Some recent contributions to CR-submersions

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Abstract

In this survey work we give some our recent contributions on submersions of CR-, or QR-submanifolds, as well as some results of other authors on the topic.

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The study of Riemannian submersions was initiated by B. O'Neill ([21]) and A. Gray ([13]). This theory was developed very much in the last thirty five years. A good reference is chapter 9 of Besse's book [7]; see also some recent papers [1], [2].
We recall that a Riemannian submersion yields a vertical distribution $\mathcal{V}$ which is integrable and a horizontal distribution $\mathcal{H}$ (see [7], p. 236). On the other hand, on a CR-submanifold M of a Kähler manifold $(\tilde{M},\tilde{g},\mathcal{J})$ there are two orthogonal complementary distributions $\mathcal{V}$ and $\mathcal{H}$, such that $\mathcal{H}$ is $\mathcal{J}$-invariant and $\mathcal{V}$ is totally real (cf. A. Beyancu [5]).
Recently, S. Kobayashi considered the similarity between the total space of a Riemannian submersion and a CR-submanifold of a Kähler manifold in terms of distributions ([15]).

Let $TM^{\perp}$ be the normal bundle of M in $\tilde{M}$. We denote by μ the orthogonal complementary vector bundle to $\mathcal{J}(\mathcal{V})$ in $TM^{\perp}$, i.e. $TM^{\perp} = \mathcal{J}(\mathcal{V}) \oplus \mu$. It is clear that μ is a holomorphic sub bundle of $TM^{\perp}$, i.e. $\mathcal{J}(\mu) = \mu$.

Definition 1. Let M be a CR-submanifold of a Kähler manifold $(\tilde{M}, g, \mathcal{J})$. A *CR-submersion* from a CR-submanifold M onto an almost Hermitian manifold $(M', g', \mathcal{J}')$ is a Riemannian submersion $\pi : M \to M'$, such that:

(i) $\mathcal{V}$ is the kernel of π_* ;

(ii) for each $x \in M$, π_*: $\mathcal{H}_x \to T_{\pi(x)}M'$ is a complex isometry, i.e. $\pi_* \circ \mathcal{J} = \mathcal{J}' \circ \pi_*$.

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The above definition is given by S. Kobayashi in the case where μ is a null sub bundle of $TM^{\perp}$(see [15]). If $\mathcal{J}\mathcal{V}_x = T_xM^{\perp}$, for any $x \in M$ we say that M is a *generic CR-submanifold of* $\tilde{M}$ (cf. [23]). For example, any real orientable hypersurface of $\tilde{M}$ is a generic CR-submanifold of $\tilde{M}$ (see also [16]). The vertical distribution $\mathcal{V}$ of a Riemannian submersion is the kernel of π_* , so that $\mathcal{V}$ is an integrable distribution. D.B. Blair and B.Y. Chen have proved that any totally real distribution $\mathcal{V}$ of a CR-submanifold M of a Kähler manifold $\tilde{M}$ is always integrable ([3]).
We have the first basic result:

Theorem 1. *Let M be a CR-submanifold of a Kähler manifold $\tilde{M}$ and let $\pi : M \to M'$ be a CR-submersion of M onto an almost Hermitian manifold M'. Then M' is a Kähler manifold.*

This theorem is proved for the generic case $\mathcal{V} = \{0\}$ in [15] and for the case $\mathcal{V} \neq \{0\}$ in [16].
In the generic case, another contribution was given in [15] on the relation between the holomorphic sectional curvatures of $\tilde{M}$ restricted to $\mathcal{H}$ and those of M'. Namely, one has shown the following formula:

$$\tilde{K}(X) = K'(\pi_*X) - 4\|B(X,X)\|^2, \tag{1}$$

for any unit horizontal vector X, where $\tilde{K}$ and K' are the holomorphic sectional curvatures of $\tilde{M}$ and M' respectively, and B is the second fundamental form of M in $\tilde{M}$.

