Associativity condition for some alternative algebras of degree three

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Abstract

In this paper we find an associativity condition for a class of alternative algebras of degree three. This is given in the last two statements of the paper.

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The algebras A over a field K satisfying the identities

$$x^2y = x(xy) \text{ and } yx^2 = (yx)x, \forall x, y \in A,$$

are called *alternative algebras.* It is obvious that an alternative algebra is power-associative: $x^2x = xx^2 = x^3$, for all $x \in A$.

An algebra A over a field K is called a *composition algebra* if we have a quadratic form $n : A \to K$ satisfying the relation $n(xy) = n(x)\, n(y)$, for every $x, y \in A$, and the associated bilineare form $f : A \times A \to K$,

$$f(x, y) = \frac{1}{2}(n(x+y) - n(x) - n(y))$$

is non-degenerate.

An alternative algebra A is of *degree three*, if, for every $x \in A$, a polynomial relation of degree three is satisfied, namely:

$$x^3 - T(x)\, x^2 + S(x)\, x - N(x) \cdot 1 = 0,$$

where T is a linear mapping, called *the generic trace*, S is a bilinear form and N is a cubic form, called *the norm form.*

The subset $A_0 = \{x \in A \,/\, T(x) = 0\}$ is a subspace of A called the *isotopic subspace* of the algebra A and $A = K \cdot 1 \oplus A_0$.

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We suppose the field K has the characteristic $\neq 2,3$. Let ω be the cubic root of unity, q be the solution of equation $x^2+3=0$ and μ be the root of the equation $3x^2-3x+1=0$.

We consider two cases:

Case 1. $\omega \in K,$ therefore $q, \mu \in K.$

We recall now some necessary results.

Proposition 1. [3] *Let A be an alternative algebra of degree three and A_0 be its isotopic subspace. The bilinear form S is nondegenerate on A if and only if S is nondegenerate on A_0.*□

Proposition 2. [3] *Let A be a finite dimensional alternative algebra of degree three over the field K. On A_0 we define the multiplication " $*$ ":*

$$a * b = \omega ab - \omega^2 ba - \frac{2\omega+1}{3} T(ab) \cdot 1. \tag{1}$$

Then S preserves the composition, that is $S(a*b) = S(a)S(b)$.

If A is a separable algebra over K (i.e. $A_F = F \otimes_K A$ is a direct sum of simple ideals*,* for every extension F of K) *then the quadratic form S is nondegenerate and we find an operation "$\acute{\nabla}$" such that (A_0, ∇) is a Hurwitz algebra. If* $\dim A = 9$, *then (A_0, ∇) is an octonion algebra. Here, the operation ∇ is*:

$$a \nabla b = (u * a) * (b * u), \text{ with } S(u) \neq 0. \square \tag{2}$$

Proposition 3. [2] *Let A be an alternative algebra of degree three over the field K with the generic minimum polynomial :*

$$P_x(\lambda) = \lambda^3 - T(x)\lambda^2 + S(x)\lambda - N(x) \cdot 1.$$

On the isotopic subspace A_0, the operation " $*$ " is defined as in Proposition 2. Then we have:

i)

$$(a * b) * a = a * (b * a) = S(a) b, \text{ for every } a, b \in A_0. \tag{3}$$

ii) S preserves composition and

$$S(x * y, z) = S(x, y * z),$$

for all $x, y, z \in A_0$.

iii) $(A_0, *)$ does not have a unit element.

iv) There exists an element $a \in A_0$, such that $\{a, a*a\}$ is a linearly independent system. A is finite dimensional and separable if and only if S is nondegenerate.□

The Proposition 3 has a converse statement and we see this in the next proposition. These two propositions show us that always in an alternative algebra of degree three we can find a flexible (i.e. $x(yx) = (xy)\,x, \forall x, y \in A$) and composition subalgebra and its symmetric associated bilinear form is associative.

