only h-components having q indices $\widehat{i},\widehat{j},\widehat{k},...$ taking values from 1 to n; the properties of v-components can be written in a similar manner with respect to indices $\widehat{a},\widehat{b},\widehat{c}...$ taking values from 1 to m):

$$(\sigma_{\widehat{i}...\widehat{j}})^{\underline{kl}} \text{ is } \left\{ \begin{array}{c} \text{symmetric on } \underline{k},\underline{l} \text{ for } n-2q\equiv 1,7\ (mod\ 8); \\ \text{antisymmetric on } \underline{k},\underline{l} \text{ for } n-2q\equiv 3,5\ (mod\ 8) \end{array} \right\} \tag{44}$$

for odd values of n, and an object $(\sigma_{\widehat{i}...\widehat{j}})^{IJ}\ \left((\sigma_{\widehat{i}...\widehat{j}})^{I'J'}\right)$

$$\text{is } \left\{ \begin{array}{c} \text{symmetric on } I,J\ (I',J') \text{ for } n-2q\equiv 0\ (mod\ 8); \\ \text{antisymmetric on } I,J\ (I',J') \text{ for } n-2q\equiv 4\ (mod\ 8) \end{array} \right\} \tag{45}$$

or

$$(\sigma_{\widehat{i}...\widehat{j}})^{IJ'} = \pm(\sigma_{\widehat{i}...\widehat{j}})^{J'I}\{ \begin{array}{c} n+2q\equiv 6(mod 8); \\ n+2q\equiv 2(mod 8), \end{array} \tag{46}$$

with vanishing of the rest of reduced components of the d-isotensor $(\sigma_{\widehat{i}...\widehat{j}})^{\underline{kl}}$ with prime / unprime sets of indices.

Transformation of d-isospinors into d-tensors; fundamental d-isospinors

We can transform every d-isospinor $\hat{\xi}^{\underline{\alpha}} = \left(\hat{\xi}^{\underline{i}},\hat{\xi}^{\underline{a}}\right)$ into a corresponding d–isotensor. For simplicity, we consider this construction only for a h–component $\hat{\xi}^{\underline{i}}$ on a h–space being of dimension n. The values

$$\hat{\xi}^{\underline{\alpha}}\hat{\xi}^{\underline{\beta}}(\sigma^{\widehat{i}...\widehat{j}})_{\underline{\alpha\beta}}\quad (n \text{ is odd}) \tag{47}$$

or

$$\hat{\xi}^I \hat{\xi}^J (\sigma^{\widehat{i}...\widehat{j}})_{IJ} \ \left(\text{or } \hat{\xi}^{I'} \hat{\xi}^{J'} (\sigma^{\widehat{i}...\widehat{j}})_{I'J'} \right) \ (n \text{ is even}) \tag{48}$$

with a different number of indices $\widehat{i}...\widehat{j}$, taken together, defines the h-spinor $\xi^{\underline{i}}$ to an accuracy to the sign. We emphasize that it is necessary to choose only those h-components of d-tensors (**??**) (or (**??**)) which are symmetric on pairs of indices $\underline{\alpha\beta}$ (or IJ (or $I'J'$)) and the number q of indices $\widehat{i}...\widehat{j}$ satisfies the condition (as a respective consequence of the properties (**??**) and/or (**??**), (**??**))

$$n - 2q \equiv 0, 1, 7 \ (mod \ 8). \tag{49}$$

Of special interest is the case when

$$q = \frac{1}{2}(n \pm 1) \ \ (n \text{ is odd}) \tag{50}$$

or

$$q = \frac{1}{2} n \ \ (n \text{ is even}). \tag{51}$$

If all expressions (**??**) and/or (**??**) are zero for all values of q with the exception of one or two ones defined by the condition (**??**) (or (**??**)), the value $\hat{\xi}^{\widehat{i}}$ (or $\hat{\xi}^I$ ($\hat{\xi}^{I'}$)) is called a fundamental h-isospinor. Defining in a similar manner the fundamental v-spinors we can introduce fundamental d-isospinors as pairs of fundamental h- and v-isospinors. Here we remark that a h(v)-isospinor $\hat{\xi}^{\widehat{i}}$ ($\hat{\xi}^{\widehat{a}}$) (we can also consider reduced components) is always a fundamental one for $n(m) < 7$, which is a consequence of (**??**)).

Finally, in this subsection, we note that the isogeometry of fundamental h- and v-isospinors is similar to that of usual fundamental spinors (see Appendix to the monograph [**?**]). We omit such details in this work, but emphasize that constructions with fundamental d-isospinors, for a locally isotropic isospace, must be adapted to the corresponding global splitting by N-connection of the isospace.

3 Isospinor Differential Isogeometry

3.1 D-isospinor Differential Isogeometry

The goal of the subsection is to formulate the differential isogeometry of d–isospinors for locally anisotropic spaces.

