Bortolotti's, Cartan's and Norden's Normalizations of Manifolds of the Projective space

S.I. Sokolovskaya

Abstract

The main result in the paper states that if there exists a nonvanishing relative invariant enveloped by the fundamental object of a submanifold of the projective space, then the field of normals of the 2-d type of the Norden normalization induces Bortolotti and Norden normalizations of the submanifold.

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1 The case of Bortolotti normalization

Consider n-dimensional projective space P_n. The structure equations of this space are:

(1) $$d\omega_{\bar{i}}^{\bar{j}} = \omega_{\bar{i}}^{\bar{k}} \wedge \omega_{\bar{k}}^{\bar{j}} \qquad (\bar{i}, \bar{j}, \bar{k} = \overline{0, n}), \qquad \omega_{\bar{i}}^{\bar{i}} = 0.$$

Consider m-dimensional submanifold S_m in the P_n equipped with a first-order frame A_0, A_1,..., A_m, A_{m+1},..., A_n, point A_0 of which coincides with the moving point A_0 of the submanifold S_m, points A_ξ (ξ=1,.., m) are located in the tangent m-plane $\mathrm{TA}_0(\mathrm{S}_m)$ to the submanifold S_m at the point A_0, and points A_u (u=m+1,.., n) are located outside of the tangent m-plane $\mathrm{TA}_0(\mathrm{S}_m)$ and are lineary independent.

Relative to such a frame, the equations of S_m are:

(2) $$\omega_0^u = 0; \qquad (u, v, w = \overline{m+1, n})$$
$$\omega_\xi^u = \Lambda_{\xi\eta}^u \omega_0^\eta; \qquad (\xi, \eta, \zeta = \overline{1, m})$$
$$(\Lambda_{\xi\eta}^u = \Lambda_{\eta\xi}^u).$$

The components of the fundamental object of the submanifold satisfy the following differential equations:

$$d\Lambda_{\xi\eta}^u + \Lambda_{\xi\eta}^u \omega_0^0 + \Lambda_{\xi\eta}^v \omega_v^u - \Lambda_{\xi\zeta}^u \omega_\eta^\zeta - \Lambda_{\zeta\eta}^u \omega_\xi^\zeta = \Lambda_{\xi\eta\zeta}^u \omega_0^\zeta;$$
$$\Lambda_{\xi[\eta\zeta]}^u = 0.$$

If there exists a nonvanishing relative invariant I [1] enveloped by the fundamental object of the submanifold S_m, then there exists the inverse fundamental object of the

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second order with components $V^{\xi\eta}_{u} = \dfrac{\partial \ln I}{\partial \Lambda^{u}_{\xi\eta}}$ which satisfy the following differential equations:

$$dV^{\xi\eta}_{u} - V^{\xi\eta}_{u}\omega^{0}_{0} - V^{\xi\eta}_{v}\omega^{v}_{u} + V^{\xi\zeta}_{u}\omega^{\eta}_{\zeta} + V^{\zeta\eta}_{u}\omega^{\xi}_{\zeta} = V^{\xi\eta}_{u\zeta}\omega^{\zeta}_{0};$$
$$V^{[\xi\eta]}_{u\zeta} = 0.$$

In addition, the following conditions are fulfilled

$$V^{\xi\eta}_{u}\Lambda^{u}_{\zeta\eta} = (n-m)\delta^{\xi}_{\zeta}; \qquad V^{\xi\eta}_{u}\Lambda^{v}_{\xi\eta} = m\delta^{v}_{u}.$$

Theorem 1 *If there exists a nonvanishing relative invariant enveloped by the fundamental object of a submanifold S_m of the projective space P_n, then the field of normals of the 2-d type of the Norden normalization induces a Bortolotti normalization of the submanifold S_m.*

Proof. To the moving point A_0 of the submanifold S_m, we put in correspondence an (m-1)-plane, belonging to the tangent m-plane to the submanifold S_m at the point A_0 and not passing through the point A_0, i.e., we construct the field of the elements of the Norden normalization of the 2-d type [2]. If the points A_1,.., A_m of the moving frame belong to the corresponding element of the 2-d type of the Norden normalization, then the equations of the field of the Norden normals of the 2-d type in the specialized frame are:

(3) $$\omega^{0}_{\eta} = L^{0}_{\eta\xi}\omega^{\xi}_{0}.$$

Exterior differentiation of the equations (3) gives differential equations for the system of the values $L^{0}_{\eta\xi}$:

$$dL^{0}_{\xi\eta} + 2L^{0}_{\xi\eta}\omega^{0}_{0} - L^{0}_{\xi\zeta}\omega^{\zeta}_{\eta} - L^{0}_{\zeta\eta}\omega^{\zeta}_{\xi} + \Lambda^{v}_{\xi\eta}\omega^{0}_{v} = L^{0}_{\xi\eta\zeta}\omega^{\zeta}_{0};$$
$$L^{0}_{\xi[\eta\zeta]} = 0.$$

Consider the geometrical object whose components $L^{0}_{u} = \frac{1}{m}V^{\xi\eta}_{u}L^{0}_{\xi\eta}$satisfy the following differential equations

