DETERMINATION OF THE RICCI STRUCTURE ON A COMPACT MANIFOLD BY SOME SPECTRA

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Abstract

Let (M, g) be a compact and orientable Riemannian manifold of dimension n. We consider one parameter family of second order differential operators D_q^{ε} from which we obtain a spectrum $Sp(M, D_q^{\varepsilon})$. The aim of the presetn paper is to determine the fluence of $Sp(M, D_q^{\varepsilon})$ on the Ricci structure on (M, g).

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1 Introduction

Let (M, g) be a compact, connected and orientable Riemannian manifold without boudary of dimension n. We denote by $\Lambda^q(M, IR)$ and the vector space of exterior q-forms on M, where $q = 0, 1, ..., n$. On $\Lambda^q(M, IR)$ there are different differential operators. We consider one parameter family of second order differential operators $D_q^{\varepsilon}, \varepsilon \in IR$. We denote by $Sp(M, D_q^{\varepsilon})$ the spectrum of D_q^{ε} for (M, g),it is obvious that $Sp(M, D_q^{\varepsilon})$ depends on ε.

The purpose of this paper is to study the influence of different $Sp(M, D_q^{\varepsilon})$ on the Ricci structure on the Riemannian manifold (M, g).

The whole paper contains four sections.

The first section is the introduction.

Some basic properties of Ricci manifold are given in the second section.

The third section includes relations between geometry and spectrum of an elliptic differential operator of second order.

Some connections between the spectra of different elliptic differential operators of second order and the Ricci structure on a compact orientable Riemannian manifold are given in the fourth section.

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2 Ricci manifolds

Let (M, g) be a Riemannian manifold of dimension n. It is known that to this manifold we can associate different tensor fields.

(I) The Riemannian curature tensor field of type $(0, 4)$ denoted by R.
(II) The curvature fensor field of type $(1, 3)$ denoted also by R.

(III) The Ricci tensor field of type $(0, 2)$ denoted by ρ.
(IV) The scalar curvature of type $(0, 2)$ denoted by T, which is function on M .
(V) The Einstein tensor field of type $(0, 0)$ denoted by G.

(VI) The Weyl conformal tensor field of type $(1, 3)$ denoted by C.

All these vector fields can be determined by the metric tensor g. We can also determine some other vector fields on M by means of g.

Let (U, φ) be a chart of (M, g) with local coordinate system $(x^1, ..., x^n)$. If the restriction of the Ricci tensor field ρ on U satisfies the relation

$$\nabla\rho = 0 \tag{1}$$

then (M, g) is called locally Ricci manifold. If the property (1) is valid for the whole manifold, then (M, g) is called Ricci manifold.

If on U we have the relation

$$\rho = 0 \tag{2}$$

then (M, g) is called locally flat Ricci manifold. If the relation (2) is valid on the whole manifold then (M, g) is called flat Ricci manifold.

The following prosositions are valid for Ricci manifolds ([6]).

Proposition 1 *Let (M, g) be a Ricci manifold. Then the scalar curvature of (M, g) is constant*

Proposition 2 *Let (M, g) be a compact and orentable Riemannian manifold of dimension n. (M, g) is Ricci manifold, if and only if, the Einstein tensor field is parallel.*

Let ρ be the Ricci tensor field on the Riemannian manifold (M, g). Then for each point $P \in M$, $\rho(P)$ is a symmetric covariant tensor of order two, obtained by the tangent space $T_p(M)$ of M at P. If $\{e_1, ..., e_n\}$ is an orthonormal base of $T_p(M)$, then $\rho(P)$ can be represented by the symmetric matrix

$$\begin{bmatrix} \rho_{11}(P) & \rho_{12}(P) & ... & \rho_{1n}(P) \\ \rho_{21}(P) & \rho_{22}(P) & ... & \rho_{2n}(P) \\ ... & ... & ... & ... \\ \rho_{n1}(P) & \rho_{n2}(P) & ... & \rho_{nn}(P) \end{bmatrix} \tag{3}$$