We shall use denotations of type

$$\hat{v}^\alpha = (\hat{v}^i, \hat{v}^a) \in \mathcal{A}^\alpha = (\mathcal{A}^i, \mathcal{A}^a) \ \text{ and } \hat{\zeta}^{\underline{\alpha}} = (\hat{\zeta}^{\underline{i}}, \hat{\zeta}^{\underline{a}}) \in \mathcal{A}^{\underline{\alpha}} = (\mathcal{A}^{\underline{i}}, \mathcal{A}^{\underline{a}})$$

for, respectively, elements of modules of d-isovector and irreduced d-isospinor fields (for usual locally anisotropic spaces see details in [**?**]). D-isotensors and d-isospinor isotensors (irreduced or reduced) will be interpreted as elements of corresponding σ -modules, for instance,

$$\hat{q}^\alpha{}_{\beta...} \in \mathcal{A}^\alpha{}_\beta, \ \hat{\psi}^{\underline{\alpha}}{}_{\underline{\beta}}{}^{\underline{\gamma}}{}_{...} \in \mathcal{A}^{\underline{\alpha}}{}_{\underline{\beta}}{}^{\underline{\gamma}}{}_{...}, \hat{\xi}^{II'}{}_{JK'N'} \in \mathcal{A}^{II'}{}_{JK'N'}, ...$$

We can establish a correspondence between the locally adapted metric $\hat{g}_{\alpha\beta}$ (19) and d-isospinor metric $\hat{\epsilon}_{\underline{\alpha\beta}}$ ($\hat{\epsilon}$-objects for both h- and v-subspaces of $\hat{\mathcal{E}}$) of a locally anisotropic space $\hat{\mathcal{E}}$ by using the relation

$$\hat{g}_{\alpha\beta} = -\frac{1}{N(n)+N(m)}((\hat{\sigma}_{(\alpha}(u))^{\underline{\alpha}_1\underline{\beta}_1}(\hat{\sigma}_{\beta)}(u))^{\underline{\beta}_2\underline{\alpha}_2})\hat{\epsilon}_{\underline{\alpha}_1\underline{\alpha}_2}\hat{\epsilon}_{\underline{\beta}_1\underline{\beta}_2}, \tag{52}$$

where

$$(\hat{\sigma}_{\alpha}(u))^{\underline{\nu\gamma}} = l_{\alpha}^{\widehat{\alpha}}(u)(\sigma_{\widehat{\alpha}})^{\underline{\nu\gamma}}, \tag{53}$$

which is a consequence of formulas (**??**)-(**??**). In brief we can write (**??**) as

$$\hat{g}_{\alpha\beta} = \hat{\epsilon}_{\underline{\alpha}_1\underline{\alpha}_2}\hat{\epsilon}_{\underline{\beta}_1\underline{\beta}_2} \tag{54}$$

if the σ-objects are considered as a fixed structure, whereas ϵ-objects are treated as caring the metric "dynamics " , on locally anisotropic isospace. This variant is used, for instance, in the so-called 2-spinor geometry [**?**, **?**] and should be preferred if we have to make explicit the algebraic symmetry properties of d-spinor objects. An alternative way is to consider as fixed the algebraic structure of ϵ-objects and to use variable components of σ-objects of type (**??**) for developing a variational d-isospinor approach to gravitational and matter field interactions on locally anisotropic isospaces (the spinor Ashtekar variables [**?**] are introduced in this manner).

We note that a d-isospinor metric

$$\hat{\epsilon}_{\underline{\nu\tau}} = \begin{pmatrix} \hat{\epsilon}_{\underline{ij}} & 0 \\ 0 & \hat{\epsilon}_{\underline{ab}} \end{pmatrix}$$

on the d–isospinor space $\hat{\mathcal{S}}= (\hat{\mathcal{S}}_{(h)}, \hat{\mathcal{S}}_{(v)})$ can have symmetric or antisymmetric h(v)–components $\hat{\epsilon}_{\underline{ij}}$ ($\hat{\epsilon}_{\underline{ab}}$). For simplicity, in this subsection (in order to avoid cumbersome calculations connected with eight-fold periodicity on dimensions n and m of a locally anisotropic space $\hat{\mathcal{E}}$ we shall develop a general d–isospinor formalism only by using irreduced isospinor spaces $\hat{\mathcal{S}}_{(h)}$ and $\hat{\mathcal{S}}_{(v)}$.

3.1.1 D-covariant derivation on locally anisotropic spaces

Let $\hat{\mathcal{E}}$ be a locally anisotropic space. We define the action on a d-isospinor of a d-covariant operator

$$\hat{\nabla}_{\alpha} = \left(\hat{\nabla}_i, \hat{\nabla}_a\right) = (\hat{\sigma}_{\alpha})^{\underline{\alpha}_1\underline{\alpha}_2}\hat{\nabla}_{\underline{\alpha}_1\underline{\alpha}_2} = \left((\hat{\sigma}_i)^{\underline{i}_1\underline{i}_2}\hat{\nabla}_{\underline{i}_1\underline{i}_2},\ (\hat{\sigma}_a)^{\underline{a}_1\underline{a}_2}\hat{\nabla}_{\underline{a}_1\underline{a}_2}\right)$$