Now we will present our investigations on the CR-submersions from an extrinsic hypersphere of an Einstein-Kähler manifold. We say that a totally umbilical hypersurface M of a Riemannian manifold $\tilde{M}$ is an *extrinsic hypersphere* iff the mean curvature vector field H is parallel and $H_x \neq 0$, for any $x \in M$. Many of the basic results concerning extrinsic spheres in Riemannian and Kählerian geometry were obtained by B.Y. Chen ([8]). We have,

$$B(E,F) = g(E,F)H, \tag{2}$$

for any couple of vector fields E, F on M. If we put $k = \|H\|$ (where the norm $\|\cdot\|$ is respect to scalar product induced by g on every tangent space to M), then $\xi = -\mathcal{J}N$ is a global unit vector on M.
We see that M is a CR- hypersurface of $\tilde{M}$ such that $\mathcal{V}$ is the one dimensional anti-invariant distribution generated by the vector field ξ.
Then, in [16] we proved the following theorem:

Theorem 2. *Let M be an orientable extrinsic hypersphere of a Kähler-Einstein manifold $\tilde{M}$. If $\pi = M \to M'$ is a CR-submersion of M onto an almost Hermitian* *manifold M', then M' is a Kähler-Einstein manifold.*

Now we suppose that $\tilde{M}$ is a complex space form. Then we may state (see [9]):

Theorem 3. *Let $\pi : M \to M'$ be a CR-submersion of a totally umbilical CR-submanifold ($dimM \geq 5$) of a complex space form $\tilde{M}(c)$ onto an almost Hermitian manifold M'. Then M' is also a complex space form.*

A CR-submanifold M of a Kähler manifold $\tilde{M}$ is said to be a *mixed foliate* if $\mathcal{H}$ is an integrable distribution and $B(U, X) = 0$ for any $U \in \mathcal{V}_x$, $X \in \mathcal{H}_x$, $x \in M$. In [9] the authors proved the following result.

Theorem 4. *Let M be a mixed foliate CR-submanifold of a Kähler-Einstein manifold $\tilde{M}$. If $\pi : M \to M'$ is a CR-submersion of M onto an almost Hermitian manifold M', then M' is also a Kähler-Einstein manifold.*

Remarks.

An extrinsic hypersphere of a Kähler-Einstein manifold $\tilde{M}$ is not a mixed foliate CR-submanifold (see Th. 2.).

In [10], the authors studied similar problems for CR-submanifolds in Hermitian, quasi-Kähler or nearly Kähler manifolds. In this cases, totally real distributions are not necessarily integrable.
To overcome this difficulty the authors consider the submersions $\pi : M \to M'$ of CR-submanifolds M with an integrable distribution $\mathcal{V}$ onto an almost Hermitian manifold M'. For example, a real hypersurface in S^6 (which is a nearly Kähler manifold) is a CR-hypersurface with a 1-dimensional distribution $\mathcal{V}$ which is always integrable.

Now we describe some results obtained by F. Narita ([20]).
Let $\tilde{M}$ be a locally conformal Kähler manifold and let L be the Lee vector field on $\tilde{M}$ (cf. [11]). Then we have

Theorem 5. *Let M be a generic CR-submanifold of a locally conformal Kähler manifold $\tilde{M}$ and let $\pi : M \to M'$ be a CR-submersion of M onto an almost Hermitian manifold. Then the Lee vector L belongs to $\mathcal{H} \oplus TM^{\perp}$ and for any horizontal unit vector $X \in \mathcal{H}$ we have*

$$\tilde{K}(X) = K'(\pi_* X) - 3\|A_X \mathcal{J} X\|^2 - \|B(X, X)\|^2,$$

where A is the integrability tensor with respect to π.
Moreover, if we assume in addition that the horizontal component hL of L is basic and $dimM' \geq 4$, then M' is also a locally conformal Kähler manifold. In particular, if $\tilde{M}$ is a generalized Hopf manifold and if the Lee vector L is basic, then M' is also

a generalized manifold.

Finally, the concept of CR-submersion was extended to semi-invariant (or contact CR-manifold) submanifolds in the Sasakian geometry by N. Papaghiuc ([22]).
He obtained basic properties of CR-submersion of a semi-invariant submanifold of a Sasakian manifold onto an almost contact manifold and he studied various relations between the sectional curvatures of $\tilde{M}$ and M'.

In the last part of this survey paper we refer to some Riemannian submersions from a hypersurface of a quaternionic Kähler manifold. First, we recall some definitions.