with respect to $\{e_1, ..., e_n\}$. The eigenvalues of (3) are real numbers

$$\lambda_1(P),\ \lambda_2(P), ..., \lambda_n(P)$$

If $\lambda_i(P) > 0$ (resp. $\lambda_i(P) < 0$) $i = 1, ..., n$ for ever $P \in M$,then the Ricci tensor field ρ is called positive definite (resp. negative deinite). If $\lambda_i(P) \geq 0$ (resp.$\lambda_i(P) \leq 0$) $i = 1, ..., n$ for every $P \in M$, then the Ricci tensor field ρ is called semi-positive (resp. semi-negative).

If $\lambda_i(P) = 0$ $i = 1, ...n$ for every $P \in M$, then $\rho = 0$ and (M, g) is Ricci flat manifold.

3 Connection between spectrum and Riemannian geometry

Let (M, g) be a compact Riemanniam manifold of dimension n. Let (U, φ) be a chart of M with local coordinate system $(x_1, ..., x_2)$. The Riemannian metric g on U takes the form

$$ds^2 = g^{ij} dx_i dx_j.$$

Let $[g^{ij}]$ be the metric on the cotangent bundle T^*M over M and let dM be the Riemannian measure of M.

Let V be a smmoth vector bundle over M. We consider

$$D : C^\infty(V) \to C^\infty(V)$$

a second order elliptic differential operator with leading symbol given by the metric tensor g. We choose a local orthonormal frame

$$\left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)$$

for V which corresponds to the chart (U, φ) whith local coordinate system $(x_1, ..., x_n)$.

Hence D in a local level can be expressed by

$$D = -(g^{ij}\partial^2/\partial x_i \partial x_j + P_k \partial/\partial x_k + S),$$

where P_k and S are square matrices which are not invariantly defined but depent on the choise of frame and local coordintes.

Let V_x be the fibre of V over x. We choose a smooth fibre metric on V. Let $L^2(V)$ be the completion of $C^\infty(V)$ with respect to global integtraded inner product, that is

$$L^2(V) = \left\{ S \in C^\infty(V) / \int_M \|S\| \, dM < \infty \right\}.$$

As a banach space $L^2(V)$ in independent on the Riemannian and fibre metric and for $t > 0$

$$\exp(-tD) : L^2(V) \to C^\infty(V)$$

is an infinitely smoothing operator of trace class. Let $K(t,x,y):V_y\to V_x$ be the Kernell of $\exp(-tD)$. K is a smooth endomorphism valued function of (t,x,y).

We define

$$f(t,D,x)=Tracev_x(K(t,x,y))$$

and

$$f(t,D)=\int_M K(t,x,x)dM$$

It is known that $f(t,D,x)$ has an asymptotic expansion, that is ([2])

$$f(t,D,x)\underset{t\to 0^+}{\cong}(4nt)^{-n/2}\sum_{m=0}^{\infty}\alpha_m(D,x)t^m$$

The coefficients, $\alpha_m(D,x)$ are smooth functions of x,which can be estimated functionally of the derivatives of the total symbols of the differential operator D. If we integrate the function

$$\alpha_m(D,x):M\to IR, m=0,1,2,...$$

on the manifold M we obtain the numbers

$$\alpha_m(D)=\int_M\alpha_m(D,x)dM$$

It is known that the numbers $\alpha_m(D), m=0,1,2,...,$ are isospectral invariants.

Let $D=\Delta_q=0,1,...,n$, be the Laplacian which is second order elliptic differential operator with leading symbol defined by the metric tensor on the cross sections of the vector bundle of exterior $q-$forms $\Lambda^q(M,R)$ over the manifold M,that is

$$\nabla_q=d\delta+\delta d:C^\infty(\Lambda^q(M)0\to C^\infty(\Lambda^q(M)),$$

where d and δ are the exterior differentiation and codifferentiation respectively.