(4) $$dL^{0}_{u} + L^{0}_{u}\omega^{0}_{0} - L^{0}_{v}\omega^{v}_{u} + \omega^{0}_{u} = L^{0}_{u\zeta}\omega^{\zeta}_{0};$$
$$L^{0}_{u\zeta} = \tfrac{1}{m}(V^{\xi\eta}_{u}L^{0}_{\xi\eta\zeta} + V^{\xi\eta}_{u\zeta}L^{0}_{\xi\eta}).$$

If the field of Norden's normals of the 2-d type is constructed on the submanifold S_m, then there exists the field of the hyperplanes, moving hyperplane α of which is passing through the points A_ξ and $\widetilde{A_u} = A_u + L^{0}_{u}A_0$ and not passing through the point A_0. The hyperplane α is invariant, since it is easy to check, that

$$dA_\xi = \omega^{\eta}_{\xi}A_\eta (\operatorname{mod} \omega^{\xi}_{0});$$
$$d\widetilde{A}_u = \omega^{v}_{u}\widetilde{A}_v + \omega^{\xi}_{u}A_\xi (\operatorname{mod} \omega^{\xi}_{0}).$$

The hyperplane α is called Bortolotti hyperplane [3]. □

2 The case of Norden normalization

Theorem 2 *If there exists a nonvanishing relative invariant enveloped by a fundamental object of the submanifold of the projective space, then the field of normals of the 2-d type of the Norden normalization induces the Norden normalization of the submanifold.*

Proof. If we specialize the moving frame of the projective space by superposing the points A_u and $\widetilde{A}_u$, then $L^{0}_{u} = 0$ and equations (4) show that

(5) $\omega^{\ 0}_{u} = L^{\ 0}_{u\zeta}\omega^{\ \zeta}_{0}$.

Exterior differentiation of the equations (5) gives differential equations for the system of the values $L^{\ 0}_{u\xi}$:

$$dL^{\ 0}_{u\xi} + 2L^{\ 0}_{u\xi}\omega^{\ 0}_{0} - L^{\ 0}_{v\xi}\omega^{\ v}_{u} - L^{\ 0}_{u\eta}\omega^{\ \eta}_{\xi} - L^{\ 0}_{\zeta\xi}\omega^{\ \zeta}_{u} = L^{\ 0}_{u\xi\eta}\omega^{\ \eta}_{0};$$
$$L^{\ 0}_{u[\eta\zeta]} = 0.$$

$J = \left|L^{\ 0}_{\xi\eta}\right|$ is relative invariant, since it satisfies the differential equation: $dJ = 2(\sum\limits_{\xi=1}^{m}\omega^{\ \xi}_{\xi} - m\omega^{\ 0}_{0})J + J_{\xi}\omega^{\ \xi}_{0}$.

Since the differential forms $\omega^{\ 0}_{\eta}$ are linearly independent, then $J \neq 0$. Moreover, there exists the inverse tensor with components $L^{\xi\eta}_{\ 0} = \frac{\partial \ln J}{\partial L^{\ 0}_{\xi\eta}}$ satisfying the differential equations $dL^{\xi\zeta}_{\ 0} - 2L^{\xi\zeta}_{\ 0}\omega^{\ 0}_{0} + L^{\xi\eta}_{\ 0}\omega^{\ \zeta}_{\eta} + L^{\eta\zeta}_{\ 0}\omega^{\ \xi}_{\eta} = L^{\xi\zeta}_{0\eta}\omega^{\ \eta}_{0}$
and $L^{\xi\eta}_{\ 0}L^{\ 0}_{\zeta\eta} = 2\delta^{\ \xi}_{\zeta}$, $\quad L^{\ 0}_{\zeta\eta}L^{\zeta\xi}_{\ 0} = 2\delta^{\ \xi}_{\eta}$.

We consider the geometrical object with components $L^{\ \zeta}_{0u} = L^{\zeta\xi}_{\ 0}L^{\ 0}_{u\xi}$, which satisfy the following differential equations:

(5) $\quad dL^{\ \zeta}_{0u} + L^{\ \eta}_{0u}\omega^{\ \zeta}_{\eta} - L^{\ \zeta}_{0v}\omega^{\ v}_{u} - \omega^{\ \zeta}_{u} = -L^{\ \zeta}_{u\eta}\omega^{\ \eta}_{0};$
$\quad -L^{\ \zeta}_{u\eta} = L^{\zeta\xi}_{0\eta}L^{\ 0}_{u\zeta} + L^{\zeta\xi}_{\ o}L^{\ 0}_{u\xi\eta}.$

It is easy to check, that the differential equations of the infinitesimal displacement of the points $\widetilde{\widetilde{A}}_u = A_u - L^{\ \zeta}_{0u}A_{\zeta}$ are: $d\widetilde{\widetilde{A}}_u = \omega^{\ v}_{u}\widetilde{\widetilde{A}}_u (\operatorname{mod}\omega^{\ \xi}_{0})$.

We specify the moving frame of the projective space by superposing point $\widetilde{\widetilde{A}}_u$ with A_u, then $L^{\ \zeta}_{0u} = 0$ and equations (5) are $\omega^{\ \zeta}_{u} = L^{\ \zeta}_{u\eta}\omega^{\ \eta}_{0}$.

This means that the field of normals 2-d type of the Norden normalization induces field of normals 1-st type of the Norden normalization. □

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Author's address:

S.I. Sokolovskaya
Department of Mathematical Calculus,
Faculty of Mathematics and Mechanics,
Lomonosov Moscow State University,
Vorobjow Gory, 119899, Moscow, Russia.