CONGRUENCES OF THE NAKANO SUPERLATTICE AND SOME OF THEIR PROPERTIES

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Abstract

We investigate properties of the *Nakano* superlattice $(H, \sqcup, \sqcap)$, where $\sqcup, \sqcap$ are the Nakano *hyperoperations* $x \sqcup y = \{z : x \vee z = y \vee z = x \vee y\}$, $x \sqcap y = \{z : x \wedge z = y \wedge z = x \wedge y\}$. We study properties of the congruences on the Nakano superlattice as well as the quotients which such congruences generate. Some new hyperoperations are introduced on the quotients and related to the quotients.

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1 Introduction

Mittas and Konstantinidou have introduced superlattices in [5]. A superlattice is a partially ordered set $(H, \leq)$ equipped with two hyperoperations (i.e. operations which to every pair of elements from H correspond a *set* of elements from H) which satisfy appropriate properties; alternatively, a superlattice is defined in terms of two hyperoperations which satisfy certain properties; these properties can be used to introduce a partial order $\leq$ on H.

Jakubik [2] studies several aspects of the theory of superlattices and defines *congruences on superlattices* (s-congruences) as a generalization of classical congruences (l-congruences). He shows that, in general, the quotient of a superlattice with respect to a s-congruence fails to be a superlattice; then he poses the following natural question: what are necessary and sufficient conditions for the quotient of a superlattice to also be a superlattice?

In this paper we specialize the above question in the context of the *Nakano superlattice*. This is a special type of superlattice, defined in terms of a hyperoperation

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$\sqcap$ first studied by Nakano [7] and its dual hyperoperation $\sqcup$ (we call these *Nakano hyperoperations*). The Nakano hyperoperations are defined on a lattice $(H, \leq)$ which is assumed in the remainder of the paper to be *modular*. In Section 2 we explore some properties of the Nakano superlattice $(H, \sqcup, \sqcap)$ and its quotient H/R with respect to a s-congruence R; if R is a s-congruence, then one can define certain hyperoperations $\overline{\curlyvee}, \overline{\curlywedge}$ on the quotient H/R. In Section 3 we find conditions which are necessary and sufficient for $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ to be a superlattice. In Section 4 we examine some order relationships on H/R.

2 Fundamental Concepts

Consider a modular lattice $(H, \leq)$, with sup and inf operations denoted by $\vee$ and $\wedge$ respectively. We define the *Nakano hyperoperations* $\sqcup, \sqcap$ on H.

Definition 1 For all $x, y \in H$ we define:

$$x \sqcup y \doteq \{z : z \vee x = z \vee y = x \vee y\}; \quad x \sqcap y \doteq \{z : z \wedge x = z \wedge y = x \wedge y\}.$$

Remark. To the best of our knowledge, the $\sqcap$ hyperoperation was first introduced by Nakano in [7], which is an investigation of hyperrings (multirings, in the author's terminology). Evidently, $\sqcup$ is the dual hyperoperation of $\sqcap$. The $\sqcup$ hyperoperation has also been studied in [3, 6] and it plays a central role in the theory of Boolean hyperrings and Boolean hyperlattices [4].

The hyperstructure $(H, \sqcup, \sqcap)$ is a *superlattice*, i.e. $\sqcup, \sqcap$ satisfy the conditions of the following proposition (note that in this proposition $\leq$ is the order of the original modular lattice). We will call $(H, \sqcup, \sqcap)$ the *Nakano superlattice*, since it makes use of the Nakano hyperoperations.

Proposition 2 $(H, \sqcup, \sqcap, \leq)$ *satisfies the following for all* $x, y, z \in H$.

S1 $x \in (x \sqcup x)$, $x \in (x \sqcap x)$.

S2 $x \sqcup y = y \sqcup x$, $x \sqcap y = y \sqcap x$.

S3 $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$, $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$.

S4 $x \in (x \sqcup y) \sqcap x$, $x \in (x \sqcap y) \sqcup x$.

S5 $x \leq y \Rightarrow y \in x \sqcup y$, $x \in x \sqcap y$.

S6 *If* $y \in x \sqcup y$ *or* $x \in x \sqcap y$, *then* $x \leq y$.