Now, we can define the reduced or Bochner Laplacian operator B_k^∇ by the following diagram

$$B_k^\nabla:C^\infty(M)\to C^\infty(TM\otimes V)\overset{\nabla_g\otimes 1+1\otimes\nabla}{\to}C^\infty(T^*M\otimes T^*M\otimes V), \tag{4}$$

where g is the Riemannian metric on M, ∇_g the Levi-Civita connection on TM,extend ∇_g on the tensor fields of all type and ∇ any connection on V. The Bochner Laplacian B_k^∇ defined by Levi-Civita connection in local coordinate system has the form

$$B_k^\nabla=-g_{ij}\nabla_i\nabla_j. \tag{5}$$

Now, we form one parameter family of second order elliptic differential operators

$$D_k^\varepsilon=\varepsilon\Delta_k+1-\varepsilon)B_k. \tag{6}$$

The coefficients, $\alpha_m(D_k^\varepsilon)$ for $m = 0, 1, 2, 3$, are given by

$$\alpha_o(D_1^\varepsilon) = nVol(M),\ \alpha_1(D_1^\varepsilon) = \frac{6\varepsilon - 1}{6}\int_M TdM, \tag{7}$$

$$\alpha_2(D_1^\varepsilon) = \frac{1}{360}\int_M \left[(5n - 6\varepsilon)T^2 - (180\varepsilon^2 - 2n)\left|\rho\right|^2 + (2n - 30)\left|R\right|^2\right] dM, \tag{8}$$

$$\begin{aligned}\alpha_3(D_1^\varepsilon) \quad = \quad & \frac{1}{360.7!}\int_M [(-98 + 588\varepsilon - 5680n)\left|\nabla T\right|^2 + (392 + 1470\varepsilon^2 - \\ & -2480n)\left|\nabla\rho\right|^2 + (49 - 280n)\left|\nabla R\right|^2 + (245 - 1400n)T^2 + \\ & +(-980 - 1470\varepsilon^2 + 1680n)T\left|\rho\right|^2 + (245 + 98\varepsilon - \\ & -1680n)T\left|R\right|^2 + (245 + 245\varepsilon - 1400n)\left|\rho\right|^3 + (393 + \\ & +800n)L_1 + (98 - 1470\varepsilon^2 + 320n)L_2 + (147 - 960)L_3]dM, \end{aligned}\tag{9}$$

where

$$L_1 = \rho_{ij}\rho_{km}R_{ijkm},\ L_2 = R_{iklm}R_{jklm}, \tag{10}$$

$$L_3 = R_{ijkm}R_{ijuu}R_{knmu}, \tag{11}$$

$|R|$ and $|\rho|$ the norm of R and ρ respectively, (ρ_{ij}) and (R_{ijkl}) are the components of ρ and R , respectively with respect to the local coordinate system $(x_1, ..., x_n)$ on the manifold M and $\bigtriangledown T, \nabla\rho, \nabla R$ are the covariant derivatives of T, ρ, R respectively.

4 Relation between spectra and Ricci manifold

In this section we study the influences of $S_p(D_1^\varepsilon)$ for different values of the parameter ε on the Ricci structure on a Riemannian manifold.

Now, we prove the theorem

Theorem 3 *Let (M, g) and (N, h) be two compact Riemannian manifold with the properties $S_p(M, D_1^\varepsilon) = S_p(N, D_1^\varepsilon)$ for four distinct values of ε. If (M, g) is Ricci, so is (N, h).*

Proof. From the assumption of the theorem we obtain

$$\mathbf{a}_k(D_1^\varepsilon, M) = \mathbf{a}_k(D_1^\varepsilon, N),\ k = 0, 1, 2, 3 \tag{12}$$

for four distinct values of ε, that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4.