(in brief, we shall write

$$\hat{\nabla}_{\alpha} = \hat{\nabla}_{\underline{\alpha}_1\underline{\alpha}_2} = \left(\hat{\nabla}_{\underline{i}_1\underline{i}_2},\ \hat{\nabla}_{\underline{a}_1\underline{a}_2}\right))$$

as a map

$$\hat{\nabla}_{\underline{\alpha}_1\underline{\alpha}_2}\ :\ \hat{\sigma}^{\underline{\beta}} \rightarrow \hat{\sigma}_{\alpha}^{\underline{\beta}} = \hat{\sigma}_{\underline{\alpha}_1\underline{\alpha}_2}^{\underline{\beta}}$$

satisfying conditions

$$\hat{\nabla}_\alpha(\hat{\xi}^{\underline{\beta}} + \hat{\eta}^{\underline{\beta}}) = \hat{\nabla}_\alpha \xi^{\underline{\beta}} + \hat{\nabla}_\alpha \eta^{\underline{\beta}},$$

and

$$\hat{\nabla}_\alpha(\hat{f}\ \hat{\xi}^{\underline{\beta}}) = \hat{f}\ \hat{\nabla}_\alpha \hat{\xi}^{\underline{\beta}} + \hat{\xi}^{\underline{\beta}} \hat{\nabla}_\alpha \hat{f}$$

for every $\hat{\xi}^{\underline{\beta}}, \hat{\eta}^{\underline{\beta}} \in \not{\sigma}^{\underline{\beta}}$ and $\hat{f}$ being an isoscalar field on $\hat{\mathcal{E}}$. It is also required that one holds the Leibnitz rule

$$(\hat{\nabla}_\alpha \hat{\zeta}_{\underline{\beta}})\hat{\eta}^{\underline{\beta}} = \hat{\nabla}_\alpha(\hat{\zeta}_{\underline{\beta}}\hat{\eta}^{\underline{\beta}}) - \hat{\zeta}_{\underline{\beta}}\hat{\nabla}_\alpha \hat{\eta}^{\underline{\beta}}$$

and that $\hat{\nabla}_\alpha$ is a real operator, i.e. it commuters with the operation of complex conjugation:

$$\overline{\nabla_\alpha \psi_{\underline{\alpha}\underline{\beta}\underline{\gamma}...}} = \hat{\nabla}_\alpha(\overline{\psi}_{\underline{\alpha}\underline{\beta}\underline{\gamma}...}).$$

Let now analyze the question on uniqueness of action on d–isospinors of an operator $\hat{\nabla}_\alpha$ satisfying necessary conditions . Denoting by $\hat{\nabla}^{(1)}_\alpha$ and $\hat{\nabla}_\alpha$ two such d-covariant operators we consider the map

$$(\hat{\nabla}^{(1)}_\alpha - \hat{\nabla}_\alpha) : \not{\sigma}^{\underline{\beta}} \to \not{\sigma}^{\underline{\beta}}_{\underline{\alpha}_\infty \underline{\alpha}_\in}. \tag{55}$$

Because the action on an isoscalar $\hat{f}$ of both operators $\hat{\nabla}^{(1)}_\alpha$ and $\hat{\nabla}_\alpha$ must be identical, i.e.

$$\hat{\nabla}^{(1)}_\alpha f = \hat{\nabla}_\alpha f, \tag{56}$$

the action (**??**) on $\hat{f} = \hat{\omega}_{\underline{\beta}} \hat{\xi}^{\underline{\beta}}$ must be written as

$$(\hat{\nabla}^{(1)}_\alpha - \hat{\nabla}_\alpha)(\hat{\omega}_{\underline{\beta}}\hat{\xi}^{\underline{\beta}}) = 0.$$

In consequence we conclude that there is an element $\hat{\Theta}_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\beta}}{}^{\underline{\gamma}} \in \not{\sigma}_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\beta}}{}^{\underline{\gamma}}$ for which

$$\hat{\nabla}^{(1)}_{\underline{\alpha}_1 \underline{\alpha}_2} \hat{\xi}^{\underline{\gamma}} = \hat{\nabla}_{\underline{\alpha}_1 \underline{\alpha}_2} \hat{\xi}^{\underline{\gamma}} + \hat{\Theta}_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\beta}}{}^{\underline{\gamma}} \xi^{\underline{\beta}} \tag{57}$$

and

$$\hat{\nabla}^{(1)}_{\underline{\alpha}_1 \underline{\alpha}_2} \hat{\omega}_{\underline{\beta}} = \hat{\nabla}_{\underline{\alpha}_1 \underline{\alpha}_2} \omega_{\underline{\beta}} - \hat{\Theta}_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\beta}}{}^{\underline{\gamma}} \omega_{\underline{\gamma}} \ .$$