Proof. $S1, S2$ are obvious. The proof of $S3$ appears in [7] and also in [1].

Regarding $S4$ let us prove that $x \in (x \sqcup y) \sqcap x = \{z : z = u \sqcap x, u \in x \sqcup y\}$. In other words, we must show that exists some $u \in x \sqcup y$ such that $x \in u \sqcap x$. But this is easy. Taking $u = x \vee y$ one obtains immediately that $x \vee y \in x \sqcup y$; and since $(x \vee y) \wedge x = x \wedge x = (x \vee y) \wedge x$, it follows that $x \in (x \vee y) \sqcap x$. One can prove dually that $x \in (x \sqcap y) \sqcup x$ and this completes the proof of $S4$.

$x \leq y \Rightarrow x \wedge y = x,\ x \vee y = y$ and these yield $S5$ immediately. Regarding $S6$: $y \in x \sqcup y \Rightarrow\ x \vee y = y \vee y =\ x \vee y \Rightarrow x \leq y$; it can be proved dually that $x \in x \sqcap y$ $\Rightarrow x \leq y$. □

Proposition 3 $(H, \sqcup, \sqcap)$ *satisfies the following for all* $x, y, z \in H$.

S6′ $y \in x \sqcup y \Leftrightarrow x \in x \sqcap y$.

S7′ $x, y \in x \sqcup y \Leftrightarrow x = y$.

S8′ $y \in x \sqcup y, z \in y \sqcup z \Rightarrow z \in x \sqcup z$.

Furthermore, we have: $(S1 - S6) \Leftrightarrow (S1 - S4, S6' - S8')$.

Proof. The proof that $(S1 - S6) \Leftrightarrow (S1 - S4, S6' - S8')$ appears in [5]. The first part of the proposition then follows from Proposition 2. □

As shown in [5], Proposition 3 is true for *any* superlattice, not only the Nakano one. In fact a superlattice can be defined in two alternative ways [5]: one may assume the underlying order $\leq$ and require that the two hyperoperations satisfy properties $S1 - S6$; or one may assume that the hyperoperations satisfy $S1 - S4, S6' - S8'$ and then define an order on H in terms of the hyperoperations, in which case the resulting order satisfies $S5, S6$.

In the remainder of the paper we will use extensively the concepts of equivalence and congruence. Let us first give the following well known definitions for the sake of completeness.

Definition 4 An equivalence on H is a relationship R which satisfies the following $\forall x, y, z \in H$:

$$\text{(i) } xRx, \quad \text{(ii) } xRy \Rightarrow yRx, \quad \text{(iii) } xRy, yRz \Rightarrow xRz.$$

Definition 5 Let R be an equivalence on H. For all $x \in H$ the class of x is denoted by $\overline{x}$ and defined by $\overline{x} \doteq \{y : xRy\}$.

Definition 6 Let R be an equivalence on H. The quotient of H with respect to R is denoted by H/R and defined by $H/R \doteq \{\overline{x} : x \in H\}$.

Notation. We use the following notation: for all $A \subseteq H$ we write $\overline{A} \doteq \{\overline{x} : x \in A\}$
Remark. It follows from the above notation that for any $A, B \subseteq H$ such that $\overline{A} = \overline{B}$ we have:

(i) $\forall x \in A \quad \exists y \in B$ such that $\overline{x} = \overline{y}$, (ii) $\forall y \in B \quad \exists x \in A$ such that $\overline{x} = \overline{y}$.

Let us now turn to the concept of congruence. In classical lattice theory, congruence is defined in terms of operations; since here we will make use of both operations and hyperoperations, we will need *two* concepts of congruence. We use the term *l-congruence* to describe what is commonly called “congruence” (with respect to the operations $\vee, \wedge$) and the term *s-congruence* to describe the analogous property with respect to the hyperoperations $\sqcup, \sqcap$.