The formulas (12) by means of the formulas (7), (8) and (9) imply

$$Vol(M) = Vol(N), \tag{13}$$

$$\int_M T_M dM = \int_N T_N dN, \tag{14}$$

$$\int_M \left[A_1(n,\varepsilon)T_M^2 + A_2(n,\varepsilon)\left|\rho_M\right|^2 + A_3(n)\left|R_M\right|^2\right] dM =$$
$$\int_N \left[A_1(n,\varepsilon)T_N^2 + A_2(n,\varepsilon)\left|\rho_N\right|^2 + A_3(n)\left|R_N\right|^2\right] dN, \tag{15}$$

where

$$A_1(n,\varepsilon) = 5n - 6\varepsilon,\ A_2(n,\varepsilon) = -180\varepsilon^2 + 2n,\ A_3(n) = 2n - 30, \tag{16}$$

$$\int_M [B_1(n,e)\left|\nabla T_M\right|^2 + B_2(n,\varepsilon)\left|\nabla \rho_M\right|^2 + B_3(n\nabla)\left|R_M\right|^2 +$$
$$+B_4(n)T_M^3 + B_5(n,\varepsilon)T_M\left|\rho_M\right|^2 + B_6(N,\varepsilon)T_M\left|R_M\right|^2 +$$
$$+B_7(n,\varepsilon)\left|\rho\right|^3 + B_8(n)L_{1M} + B_9(n,\varepsilon)_{2M} + B_{10}(n)L_{3M}]dM$$

$$= \int_N [B_1(n,\varepsilon)\left|\nabla T_N\right|^2 + B_2(n,\varepsilon)\left|\nabla \rho\right|^2 + B_3(n) + B_3(n)\left|\nabla R_N\right|^2 +$$
$$+B_4(n)T_N^3 + B_5(n,\varepsilon)\left|\rho_N\right|^2 T_N + B_6 + (n,\varepsilon)\left|R_N\right|^2 T_N + B_7(n,\varepsilon)\left|\rho_N\right|^3 +$$
$$+B_8(N)L_{1N} + B_9(n,\varepsilon)L_{2N} + B_{10}(n)L_{3N}]dN, \tag{17}$$

$$B_1(n,\varepsilon) = -98 + 588\varepsilon - 5680n,\ B_2(n,\varepsilon) = 392 - 1470\varepsilon^2 - 2480n, \tag{18}$$

$$B_3(n) = 49 - 280n, B_4(n) = 245 - 1400n, \tag{19}$$

$$B_5(n,\varepsilon) = -980 - 1470\varepsilon^2 + 1680n,\ B_6(n,\varepsilon) = 245 + 98\varepsilon - 1680n, \tag{20}$$

$$B_7(n,\varepsilon) = 245 + 245\varepsilon - 1400n,\ B_8(n) = 392 + 800n, \tag{21}$$

$$B_9(n,\varepsilon) = 98 - 1470\varepsilon^2 + 320n,\ B_{10}(n) = 147 - 960n. \tag{22}$$

If we put

$$Y_1 = \left(\int_M T_M^2 dM - \int_N T_N^2 dN\right),\ Y_2 = \left(\int_M \left|\rho_M\right|^2 dM - \int_N \left|\rho_N\right|^2 dN\right), \tag{23}$$

$$Y_3 = \int_M \left|R_M\right|^2 dM - \int_N \left|R_N\right|^2 dN, \tag{24}$$

then from the relation (15) for three distinct values $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of the parameter ε, we obtain the following system

$$A_1(n,\varepsilon_1)Y_1 + A_2(n,\varepsilon_1)Y_2 + A_3(n)Y_3 = 0,$$

$$A_1(n,\varepsilon_2)Y_1 + A_2(n,\varepsilon_2)Y_2 + A_3(n)Y_3 = 0, \tag{25}$$

$$A_1(n,\varepsilon_3) + A_2(n,\varepsilon_3)Y_2 + A_3(n)Y_3 = 0.$$

If we choose the numbers $\varepsilon_1, \varepsilon_2$ and ε_3 such that the determinant

$$\begin{vmatrix} A_1(n,\varepsilon_1) & A_2(n,\varepsilon_1) & A_3(n) \\ A_1(n,\varepsilon_2) & A_2(n,\varepsilon_2) & A_3(n) \\ A_1(n,\varepsilon_3) & A_2(n,\varepsilon_3) & A_3(n) \end{vmatrix} \neq 0, \tag{26}$$

then the system (25) has the trivial solution, that means:

$$Y_1 = 0,\ Y_2 = 0,\ Y_3 = 0, \tag{27}$$

which imply

$$\int_M T_M^2 dM = \int_N T_N^2 dN, \tag{28}$$

$$\int_M \rho_M^2 dM = \int_N \rho_N^2 dN,\ \int_M R_M^2 dM = \int_N R_N^2 dN. \tag{29}$$