The action of the operator (**??**) on a d-isovector $\hat{v}^\beta = \hat{v}^{\underline{\beta}_1 \underline{\beta}_2}$ can be written by using formula (**??**) for both indices $\underline{\beta}_1$ and $\underline{\beta}_2$:

$$\begin{aligned}
(\hat{\nabla}^{(1)}_\alpha - \hat{\nabla}_\alpha)\hat{v}^{\underline{\beta}_1 \underline{\beta}_2} &= \hat{\Theta}_{\alpha\underline{\gamma}}{}^{\underline{\beta}_1} \hat{v}^{\underline{\gamma}\underline{\beta}_2} + \hat{\Theta}_{\alpha\underline{\gamma}}{}^{\underline{\beta}_2} \hat{v}^{\underline{\beta}_1\underline{\gamma}} \\
&= (\hat{\Theta}_{\alpha\underline{\gamma}_1}{}^{\underline{\beta}_1} \hat{\delta}_{\underline{\gamma}_2}{}^{\underline{\beta}_2} + \hat{\Theta}_{\alpha\underline{\gamma}_1}{}^{\underline{\beta}_2} \hat{\delta}_{\underline{\gamma}_2}{}^{\underline{\beta}_1})\hat{v}^{\underline{\gamma}_1 \underline{\gamma}_2} = \hat{Q}^\beta{}_{\alpha\gamma}\hat{v}^\gamma,
\end{aligned}$$

where

$$\hat{Q}^\beta{}_{\alpha\gamma} = \hat{Q}^{\underline{\beta}_1 \underline{\beta}_2}{}_{\underline{\alpha}_1 \underline{\alpha}_2\ \underline{\gamma}_1 \underline{\gamma}_2} = \hat{\Theta}_{\alpha\underline{\gamma}_1}{}^{\underline{\beta}_1} \delta_{\underline{\gamma}_2}{}^{\underline{\beta}_2} + \hat{\Theta}_{\alpha\underline{\gamma}_1}{}^{\underline{\beta}_2} \delta_{\underline{\gamma}_2}{}^{\underline{\beta}_1}. \tag{58}$$

The d-isocommutator $\hat{\nabla}_{[\alpha}\hat{\nabla}_{\beta]}$ defines the d-torsion .

The action of operator $\nabla^{(1)}_\alpha$ on d-spinor tensors of type $\chi_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\alpha}_3 ...}{}^{\underline{\beta}_1 \underline{\beta}_2 ...}$ must be constructed by using formula (2.83) for every upper index $\underline{\beta}_1 \underline{\beta}_2 ...$ and formula (2.84) for every lower index $\underline{\alpha}_1 \underline{\alpha}_2 \underline{\alpha}_3 ...$.

3.1.2 Infeld - van der Waerden isocoefficients and d-isoconnections

Let

$$\delta_{\underline{\boldsymbol{\alpha}}}{}^{\underline{\alpha}} = \left(\delta_{\underline{\mathbf{1}}}{}^{\underline{i}}, \delta_{\underline{\mathbf{2}}}{}^{\underline{i}}, ..., \delta_{\underline{\mathbf{N(n)}}}{}^{\underline{i}}, \delta_{\underline{\mathbf{1}}}{}^{\underline{a}}, \delta_{\underline{\mathbf{2}}}{}^{\underline{a}}, ..., \delta_{\underline{\mathbf{N(m)}}}{}^{\underline{i}}\right)$$

be a d-isospinor basis (for simplicity, we omit hats. The dual to it basis is denoted

$$\delta_{\underline{\alpha}}{}^{\underline{\boldsymbol{\alpha}}} = \left(\delta_{\underline{i}}{}^{\underline{\mathbf{1}}}, \delta_{\underline{i}}{}^{\underline{\mathbf{2}}}, ..., \delta_{\underline{i}}{}^{\underline{\mathbf{N(n)}}}, \delta_{\underline{i}}{}^{\underline{\mathbf{1}}}, \delta_{\underline{i}}{}^{\underline{\mathbf{2}}}, ..., \delta_{\underline{i}}{}^{\underline{\mathbf{N(m)}}}\right).$$

A d-isospinor $\hat{\kappa}^{\underline{\alpha}} \in \hat{\kappa}^{\underline{\alpha}}$ has components $\hat{\kappa}^{\underline{\boldsymbol{\alpha}}} = \hat{\kappa}^{\underline{\alpha}} \delta_{\underline{\alpha}}{}^{\underline{\boldsymbol{\alpha}}}$. Taking into account that

$$\delta_{\underline{\boldsymbol{\alpha}}}{}^{\underline{\alpha}} \delta_{\underline{\boldsymbol{\beta}}}{}^{\underline{\beta}} \hat{\nabla}_{\underline{\alpha\beta}} = \hat{\nabla}_{\underline{\boldsymbol{\alpha\beta}}},$$

we write out the components $\nabla_{\underline{\alpha\beta}}\ \kappa^{\underline{\gamma}}$ as