Definition 7 An equivalence R on H is called a l-congruence on a lattice $(H, \vee, \wedge)$ iff for all $x, y, z \in H$ we have

$$\overline{x} = \overline{y} \Rightarrow \begin{cases} \overline{x \vee z} = \overline{y \vee z} \\ \overline{x \wedge z} = \overline{y \wedge z} \end{cases}$$

Definition 8 An equivalence R on H is called a s-congruence on a superlattice $(H, \sqcup, \sqcap)$ iff for all $x, y, z \in H$ we have

$$\overline{x} = \overline{y} \Rightarrow \begin{cases} \overline{x \sqcup z} = \overline{y \sqcup z} \\ \overline{x \sqcap z} = \overline{y \sqcap z} \end{cases} .$$

In the remaining part of this work we will use the expression "s-congruence R on H" instead of the more correct, but also longer, "s-congruence R on a Nakano superlattice $(H, \sqcup, \sqcap)$".

We now define two new hyperoperations $\overline{\curlyvee}, \overline{\curlywedge}$ on the *quotient* H/R.

Definition 9 Given a s-congruence R on H we define hyperoperations $\overline{\curlyvee}, \overline{\curlywedge}$ as follows: for all $x, y \in H$

$$\overline{x}\overline{\curlyvee}\overline{y} \doteq \overline{x \sqcup y}; \quad \overline{x}\overline{\curlywedge}\overline{y} \doteq \overline{x \sqcap y}.$$

Remark. The above definition makes sense only if R is a s-congruence. Because, if for some $x, y \in H$ such that $\overline{x} = \overline{y}$ we had some $z \in H$ such that $\overline{x \sqcup z} \neq \overline{y \sqcup z}$, then we would have the following contradiction: $\overline{y}\overline{\curlyvee}\overline{z} = \overline{x}\overline{\curlyvee}\overline{z} = \overline{x \sqcup z} \neq \overline{y \sqcup z} = \overline{y}\overline{\curlyvee}\overline{z}$.

Remark. If the hyperoperations $\overline{\curlyvee}$, $\overline{\curlywedge}$ are well defined, then an s-congruence can be equivalently defined as follows.

Definition 10 An equivalence R on H is called a s-congruence on H iff for all $x, y, z \in H$ we have

$$\overline{x} = \overline{y} \Rightarrow \begin{cases} \overline{x}\overline{\curlyvee}\overline{z} = \overline{y}\overline{\curlyvee}\overline{z} \\ \overline{x}\overline{\curlywedge}\overline{z} = \overline{y}\overline{\curlywedge}\overline{z} \end{cases} .$$

It is worth noting that $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ is in general a proper hyperstructure, i.e. the $\overline{\curlyvee}, \overline{\curlywedge}$ hyperoperations yield non-singleton sets. This is seen from the following proposition.

Proposition 11 *Let $(H, \vee, \wedge)$ have either a maximum or a minimum element and R be a s- congruence on H. If $card(H/R) \geq 2$, then $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ is a proper hyperstructure.*

Proof. Assume H has maximum element denoted by 1. Then $1 \sqcup 1 = \{z : z \leq 1\} = H$ and $\overline{1 \sqcup 1} = \{\overline{z} : z \in H\} = H/R$. Hence $card(\overline{1 \sqcup 1}) \geq 2$ and so H/R is a proper hyperstructure. The same can be proved dually if H has minimum element denoted by 0. $\square$

Proposition 12 *Let R be a s-congruence on H, then $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ satisfies for all $x, y, z \in H$ the following:*

T1 $\overline{x} \in (\ \overline{x}\overline{\curlyvee}\overline{x}), \overline{x} \in (\ \overline{x}\overline{\curlywedge}\overline{x})$;

T2 $\overline{x}\overline{\curlyvee}\overline{y} = \overline{y}\overline{\curlyvee}\overline{x},\ \overline{x}\overline{\curlywedge}\overline{y} = \overline{y}\overline{\curlywedge}\overline{x}$;

T3 $(\overline{x}\overline{\curlyvee}\overline{y})\overline{\curlyvee}\overline{z} = x\overline{\curlyvee}(\overline{y}\overline{\curlyvee}\overline{z}),\ (\overline{x}\overline{\curlywedge}\overline{y})\overline{\curlywedge}\overline{z} = x\overline{\curlywedge}(\overline{y}\overline{\curlywedge}\overline{z})$;

T4 $\overline{x} \in (\ \overline{x}\overline{\curlyvee}\overline{y})\overline{\curlywedge}\overline{x}, \overline{x} \in (\ \overline{x}\overline{\curlywedge}\overline{y})\overline{\curlyvee}\overline{x}$.