We say that a $4(m+1)$-dimensional manifold $\tilde{M}$ with a metric g is a *quaternionic Kähler manifold* ($m \geq 1$), if there exists a 3-dimensional vector bundle $\mathcal{V}$ of tensors of type $(1,1)$ on $\tilde{M}$ satisfying the following conditions:

a) In any coordinate neighborhood $\tilde{U}$ on $\tilde{M}$ there is a local basis with almost Hermitian structures $\{\mathcal{J}_a, g\}$ such that $\mathcal{J}_a^2 = -Id$, $a \in \{1,2,3\}$ and $\mathcal{J}_a \circ \mathcal{J}_b = -\mathcal{J}_b \circ \mathcal{J}_a = \mathcal{J}_c$, for any cyclic permutation (a,b,c) of $(1,2,3)$.

b) For any local section φ of V and any tangent vector X to $\tilde{M}$, $\tilde{\nabla}_X \varphi$ is also a local section in V, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$ ([14], [17]).

Let M be an orientable hypersurface of $\tilde{M}$ and le ξ be a unit normal field defined on M. One obtains a distribution $\mathcal{V}$ on M which is locally represented by $\{\xi_a\}$, $1 \leq a \leq 3$, on $\tilde{U}$, where $\xi_a = -\mathcal{J}_a(\xi)$, $a \in \{1,2,3\}$. Let $\mathcal{H}$ be the orthogonal complementary distribution to $\mathcal{V}$ with respect to g on M.
We say that M is a QR-hypersurface of $\tilde{M}$ (cf. [6]).
Firstly, we remark that a real hypersurface of a quaternionic Kähler manifold is not a CR-hypersurface of $\tilde{M}$.

Definition 2. Let M be a QR-hypersurface of a quaternionic Kähler manifold $\tilde{M}$, such that the vertical distribution $\mathcal{V}$ is integrable. We say that a Riemannian submersion $\pi : M \to M'$ is a *QR-submersion* if the following conditions are satisfied:

i) $\mathcal{V}$ is the kernel of π_*;

ii) for each x, $\pi_* : \mathcal{H}_x \to T_{\pi(x)}M'$ is an isometry with respect to each complex structure of $\mathcal{H}_x$ and $T_{\pi(x)}M'$.

It is well known that the vertical distribution $\mathcal{V}$ is integrable if and only if $B(U,X) = 0$ for any $U \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$. In this case we say that M is a *mixed geodesic* QR-hypersurface.
We proved the following result ([17]).

Theorem 6. *Let M be a mixed geodesic QR-hypersuface of a quaternionic Kähler manifold $\tilde{M}$. If $\pi : M \to M'$ is a QR-submersion of M on an almost quaternionic*

Hermitian manifold, then M' is a quaternionic Kähler manifold .

Theorem 7. *Let M be a totally umbilical, but not totally geodesic, QR-hypersurface of a quaternionic Kähler manifold. Then,*

a) $\tilde{K}(U,V) = K(U,V) - \|H\|^2$, *where $\{U,V\}$ is an orthonormal basis of the vertical $2-$plane α, $\alpha \in \mathcal{V}_x, x \in M$ and $\tilde{K}, K$ denote the sectional curvatures of α on $\tilde{M}, M$, respectively.*

b) $K(X,Y) = K'(X',Y') - 3\|H\|^2 \sum_{a=1}^{3} < X, \mathcal{J}_a Y >^2$, *where X, Y is an orthonormal basis of a horizontal $2-$plane $\alpha \in \mathcal{H}_x$, $K(X,Y)$ denotes the sectional curvature of α, and $K'(X',Y')$ denotes the sectional curvature in M' of the $2-$plane spanned by $X' = \pi_* X$ and $Y' = \pi_* Y$.*

Theorem 8. *Let M be an extrinsic hypersurface of a flat quaternionic Kähler manifold $\tilde{M}$ and let $\pi : M \to M'$ be a QR-submersion of M on a quaternionic Kähler manifold M'.*
Then M' is a quaternionic Kähler manifold with constant quaternionic sectional curvature $c > 0$, ($c = \|H\|$).

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