$$\begin{aligned} \delta_{\underline{\boldsymbol{\alpha}}}{}^{\underline{\alpha}}\ \delta_{\underline{\boldsymbol{\beta}}}{}^{\underline{\beta}}\ \delta_{\underline{\gamma}}{}^{\underline{\boldsymbol{\gamma}}}\ \hat{\nabla}_{\underline{\alpha\beta}} \kappa^{\underline{\gamma}} &= \delta_{\underline{\boldsymbol{\epsilon}}}{}^{\underline{\tau}}\ \delta_{\underline{\tau}}{}^{\underline{\boldsymbol{\gamma}}}\ \hat{\nabla}_{\underline{\boldsymbol{\alpha\beta}}} \kappa^{\underline{\boldsymbol{\epsilon}}} + \kappa^{\underline{\boldsymbol{\epsilon}}}\ \delta_{\underline{\epsilon}}{}^{\underline{\boldsymbol{\gamma}}}\ \hat{\nabla}_{\underline{\boldsymbol{\alpha\beta}}} \delta_{\underline{\boldsymbol{\epsilon}}}{}^{\underline{\epsilon}} \\ &= \hat{\nabla}_{\underline{\boldsymbol{\alpha\beta}}} \kappa^{\underline{\boldsymbol{\gamma}}} + \kappa^{\underline{\boldsymbol{\epsilon}}} \hat{\gamma}^{\underline{\boldsymbol{\gamma}}}{}_{\underline{\boldsymbol{\alpha\beta\epsilon}}}, \end{aligned} \tag{59}$$

where the coordinate components of the d-isospinor connection $\gamma^{\underline{\boldsymbol{\gamma}}}{}_{\underline{\boldsymbol{\alpha\beta\epsilon}}}$ are defined as

$$\hat{\gamma}^{\underline{\boldsymbol{\gamma}}}{}_{\underline{\boldsymbol{\alpha\beta\epsilon}}} \doteq \delta_{\underline{\tau}}{}^{\underline{\boldsymbol{\gamma}}}\ \hat{\nabla}_{\underline{\boldsymbol{\alpha\beta}}} \delta_{\underline{\boldsymbol{\epsilon}}}{}^{\underline{\tau}}. \tag{60}$$

We call the Infeld - van der Waerden d-isosymbols a set of σ-objects $(\sigma_{\boldsymbol{\alpha}})^{\underline{\boldsymbol{\alpha\beta}}}$ parametrized with respect to a coordinate d-isospinor basis. Defining

$$\hat{\nabla}_{\boldsymbol{\alpha}} = (\hat{\sigma}_{\boldsymbol{\alpha}})^{\underline{\boldsymbol{\alpha\beta}}}\ \hat{\nabla}_{\underline{\boldsymbol{\alpha\beta}}},$$

introducing denotations

$$\hat{\gamma}^{\underline{\boldsymbol{\gamma}}}{}_{\boldsymbol{\alpha}\underline{\boldsymbol{\tau}}} \doteq \hat{\gamma}^{\underline{\boldsymbol{\gamma}}}{}_{\underline{\boldsymbol{\alpha\beta\tau}}}\ (\hat{\sigma}_{\boldsymbol{\alpha}})^{\underline{\boldsymbol{\alpha\beta}}}$$

and using properties (**??**) we can write relations

$$\hat{l}^{\alpha}_{\boldsymbol{\alpha}}\ \delta_{\underline{\beta}}{}^{\underline{\boldsymbol{\beta}}}\ \hat{\nabla}_{\alpha} \kappa^{\underline{\beta}} = \hat{\nabla}_{\boldsymbol{\alpha}} \kappa^{\underline{\boldsymbol{\beta}}} + \kappa^{\underline{\boldsymbol{\delta}}}\ \hat{\gamma}^{\underline{\boldsymbol{\beta}}}{}_{\boldsymbol{\alpha}\underline{\boldsymbol{\delta}}} \tag{61}$$

and

$$\hat{l}^{\alpha}_{\boldsymbol{\alpha}}\ \delta_{\underline{\boldsymbol{\beta}}}{}^{\underline{\beta}}\ \hat{\nabla}_{\alpha}\ \hat{\mu}_{\underline{\beta}} = \hat{\nabla}_{\boldsymbol{\alpha}}\ \hat{\mu}_{\underline{\boldsymbol{\beta}}} - \hat{\mu}_{\underline{\boldsymbol{\delta}}} \hat{\gamma}^{\underline{\boldsymbol{\delta}}}{}_{\boldsymbol{\alpha}\underline{\boldsymbol{\beta}}} \tag{62}$$

for d–isocovariant derivations $\hat{\nabla}_{\underline{\alpha}} \hat{\kappa}^{\underline{\beta}}$ and $\hat{\nabla}_{\underline{\alpha}}\ \hat{\mu}_{\underline{\beta}}$.