Proposition 4. [2] *Let $(B, *)$ be a nonunitary algebra over the field K with quadratic form S satisfying the condition* (3) *from above. If B has an element b_0 such that the system $\{b_0, b_0 * b_0\}$ is linearly independent, then we find an alternative algebra A of degree three over K such that $(B, *)$ is isomorphic to the algebra $(A_0, *)$ defined in* Proposition 2.□

If $S\,(x, y)$ is the symmetric bilinear form associated to the quadratic form S and $A = K \cdot 1 \oplus B$, for $x \in A$, $x = \alpha + a, \alpha \in K$, $a \in B$, then define the multiplication on A:

$$ab = -\frac{2S\,(a,b)}{3} \cdot 1 + \frac{1}{3}[\left(\omega^2\text{-}1\right) a * b\text{-}\,(\omega\text{-}1)\, b * a]\ , \forall a, b \in B, \tag{4}$$

and $1x = x1 = x, \forall x \in A$. We show, by straightforward calculation, that A is an alternative algebra of degree three over K.

If we take $T\,(x) = 3\alpha,\ S\,(x) = 3\alpha^2 + S\,(a)\,,\ N\,(x) = \alpha^3 + S\,(a)\,\alpha\text{-}\rho\dfrac{2S(a*a,a)}{3}$, we obtain

$$x^3 - T\,(x)\,x^2 + S\,(x)\,x - N\,(x) \cdot 1 = 0,\ \forall x \in A$$

and $B = A_0 = \{x \in A /\, T\,(x) = 0\}$. Then A is the asked algebra of degree three.

Proposition 5. [2] *With the assumptions and the notations from* Proposition 4, *the algebras $(B_1, *)\,, (B_2, *)$ are isomorphic if and only if the corresponding alternative algebras of degree three are isomorphic.*□

We need some more properties of the computation " $\acute{*}$ ".

Proposition 6. *With the notations from Proposition 4., the algebra $A = K \cdot 1 \oplus B$ is associative if and only if the following condition holds* :

$$(a, c, b)^* + (b, a, c)^* = (a, b, c)^*\,, \forall a, b, c \in B, \tag{5}$$

where $(a, b, c)^* = (a * b) * c - a * (b * c)\,.$

Proof. Let $a, b, c \in B$, then we have:

$$c\,(ab) = c\left[\text{-}\frac{2S\,(a,b)}{3} \cdot 1 + \frac{1}{3}\left(\left(\omega^2\text{-}1\right) a * b\text{-}\,(\omega\text{-}1)\, b * a\right)\right] =$$

$$= -\frac{2S\,(a,b)}{3}c + \frac{\omega^2\text{-}1}{3}c\,(a*b) + \frac{\omega\text{-}1}{3}c\,(b*a) = -\frac{2S\,(a,b)}{3}c +$$

$$+\frac{\omega^2\text{-}1}{3}\left[\text{-}\frac{2S\,(c, a*b)}{3} \cdot 1 + \frac{\omega^2\text{-}1}{3}c*(b*a)\text{-}\frac{\omega\text{-}1}{3}\,(b*a)*c\right]\text{-}$$

$$-\frac{\omega\text{-}1}{3}\left[-\frac{2S\left(c,b*a\right)}{3}+\frac{\omega^{2}\text{-}1}{3}c*\left(b*a\right)-\frac{\omega\text{-}1}{3}\left(b*a\right)*c\right].$$

$$\left(ca\right)b=\left[-\frac{2S\left(c,a\right)}{3}\cdot 1+\frac{1}{3}\left(\left(\omega^{2}-1\right)c*a-\left(\omega-1\right)a*c\right)\right]b=$$

$$=-\frac{2S\left(c,a\right)}{3}b+\frac{\omega^{2}\text{-}1}{3}\left(c*a\right)b-\frac{\omega\text{-}1}{3}\left(a*c\right)b=-\frac{2S\left(c,a\right)}{3}b+$$

$$+\frac{\omega^{2}\text{-}1}{3}\left[-\frac{2S\left(c*a,b\right)}{3}\cdot 1+\frac{\omega^{2}\text{-}1}{3}\left(c*a\right)*b-\frac{\omega\text{-}1}{3}b*\left(c*a\right)\right]-$$

$$-\frac{\omega\text{-}1}{3}\left[-\frac{2S\left(a*c,b\right)}{3}\cdot 1+\frac{\omega^{2}\text{-}1}{3}\left(a*c\right)*b-\frac{\omega\text{-}1}{3}b*\left(a*c\right)\right].$$