Proof. It appears in [2]. □

Remark. In other words, if R is a s-congruence, then $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ satisfies the first four properties of a superlattice. A natural question is the following: what are necessary and sufficient conditions for $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ to actually be a superlattice? This question can be formulated more precisely in terms of the following properties $(T6' - T8')$.

$T6'$ $\overline{y} \in (\ \overline{x}\overline{\curlyvee}\overline{y}) \Leftrightarrow \overline{x} \in (\ \overline{x}\overline{\curlywedge}\overline{y})$.

$T7'$ $\overline{x}, \overline{y} \in (\ \overline{x}\overline{\curlyvee}\overline{y}) \Rightarrow \overline{x} = \overline{y}$.

$T8'$ $\overline{y} \in (\ \overline{x}\overline{\curlyvee}\overline{y}), \overline{z} \in (\overline{y}\overline{\curlyvee}\overline{z}) \Rightarrow \overline{z} \in (\ \overline{x}\overline{\curlyvee}\overline{z})$.

Question. What conditions must R satisfy so that $(T6' - T8')$ hold?

If $(T6' - T8')$ hold, then $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ is a superlattice and we can define the *order* relationship $\precsim$ as follows.

Definition 13 Let R be a s-congruence on H, such that $(T1 - T4, T6' - T8')$ hold. We write $\overline{x} \precsim \overline{y}$ iff $\overline{y} \in \overline{x}\overline{\curlyvee}\overline{y}$.

Furthermore, if we prove that $(T1 - T4, T6' - T8')$ hold, then we will know that the following conditions also hold.

$T5$ $\overline{x} \precsim \overline{y} \Rightarrow \overline{y} \in (\ \overline{x}\overline{\curlyvee}\overline{y}), \overline{x} \in (\ \overline{x}\overline{\curlywedge}\overline{y})$.

$T6$ $(\overline{y} \in (\ \overline{x}\overline{\curlyvee}\overline{y})$ or $\overline{x} \in (\ \overline{x}\overline{\curlywedge}\overline{y})\ ) \Rightarrow \overline{x} \precsim \overline{y}$.

3 Convex Congruences

We now explore the connection between convexity and conditions $(T6' - T8')$. Let us first give two definitions of convexity on ordered sets.

Definition 14 Given $A \subseteq H$, we say that A is w-convex iff: $\forall x, y \in A$ with $x \leq y$ and $\forall z$ such that $x \leq z \leq y$, we have $z \in A$.

Definition 15 Given $A \subseteq H$, we say that A is s-convex iff: $\forall x, y \in A$ and $\forall z$ such that $x \wedge y \leq z \leq x \vee y$, we have $z \in A$.

Remark. In other words, s-convexity is a stronger property than w-convexity: A is s-convex if it is a w-convex *sublattice* of H.

Let us now define convexity of congruence relationships.

Definition 16 Let R be an s-congruence on H. We say that R is w-convex iff $\forall x \in H$ we have that $\overline{x}$ is w-convex.

Definition 17 Let R be an s-congruence on H. We say that R is s-convex iff $\forall x \in H$ we have that $\overline{x}$ is s-convex.

We will now show that: R is w-convex iff $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ is a superlattice. To this end we will first prove the auxiliary Propositions 18 - 21.

Proposition 18 *Let R be a w-convex s-congruence on H. Then for all $x, y \in H$ we have:*

$$\textit{(i)}\ \overline{x} \in \overline{x}\overline{\curlyvee}\overline{y} \Leftrightarrow \overline{x} = \overline{x \vee y}; \quad \textit{(ii)}\ \overline{y} \in \overline{x}\overline{\curlywedge}\overline{y} \Leftrightarrow \overline{y} = \overline{x \wedge y}.$$

Proof. We only prove (i), since (ii) is proved dually.