We can consider expressions similar to (**??**) and (**??**) for values having both types of d–isospinor and d-isotensor indices, for instance,

$$\hat{l}^{\alpha}_{\boldsymbol{\alpha}}\ \hat{l}^{\boldsymbol{\gamma}}_{\gamma}\ \delta_{\underline{\boldsymbol{\delta}}}{}^{\underline{\delta}}\ \hat{\nabla}_{\alpha} \theta_{\underline{\delta}}{}^{\gamma} = \hat{\nabla}_{\boldsymbol{\alpha}} \hat{\theta}_{\underline{\boldsymbol{\delta}}}{}^{\boldsymbol{\gamma}} - \hat{\theta}_{\underline{\boldsymbol{\epsilon}}}{}^{\boldsymbol{\gamma}} \hat{\gamma}^{\underline{\boldsymbol{\epsilon}}}{}_{\boldsymbol{\alpha}\underline{\boldsymbol{\delta}}} + \hat{\theta}_{\underline{\boldsymbol{\delta}}}{}^{\boldsymbol{\tau}}\ \hat{\Gamma}\ {}^{\boldsymbol{\gamma}}{}_{\boldsymbol{\alpha\tau}}$$

(we can prove this by a straightforward calculation of the derivation $\hat{\nabla}_{\alpha}(\hat{\theta}_{\underline{\boldsymbol{\epsilon}}}{}^{\boldsymbol{\tau}}\ \ \delta_{\underline{\delta}}{}^{\underline{\boldsymbol{\epsilon}}}\ \hat{l}^{\gamma}_{\boldsymbol{\tau}})$).

Now we shall consider some possible relations between components of d-isoconnections $\hat{\gamma}^{\boldsymbol{\epsilon}}_{\ \underline{\boldsymbol{\alpha\delta}}}$ and $\hat{\Gamma}^{\ \gamma}_{\ \ \boldsymbol{\alpha\tau}}$ and derivations of $(\sigma_{\boldsymbol{\alpha}})^{\underline{\boldsymbol{\alpha\beta}}}$. According to definitions (19) we can write

$$\begin{aligned}\hat{\Gamma}^{\boldsymbol{\alpha}}_{\ \boldsymbol{\beta\gamma}} &= \hat{l}^{\boldsymbol{\alpha}}_{\alpha}\hat{\nabla}_{\boldsymbol{\gamma}}\hat{l}^{\alpha}_{\boldsymbol{\beta}} = \hat{l}^{\boldsymbol{\alpha}}_{\alpha}\hat{\nabla}_{\boldsymbol{\gamma}}(\hat{\sigma}_{\boldsymbol{\beta}})^{\underline{\epsilon\tau}} = \hat{l}^{\boldsymbol{\alpha}}_{\alpha}\hat{\nabla}_{\boldsymbol{\gamma}}((\hat{\sigma}_{\boldsymbol{\beta}})^{\underline{\boldsymbol{\epsilon\tau}}}\delta_{\underline{\boldsymbol{\epsilon}}}^{\ \underline{\epsilon}}\delta_{\underline{\boldsymbol{\tau}}}^{\ \underline{\tau}})\\ &= \hat{l}^{\boldsymbol{\alpha}}_{\alpha}\delta_{\underline{\boldsymbol{\alpha}}}^{\ \underline{\alpha}}\delta_{\underline{\boldsymbol{\epsilon}}}^{\ \underline{\epsilon}}\hat{\nabla}_{\boldsymbol{\gamma}}(\hat{\sigma}_{\boldsymbol{\beta}})^{\underline{\boldsymbol{\alpha\epsilon}}} + \hat{l}^{\boldsymbol{\alpha}}_{\alpha}(\hat{\sigma}_{\boldsymbol{\beta}})^{\underline{\boldsymbol{\epsilon\tau}}}(\delta_{\underline{\boldsymbol{\tau}}}^{\ \underline{\tau}}\hat{\nabla}_{\boldsymbol{\gamma}}\delta_{\underline{\boldsymbol{\epsilon}}}^{\ \underline{\epsilon}} + \delta_{\underline{\boldsymbol{\epsilon}}}^{\ \underline{\epsilon}}\hat{\nabla}_{\boldsymbol{\gamma}}\delta_{\underline{\boldsymbol{\tau}}}^{\ \underline{\tau}})\\ &= \hat{l}^{\boldsymbol{\alpha}}_{\underline{\boldsymbol{\epsilon\tau}}}\ \hat{\nabla}_{\boldsymbol{\gamma}}(\hat{\sigma}_{\boldsymbol{\beta}})^{\underline{\boldsymbol{\epsilon\tau}}} + \hat{l}^{\boldsymbol{\alpha}}_{\underline{\boldsymbol{\mu\nu}}}\delta_{\underline{\epsilon}}^{\ \underline{\boldsymbol{\mu}}}\delta_{\underline{\tau}}^{\ \underline{\boldsymbol{\nu}}}(\hat{\sigma}_{\boldsymbol{\beta}})^{\underline{\epsilon\tau}}(\delta_{\underline{\boldsymbol{\tau}}}^{\ \underline{\tau}}\hat{\nabla}_{\boldsymbol{\gamma}}\delta_{\underline{\boldsymbol{\epsilon}}}^{\ \underline{\epsilon}} + \delta_{\underline{\boldsymbol{\epsilon}}}^{\ \underline{\epsilon}}\hat{\nabla}_{\boldsymbol{\gamma}}\delta_{\underline{\boldsymbol{\tau}}}^{\ \underline{\tau}}),\end{aligned}$$

where $\hat{l}^{\boldsymbol{\alpha}}_{\alpha} = (\hat{\sigma}_{\underline{\boldsymbol{\epsilon\tau}}})^{\boldsymbol{\alpha}}$, from which it follows