In the following, we use the relations:
$\left(a*b\right)*c+\left(c*b\right)*a=2S\left(a,c\right)b=2S\left(c,a\right)b$
$\left(a*c\right)*b+\left(b*c\right)*a=2S\left(a,b\right)c$, $\left(\omega^{2}-1\right)^{2}=-3\omega^{2}$, $\left(\omega^{2}-1\right)\left(\omega-1\right)=3$,
$\left(\omega-1\right)^{2}=-3\omega$.
Then:

$$c\left(ab\right)-\left(ca\right)b=-\frac{2S\left(a,b\right)}{3}c-\frac{2\left(\omega^{2}\text{-}1\right)}{9}S\left(c,a*b\right)+\frac{\left(\omega^{2}\text{-}1\right)^{2}}{9}c*\left(a*b\right)-$$

$$-\frac{\left(\omega^{2}\text{-}1\right)\left(\omega\text{-}1\right)}{9}\left(a*b\right)*c+\frac{2\left(\omega\text{-}1\right)}{9}S\left(c,b*a\right)-\frac{\left(\omega^{2}\text{-}1\right)\left(\omega\text{-}1\right)}{9}c*\left(b*a\right)+$$

$$+\frac{\left(\omega\text{-}1\right)^{2}}{3}\left(b*a\right)*c+\frac{2S\left(c,a\right)}{3}b+\frac{2\left(\omega^{2}\text{-}1\right)}{9}S(c*a,b)-$$

$$-\frac{\left(\omega^{2}\text{-}1\right)^{2}}{9}(c*a)*b+\frac{\left(\omega^{2}\text{-}1\right)\left(\omega\text{-}1\right)}{9}b*\left(c*a\right)-\frac{2\left(\omega\text{-}1\right)}{9}S\left(a*c,b\right)+$$

$$+\frac{\left(\omega^{2}\text{-}1\right)\left(\omega\text{-}1\right)}{9}\left(a*c\right)*b-\frac{\left(\omega\text{-}1\right)^{2}}{9}b*\left(a*c\right)=$$

$$=-\frac{1}{3}\left(a*c\right)*b-\frac{1}{3}\left(b*c\right)*a-\frac{2\left(\omega^{2}\text{-}1\right)}{9}S\left(c,a*b\right)-$$

$$-\frac{\omega^2}{3}c*(a*b)-\frac{1}{3}(a*b)*c+\frac{2(\omega-1)}{9}S(c,b*a)-\frac{1}{3}c*(b*a)-$$

$$-\frac{\omega}{3}(b*a)*c+\frac{1}{3}(a*b)*c+\frac{1}{3}(c*b)*a+\frac{2(\omega^2-1)}{9}S(c*a,b)+$$

$$+\frac{\omega^2}{3}(c*a)*b+\frac{1}{3}b*(c*a)-\frac{2(\omega-1)}{9}S(a*c,b)+$$

$+\frac{1}{3}(a*c)*b+\frac{\omega}{3}b*(a*c)$. Since S is associative over B, we have: $S(c,a*b)=S(c*a,b)$ şi $S(c,b*a)=S(b*a,c)=S(b,a*c)$. We get :

$$c(ab)-(ca)b=-\frac{1}{3}[(b*c)*a-b*(c*a)]-$$

$$-\frac{\omega}{3}[(b*a)*c-b*(a*c)]+\frac{1}{3}[(c*b)*a-c*(b*a)]-$$

$$-\frac{\omega^2}{3}c*(a*b)+\frac{\omega^2}{3}(c*a)*b=$$

$$=-\frac{1}{3}(b,c,a)^*-\frac{\omega}{3}(b,a,c)^*+\frac{1}{3}(c,b,a)^*+$$

$$+\frac{\omega^2}{3}(c,a,b)^*-\frac{\omega^2}{3}(c*a)*b+\frac{\omega^2}{3}(c*a)*b=$$

$$=-\frac{1}{3}(b,c,a)^*+\frac{1}{3}(c,b,a)^*+\frac{1}{3}(b,a,c)^*.$$

Here, we used for the flexible algebra, A that $(a,b,c)=-(c,b,a)$.□

Case 2.