Since the Riemannian manifold M is Ricci then we obtain

$$T_M = \text{ constant}. \tag{30}$$

From the relations (15) and (28) we have

$$T_N = T_M = \text{ constant}. \tag{31}$$

From the relations (29) and (31) we conclude that

$$\int_M T_M \left|\rho_M\right|^2 dM = \int_N T_N \left|\rho_N\right|^2 dN,\ \int_M T_M \left|R_M\right|^2 dM = \int T_N \left|R_N\right|^2 dN. \tag{32}$$

The relation (17) by means of (31) and (32) takes the form

$$\int [B_2(n,\varepsilon) \left|\nabla R_M\right|^2 + B_3(n) \left|\nabla \rho_M\right|^2 + B_7(n,\varepsilon) \left|\rho_M\right|^3 +$$
$$B_8(n)L_{1M} + B_9(n,\varepsilon)L_{2M} + B_{10}(n)L_{3M}]dM=$$

$$= \int [B_1(n,\varepsilon)\,|\nabla\rho_N|^2 + B_3(n)\,|\nabla R_N|^2 + B_7(n,\varepsilon)\,|\rho_N|^3 + B_8(n)L_{1M} + B_9(n,\varepsilon)L_{2M} + B_{10}(n)]L_{3M}dN. \tag{33}$$

If we set

$$Z_1 = \int_N (|\nabla\rho_M|^2\,dM - \int_N |\nabla\rho_N|^2\,dN),\ Z_2 = (\int_M |\nabla R_M|^2\,dM - \int_N |\nabla R_N|^2\,dN), \tag{34}$$

$$Z_3 = (\int_M |\rho_M|^3\,dM - \int_N |\rho_N|^3\,dN),\ Z_4 = (\int_M L_{1M}dM - \int_N L_{1N}dN), \tag{35}$$

$$Z_5 = (\int_M L_{2M}dM - \int_N L_{2N}dN),\ Z_6 = (\int_M L_{3M}dM - \int_N L_{3N}dN), \tag{36}$$

then the relation (33) for the values $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and another value ε_4 of the parameter ε, we obtain

$$B_2(n,\varepsilon_1)Z_1 + B_3(n)Z_2 + B_7(n,\varepsilon_1)Z_3 + B_8(n)Z_4) + B_9(n,\varepsilon_1)Z_5 + B_{10}(n)Z_6 = 0, \tag{37}$$

$$B_2(n,\varepsilon_2)Z_1 + B_3(n)Z_2 + B_7(n,\varepsilon_2)Z_3 + B_8(n)Z_4 + B_9(n,\varepsilon_2)Z_5 + B_{10}(n)Z_6 = 0, \tag{38}$$

$$B_2(n,\varepsilon_3)Z_1 + B_3(n)Z_2 + B_7(n,\varepsilon_3)Z_3 + B_8(n)Z_4 + B_9(n,\varepsilon_3)Z_5 + B_{10}(n)Z_6 = 0, \tag{39}$$

$$B_2(n,\varepsilon_4)Z_1 + B_3(n)Z_2 + B_7(n,\varepsilon_4)Z_3 + B_8(n)Z_4 + B_9(n,\varepsilon_4)Z_5 + B_{10}(n)Z_6 = 0, \tag{40}$$