$$(\hat{\sigma}_{\boldsymbol{\alpha}})^{\underline{\boldsymbol{\mu\nu}}}(\hat{\sigma}_{\underline{\boldsymbol{\alpha\beta}}})^{\boldsymbol{\beta}}\hat{\Gamma}^{\boldsymbol{\alpha}}_{\ \boldsymbol{\gamma\beta}} = (\hat{\sigma}_{\underline{\boldsymbol{\alpha\beta}}})^{\boldsymbol{\beta}}\hat{\nabla}_{\boldsymbol{\gamma}}(\hat{\sigma}_{\boldsymbol{\alpha}})^{\underline{\boldsymbol{\mu\nu}}} + \delta_{\underline{\boldsymbol{\beta}}}^{\ \underline{\boldsymbol{\nu}}}\hat{\gamma}^{\underline{\boldsymbol{\mu}}}_{\ \boldsymbol{\gamma}\underline{\boldsymbol{\alpha}}} + \delta_{\underline{\boldsymbol{\alpha}}}^{\ \underline{\boldsymbol{\mu}}}\hat{\gamma}^{\underline{\boldsymbol{\nu}}}_{\ \boldsymbol{\gamma}\underline{\boldsymbol{\beta}}}.$$

Connecting the last expression on $\underline{\boldsymbol{\beta}}$ and $\underline{\boldsymbol{\nu}}$ and using an orthonormalized d-spinor basis when $\hat{\gamma}^{\underline{\boldsymbol{\beta}}}_{\ \boldsymbol{\gamma}\underline{\boldsymbol{\beta}}} = 0$ (a consequence from (**??**)) we have

$$\hat{\gamma}^{\underline{\boldsymbol{\mu}}}_{\ \boldsymbol{\gamma}\underline{\boldsymbol{\alpha}}} = \frac{1}{N(n)+N(m)}(\hat{\Gamma}^{\underline{\boldsymbol{\mu\beta}}}_{\ \ \boldsymbol{\gamma}\ \underline{\boldsymbol{\alpha\beta}}} - (\hat{\sigma}_{\underline{\boldsymbol{\alpha\beta}}})^{\boldsymbol{\beta}}\hat{\nabla}_{\boldsymbol{\gamma}}(\hat{\sigma}_{\boldsymbol{\beta}})^{\underline{\boldsymbol{\mu\beta}}}), \tag{63}$$

where

$$\hat{\Gamma}^{\underline{\boldsymbol{\mu\beta}}}_{\ \ \boldsymbol{\gamma}\ \underline{\boldsymbol{\alpha\beta}}} = (\hat{\sigma}_{\boldsymbol{\alpha}})^{\underline{\boldsymbol{\mu\beta}}}(\hat{\sigma}_{\underline{\boldsymbol{\alpha\beta}}})^{\boldsymbol{\beta}}\hat{\Gamma}^{\boldsymbol{\alpha}}_{\ \boldsymbol{\gamma\beta}}. \tag{64}$$

3.1.3 D-isospinors of isocurvature and isotorsion

The d-tensor indices of the commutator $\hat{\Delta}_{\alpha\beta}$, can be transformed into d-spinor ones:

$$\hat{\square}_{\underline{\alpha\beta}} = (\hat{\sigma}^{\alpha\beta})_{\underline{\alpha\beta}}\hat{\Delta}_{\alpha\beta} = (\hat{\square}_{\underline{ij}}, \hat{\square}_{\underline{ab}}), \tag{65}$$

with h- and v-components,

$$\hat{\square}_{\underline{ij}} = (\hat{\sigma}^{\alpha\beta})_{\underline{ij}}\hat{\Delta}_{\alpha\beta} \text{ and } \hat{\square}_{\underline{ab}} = (\hat{\sigma}^{\alpha\beta})_{\underline{ab}}\hat{\Delta}_{\alpha\beta},$$

being symmetric or antisymmetric in dependence of corresponding values of dimensions n and m (see eight-fold parametizations (**??**), (**??**) and (**??**)). Considering the actions of operator (**??**) on d-isospinors $\hat{\pi}^{\underline{\gamma}}$ and $\hat{\mu}_{\underline{\gamma}}$ we introduce the d-isospinor curvature $\hat{X}_{\underline{\delta}}^{\ \ \underline{\gamma}}{}_{\underline{\alpha\beta}}$ as to satisfy equations

$$\hat{\square}_{\underline{\alpha\beta}}\ \hat{\pi}^{\underline{\gamma}} = \hat{X}_{\underline{\delta}}^{\ \ \underline{\gamma}}{}_{\underline{\alpha\beta}}\hat{\pi}^{\underline{\delta}} \tag{66}$$

and

$$\hat{\square}_{\underline{\alpha\beta}}\ \hat{\mu}_{\underline{\gamma}} = \hat{X}_{\underline{\gamma}}^{\ \ \underline{\delta}}{}_{\underline{\alpha\beta}}\hat{\mu}_{\underline{\delta}}.$$

The gravitational d-isospinor $\hat{\Psi}_{\underline{\alpha\beta\gamma\delta}}$ is defined by a corresponding symmetrization of d-isospinor indices:

$$\hat{\Psi}_{\underline{\alpha\beta\gamma\delta}} = \hat{X}_{(\underline{\alpha}|\underline{\beta}|\underline{\gamma\delta})}. \tag{67}$$

We note that d-ispspinor tensors $\hat{X}_{\underline{\delta}}\ {}^{\underline{\gamma}}{}_{\underline{\alpha\beta}}$ and $\hat{\Psi}_{\underline{\alpha\beta\gamma\delta}}$ are transformed into similar 2-isospinor objects on locally isotropic isospaces [?, ?] if we consider vanishing of the N-connection structure and a limit to a locally isotropic isospace.