We consider the field K such that $\omega\notin K$; therefore $q\notin K$. Let $F=K(\omega)=$ $=K(q)$ and A be an alternative algebra of degree three over the field K, equipped with an involution J of second kind. Then J is an automorphism

$$J:A\to A,\ J(xy)=J(y)J(x),\ J(\alpha x)=\alpha^J J(x),\ J(k)=k,\ \forall k\in K,\ \alpha\in F.$$

Since $K\subset F$ is an extension of second degree, the associated Galois group has the degree two, and $\alpha\to\alpha^J$ is a K-morphism over F of second degree, which differs of identity, since $\omega^J = \omega^2$ and $q^J = -q$. Let

$$\bar{A}_0 = \{x \in A/\ T(x) = 0 \text{ and } J(x) = -x\}.$$

First we recall a result given by Elduque and Myung:

Proposition 7. [2] *Let $a, b \in \bar{A}_0$, such that*

$$a * b = \omega ab - \omega^2 ba - \frac{2\omega + 1}{3} T(ab) \cdot 1.$$

Then $(\bar{A}_0, *)$ is a K-algebra, $S\mid_{\bar{A}_0}$ is a nondegenerate quadratic form over K and S permits composition over $(\bar{A}_0, *)$.□

Proposition 8. *With the same assumptions and notations as above, the following equalities hold:*

i) $A = \bar{A} \oplus q\bar{A} \simeq F \otimes_K \bar{A}$.

ii) $A_0 = F \otimes_K \bar{A}_0$.□

Remark 9. i) By Proposition 7 we have that $S/_{\bar{A}}$ and $S/_{\bar{A}_0}$ are quadratic forms over K. Since the F-algebra $(A_0, *)$ satisfies the conditions *i)-iii)* in Proposition 3, $(\bar{A}_0, *)$ satisfies the same conditions, $\bar{A}_0$ being a K-subalgebra of A_0. Then we have:

1) $(a * b) * a = a * (b * a) = S(a) b, \forall a, b \in (\bar{A}_0, *)$.

2) S permits composition and is associative over $(\bar{A}_0, *)$.

3) $(\bar{A}_0, *)$ does not have the unity element.

We prove a property of the alternative algebras in both cases 1 and 2.

Propozition 10. *Let $(A, *)$ be a nonunitary algebra over the field K, which has a bilinear nondegenerate form S satisfying the condition (3).*

If the elements $\{x, x * x\}$ are linearly dependent for each $x \in A$, then

$$x * x = \alpha(x) x,$$

and $\alpha : A \to K$ defined by this relations is a K-algebra morphism.

Proof. If in the relation (3) we take $a = b = x$, we get:

$(x * x) * x = S(x) x$. It results $(\alpha(x) x) * x = S(x) x$ so $\alpha^2(x) x = S(x) x$, and we have

$$\alpha^2(x) = S(x) \tag{6}$$

Let $x, y \in A$ be arbitrary elements. Then:

$(x + y) * (x + y) = \alpha(x + y)(x + y)$, therefore

$x * x + x * y + y * x + y * y = \alpha(x + y) x + \alpha(x + y) y$, hence

$\alpha(x) x + \alpha(y) y + x * y + y * x = \alpha(x + y) x + \alpha(x + y) y$, and we obtain:

$$(\alpha(x+y) - \alpha(x)) x + (\alpha(x+y) - \alpha(y)) y = x * y + y * x, \forall x, y \in A. \tag{7}$$

Since

$S(x,y) = \frac{1}{2}(S(x+y) - S(x) - S(y)) = \frac{1}{2}\left(\alpha^2(x+y) - \alpha^2(x) - \alpha^2(y)\right),$

we have:

$$\alpha^2(x+y) = \alpha^2(x) + \alpha^2(y) + 2S(x,y), \forall x, y \in A. \tag{8}$$

Applying $S(\,.\,, x)$ to the relation (7), we obtain:

$(\alpha(x+y) - \alpha(x))\, S(x,x) + (\alpha(x+y) - \alpha(y))\, S(x,y) =$

$= S(x*y, x) + S(y*x, x)$. The conditions from the hypothesis involve that $S(.\,,.)$ is associative and permits composition, no matter whether ω is in K or not (we use the Proposition 3 and the Remark 9 i)). So

$S(x*y, x) = S(x, x*y) = S(x*x, y) = \alpha(x)\, S(x,y)$, and

$S(y*x, x) = S(y, x*x) = \alpha(x)\, S(x,y)$.