The above system $(37), (38), (39)$ and (40) is equivalent to the following system

$$B_2(n,\varepsilon_1)Z_1 + B_3(n)Z_2 + B_7(n,\varepsilon_1)Z_3 + B_8(n)Z_7) + B_9(n,\varepsilon_1)Z_5 + B_{10}(n)Z_6 = 0, \tag{41}$$

$$[B_2(n,\varepsilon_1) - B_2(n,\varepsilon_2)]Z_1 + [B_7(n,\varepsilon_1) - B_7(n,\varepsilon_2)]Z_3 + [B_9(n,\varepsilon_1) - B_9(n,\varepsilon_2)]Z_6 = 0, \tag{42}$$

$$[B_2(n,\varepsilon_1) - B_2(n,\varepsilon_3)]Z_1 + [B_7(n,\varepsilon_1) - B_7(n,\varepsilon_3)]Z_3 + [B_9(n,\varepsilon_1) - B_9(n,\varepsilon_3)]Z_6 = 0 \tag{43}$$

$$[B_2(n,\varepsilon_1)-B_2(n,\varepsilon_4)]Z_1+[B_7(n,\varepsilon_1)-B_7(n,\varepsilon_4)]Z_3+[B_9(n,\varepsilon_1)-B_9(n,\varepsilon_4)]Z_6=0. \tag{44}$$

The equations $(42),(43)$ and (44) form a homogeneous linear system of three equatinons with unknown Z_1, Z_3 and Z_5. If we choses the value ε_4, the values $\varepsilon_1, \varepsilon_2$ and ε_3 have been chosen previous, such that the determinant

$$\begin{vmatrix} B_1(n,\varepsilon_1)-B_2(n,\varepsilon_2) & B_7(n,\varepsilon_1)-B_7(n,\varepsilon_2) & B_9(n,\varepsilon_1)-B_9(n,\varepsilon_2) \\ B_2(n,\varepsilon_1)-B_2(n,\varepsilon_3) & B_2(n,\varepsilon_1)-B_2(n,\varepsilon_3) & B_9(n,\varepsilon_1)-B_9(n,\varepsilon_3) \\ B_2(n,\varepsilon_1)-B_2(n,\varepsilon_4) & B_7(n,\varepsilon_1)-B_7(n,\varepsilon_4) & B_9(n,\varepsilon_1)-B_9(n,\varepsilon_4) \end{vmatrix} \neq 0, \tag{45}$$

then this system has the obious solution, that is

$$Z_1=\int_N |\nabla\rho_M|^2\, dM-\int_N |\nabla\rho_N|^2\, dN=0, \tag{46}$$

$$Z_2=\int_N |\nabla\rho_M|^2\, dM-\int_N |\nabla\rho_N|^2\, dN=0, \tag{47}$$

$$Z_3=\int_N L_{2M} dM-\int_N \nabla L_{2N}\ dN=0. \tag{48}$$

Since the Riemannian manifold (M,g) is Ricci, that is

$$\nabla\rho_M=0 \tag{49}$$

The relation (46) by virtue of (49) implies

$$\nabla\rho_N=0, \tag{50}$$

which implies (N,h) is Ricci.

References

[1] P.Gilkey, *The spectral Geometry of real and complexes manifolds*, Proc. Sympos. Pure Math. vol.27, Amer. Math. Soc. Providence, R.I (1973) 269-280.

[2] P.Gilkey, *The spectral Geometry of Riemannian manifolds*, J. Diff. Geom., 10 (1975) l601-618.

[3] T. Sakai, *On eigenvalues of Laplacian and Curvature of Riemannian manifold*, Tohoku Math. J. 23 (1971) 27-32.

[4] Gr. Tsagas, *The spectrum of Laplace operator for Einstein manifold*, Bull. Cal. Math. Soc., 78 (1986) 27-32.

[5] Gr. Tsagas and G. Dimou, *The influence of the Spectrum of the Laplace operator on the symmetric structure on a manifold*, Algebras, Groups and Geometries, vol. 9, 2 (1992) 99-110.

[6] Gr. Tsagas, *Determination of the Ricci structure on a compact manifold by some spectra*, Proccedings of the Workshop on Global Analysis, Diffential Geometry and Lie Algebras, 1995, 95-108.

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