Putting $\delta_{\underline{\gamma}}\ {}^{\underline{\gamma}}$ instead of $\hat{\mu}_{\underline{\gamma}}$ in (??) and using (??) we can express respectively the curvature and gravitational d-spinors as

$$\hat{X}_{\underline{\gamma\delta\alpha\beta}} = \delta_{\underline{\delta\tau}}\Box_{\underline{\alpha\beta}}\delta_{\underline{\gamma}}\ {}^{\underline{\tau}} \text{ and } \hat{\Psi}_{\underline{\gamma\delta\alpha\beta}} = \delta_{\underline{\delta\tau}}\Box_{(\underline{\alpha\beta}}\delta_{\underline{\gamma})}\ {}^{\underline{\tau}}.$$

The d-spinor torsion $T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}}$ is defined similarly as for d-isotensors) by using the d-isospinor commutator (??) and equations

$$\Box_{\underline{\alpha\beta}}\hat{f} = \hat{T}^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}}\hat{\nabla}_{\underline{\gamma}_1\underline{\gamma}_2}\hat{f}.$$

The d-isospinor components $\hat{R}_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}}$ of the curvature d-isotensor $\hat{R}_{\gamma}\ {}^{\delta}{}_{\alpha\beta}$ can be computed by using relations (??), and (??) and (??) as to satisfy the equations

$$(\Box_{\underline{\alpha\beta}} - \hat{T}^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}}\hat{\nabla}_{\underline{\gamma}_1\underline{\gamma}_2})\hat{V}^{\underline{\delta}_1\underline{\delta}_2} = R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}}\hat{V}^{\underline{\gamma}_1\underline{\gamma}_2}, \tag{68}$$

the d-isovector $\hat{V}^{\underline{\gamma}_1\underline{\gamma}_2}$ is considered as a product of d–isospinors, i.e. $\hat{V}^{\underline{\gamma}_1\underline{\gamma}_2} = \hat{\nu}^{\underline{\gamma}_1}\hat{\mu}^{\underline{\gamma}_2}$. We find

$$\begin{aligned} R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}} &= \left(X_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}{}_{\underline{\alpha\beta}} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha\beta}}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1}\right)\delta_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2} \\ &+ \left(X_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2}{}_{\underline{\alpha\beta}} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha\beta}}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}\right)\delta_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}. \end{aligned} \tag{69}$$

It is convenient to use this d–isospinor expression for the curvature d–isotensor

$$\begin{aligned} R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} &= \left(X_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1}\right)\delta_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2} \\ &+ \left(X_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}\right)\delta_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1} \end{aligned}$$

in order to get the d-isospinor components of the Ricci d-isotensor

$$\begin{aligned} R_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} &= R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\delta}_2} = X_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} \\ &+T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} + X_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2} \end{aligned} \tag{70}$$

and this d–isospinor decomposition of the isoscalar curvature:

$$\begin{aligned} q\overleftarrow{R} &= R^{\underline{\alpha}_1\underline{\alpha}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2} = X^{\underline{\alpha}_1\underline{\delta}_1}{}_{\underline{\alpha}_1}{}^{\underline{\alpha}_2}{}_{\underline{\delta}_1\underline{\alpha}_2} + T^{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_1}{}_{\underline{\alpha}_2\underline{\delta}_1}{}^{\underline{\alpha}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_1} \\ &+X^{\underline{\alpha}_2\underline{\delta}_2\underline{\alpha}_1}{}_{\underline{\alpha}_2\underline{\delta}_2\underline{\alpha}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1}{}^{\underline{\alpha}_2\underline{\alpha}_1}{}_{\underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_2}. \end{aligned} \tag{71}$$

Using (**??**) and (**??**) we find the d-spinor components of the Einstein d–isotensor:

$$\begin{aligned}
\overleftarrow{G}_{\gamma\alpha} = & \overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} \\
& + X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2} - \\
& \frac{1}{2}\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2}[X^{\underline{\beta}_1\underline{\mu}_1}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2}{}_{\underline{\mu}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1}{}_{\underline{\beta}_2\underline{\mu}_1}{}^{\underline{\beta}_2}\ \gamma^{\underline{\mu}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1} + \\
& X^{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1}{}_{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2\underline{\beta}_1}{}_{\underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_2}].
\end{aligned}$$

The presented in this subsection isogeometric background allows us to formulate a classical variant of isotopic and locally anisotropic gravitational and matter field equations.

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