We have, for all $x, y \in A$:

$$(\alpha(x+y)\text{-}\alpha(x))\alpha^2(x) + (\alpha(x+y)\text{-}\alpha(y))S(x,y) = 2\alpha(x)S(x,y). \tag{9}$$

We note that $\alpha(x+y) = z, \alpha(x) = a, \alpha(y) = b$.

Then (8) and (9) become respectively:

$$z^2 = a^2 + b^2 + 2S(x,y), \tag{10}$$

$$(z-a)\, a^2 = (b + 2a - z)\, S(x,y). \tag{11}$$

We suppose that $\alpha(x) \neq 0$. If $b + 2a - z = 0$, then $\alpha(y) + 2\alpha(x) = \alpha(x+y)$, for all $x, y \in A, x \neq 0$. If we take $y = 0$, we obtain $2\alpha(x) = \alpha(x)$, so $\alpha(x) = 0$, false. Then $b + 2a - z \neq 0$.

By the relation (11), we have $S(x,y) = \dfrac{(z-a)\, a^2}{b + 2a - z}$ and replace it in the relation (10). We get $z^2 = a^2 + b^2 + 2\dfrac{(z-a)\, a^2}{b + 2a - z}$, so $(b\text{+}2a\text{-}z)\, z^2 =$

$= \left(a^2 + b^2\right)(b\text{+}2a\text{-}z) + 2a^2(z\text{-}a)$ and we have

$$-z^3 + (b + 2a)\, z^2 + \left(b^2 - a^2\right) z - b\,(a+b)^2 = 0.$$

The polynomial $P(Z) = \text{-}Z^3 + (b\text{+}2a)\, Z^2 + \left(b^2\text{-}a^2\right) Z\text{-}b\,(a\text{+}b)^2$ has the decomposition in irreducible factors: $P(Z) = \text{-}(Z\text{-}a\text{-}b)^2\,(Z\text{+}b)$. It results that $z_1 = a + b$ and $z_2 = -b$ are the different roots of the polynomial P. If $z = a + b$, we have $\alpha(x+y) = \alpha(x) + \alpha(y)$. If $z = -b$, we obtain $\alpha(x+y) = -\alpha(y)$ and, for $y = 0$, it result $\alpha(x) = 0$, which is a contradiction. Therefore

$$\alpha(x+y) = \alpha(x) + \alpha(y), \forall x, y \in A,$$

with $\alpha(x) \neq 0$. If $\alpha(x) = 0$, by relation (9), we have $(z - b) S(x, y) = 0$, for all $y \in A$ and $x \in A$ with $\alpha(x) = 0$. If $z = b$, then

$$\alpha(x + y) = \alpha(y) = \alpha(y) + \alpha(x).$$

If $S(x, y) = 0, \forall y \in A,$ we obtain $x = 0$ and $\alpha(0) = 0,$ therefore

$$\alpha(x + y) = \alpha(x) + \alpha(y), \forall x, y \in A. \tag{12}$$

By the relation (7), for $y = ax$, with $a \in K$, we have
$\alpha((a + 1)x)x - \alpha(x)x + a\alpha((a + 1)x)x - a\alpha(ax)x = 2a\alpha(x)x$, hence
$\alpha(ax)x + \alpha(x)x - \alpha(x)x + a\alpha(ax)x + a\alpha(x)x - a\alpha(ax)x = 2a\alpha(x)x$, and we get

$$\alpha(ax) = a\alpha(x). \tag{13}$$

Since $S(.,.)$ permits composition, it results that $S(x * y, x * y) =$
$= S(x, x) S(y, y)$, hence $S(x * y) = S(x, x) S(y, y)$.Therefore

$$\alpha(x * y) = \alpha(x) \alpha(y). \tag{14}$$

By relations $(12), (13), (14)$ it results that α is a K-morphism .□

With the notations in the Proposition 10, we note that S is nondegenerate over $\bar{A}$ if and only if S is nondegenerate over A. Moreover, we have:

Proposition 11. [2] *Let $A, A_0, \bar{A}, \bar{A}_0$ be the algebras defined above. Then there exists $a \in \bar{A}_0$ such that $\{a, a * a\}$ is linearly independent and A is finite-dimensional and separable if and only if S is nondegenerate.*□

Proposition 12. [2] *Let $(B, *)$ be an algebra over the field K , $q \notin K$. If B has a quadratic form S over K satisfying the relation (3), for all $a, b \in B$, and there exists $x_0 \in B$ such that $\{x_0, x_0 * x_0\}$ is a linearly independent system, then there exists an alternative algebra A of degree three over $F = K(q)$ equipped with an involution J of second kind such that $(B, *)$ is isomorphic with the algebra $\left(\bar{A}_0, *\right)$ defined above.*□

Proposition 13. [2] *The algebras $(B_1, *), (B_2, *)$ in Proposition 12 are isomorphic if and only if the alternative algebras of degree three with the corresponding involution are isomorphic.*□

Now we can state the main results of the paper.

Proposition 14. *The algebra A in Proposition 12 is associative if and only if $(a, c, b)^* + (b, a, c)^* = (a, b, c)^*, \forall a, b, c \in B$.*

Proof. If the relation in the hypothesis is true for all $a, b, c \in B$, it results that the this proposition is true and for all $x, y, z \in F \otimes_K B$. Then we use the Proposition 6 and A is an associative F-algebra. Indeed, let
$x, y, z \in F \otimes_K B, x = \alpha \otimes a,\ y = \beta \otimes b,\ z = \gamma \otimes c,\ \alpha, \beta, \gamma \in F,\ a, b, c \in B$. Then in $F \otimes_K B$, we have
$(x, y, z)^*$=$((\alpha \otimes a) * (\beta \otimes b)) * (\gamma \otimes c)$-$(\alpha \otimes a) * ((\beta \otimes b) * (\gamma \otimes c))$=

$= \alpha\beta\gamma \otimes (a * b) * c - \alpha\beta\gamma \otimes a * (b * c) = \alpha\beta\gamma \otimes (a, b, c)^* .\square$

Proposition 15. *Let K be a field , $\omega \notin K$ and $F{=}K(\omega)$. If A is a finite dimensional algebra of degree three equipped with an involution J of second kind, $\bar{A}_0 = \{x \in A /\ T(x) = 0,\ J(x) = -x\}$ then the multiplication:*

$$a * b = \omega ab - \omega^2 ba - \frac{2\omega + 1}{3} T(ab) \cdot 1, \forall\, a, b \in \bar{A}_0 \textit{ can be defined.}$$

If A is separable, then the quadratic form S is nondegenerate and $(\bar{A}_0, *)$ becomes a composition K-algebra . If dim $A = 9$, then we may define the multiplication ∇ on $\bar{A}_0$ such that $(\bar{A}_0, \nabla)$ is an octonionic K-algebra. If $(a, c, b)^* + (b, a, c)^* = (a, b, c)^*, \forall a, b, c \in \bar{A}_0$, then A is associative.

Proof. By Proposition 7, we have that $\bar{A}_0$ is a K-algebra and $S/_{\bar{A}_0}$ is a quadratic form over K permitting composition. By Proposition 11, we have that $(\bar{A}_0, *)$ is flexible, and the associated bilinear form to S is associative. If S is nondegenerate, then $(\bar{A}_0, *)$ is a non-unitary composition algebra and there exists $u \in \bar{A}_0$, such that $S(u) \neq 0$. Then we define:

$$a \nabla b = (u * a) * (b * u),$$

with $a, b \in \bar{A}_0$ and we get that $(\bar{A}_0, \nabla)$ is a unitary composition algebra, with the unity $e = u * u$, therefore it is a Hurwitz algebra. If dim $A = 9$, then, by Proposition 8, we have that *dim* $A =$ *dim* $\bar{A} =$ *dim* $A_0 + 1 =$ *dim* $\bar{A}_0 + 1$,
therefore dim $\bar{A}_0 = 8$ and $\bar{A}_0$ is an octonion algebra.$\square$

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