Suppose that $\overline{x} \in \overline{x}\overline{\curlyvee}\overline{y} = \overline{x \sqcup y}$. This implies that exists some z such that $z \in x \sqcup y$ and $\overline{z} = \overline{x}$. Since $z \in x \sqcup y$ it follows: $z \vee x = z \vee y = x \vee y \Rightarrow \ z \vee x = z \vee x \vee y = x \vee x \vee y \Rightarrow \ x \vee y \in z \sqcup x \Rightarrow \overline{x \vee y} \in \overline{z \sqcup x} = \overline{x \sqcup x}$. Now $\overline{x \vee y} \in \overline{x \sqcup x}$ implies that exists some w such that $w \in x \sqcup x$ and $\overline{w} = \overline{x \vee y}$. Since $w \in x \sqcup x$ it follows: $x \vee w = x \vee x = x \Rightarrow w \leq x$. Hence we have $w \leq x \leq x \vee y$; since $\overline{w} = \overline{x \vee y}$, by w-convexity of R it follows that $\overline{x} = \overline{x \vee y}$.

Conversely, suppose $\overline{x} = \overline{x \vee y}$; we also have $x \vee y \in x \sqcup y$ and hence $\overline{x \vee y} \in \overline{x \sqcup y}$ $= \overline{x}\overline{\curlyvee}\overline{y}$. It follows that $\overline{x} \in \overline{x}\overline{\curlyvee}\overline{y}$. □

Proposition 19 *Let R be a w-convex s-congruence on H. Then for all $x, y \in H$ we have:*

$$\overline{x} \in \overline{x}\overline{\curlyvee}\overline{y} \Leftrightarrow \overline{y} \in \overline{x}\overline{\curlywedge}\overline{y}.$$

Proof. By Proposition 18 we have: $\overline{x} \in \overline{x}\overline{\curlyvee}\overline{y} \Rightarrow \overline{x} = \overline{x \vee y}$. Now, it is easy to check that for all $u, w \in H$ we have $u \wedge w \in u \sqcap w$. In particular, $y = (x \vee y) \wedge y \in (x \vee y) \sqcap y \Rightarrow \overline{y} \in \overline{(x \vee y) \sqcap y} = (\overline{x \vee y})\overline{\curlywedge}\overline{y} = \overline{x}\overline{\curlywedge}\overline{y}$. So we have shown that $\overline{x} \in \overline{x}\overline{\curlyvee}\overline{y} \Rightarrow \overline{y} \in \overline{x}\overline{\curlywedge}\overline{y}$. It can be shown dually that $\overline{y} \in \overline{x}\overline{\curlywedge}\overline{y} \Rightarrow \ \overline{x} \in \overline{x}\overline{\curlyvee}\overline{y}$. □

Proposition 20 *Let R be a w-convex s-congruence on H. Then for all $x, y \in H$ we have:*

$$\textit{(i)}\ \ \overline{x}, \overline{y} \in \overline{x}\overline{\curlyvee}\overline{y} \Rightarrow \overline{x} = \overline{y}; \quad \textit{(ii)}\ \ \overline{x}, \overline{y} \in \overline{x}\overline{\curlywedge}\overline{y} \Rightarrow \overline{x} = \overline{y}.$$

Proof. We only prove (i), since (ii) is proved dually. From Proposition 18 we have

$$\left.\begin{array}{l} \overline{x} \in \overline{x}\overline{\curlyvee}\overline{y} \Rightarrow \overline{x} = \overline{x \vee y} \\ \overline{y} \in \overline{x}\overline{\curlyvee}\overline{y} \Rightarrow \overline{y} = \overline{x \vee y} \end{array}\right\} \Rightarrow \overline{x} = \overline{y}.$$

This completes the proof. □

Proposition 21 *Let R be a w-convex s-congruence on H. Then for all $x, y, z \in H$ we have:*

$$\textit{(i)}\ \overline{y} \in \overline{x}\overline{\curlyvee}\overline{y}, \overline{z} \in \overline{y}\overline{\curlyvee}\overline{z} \Rightarrow \overline{z} \in \overline{x}\overline{\curlyvee}\overline{z}; \quad \textit{(ii)}\ \overline{x} \in \overline{x}\overline{\curlywedge}\overline{y}, \overline{y} \in \overline{y}\overline{\curlywedge}\overline{z} \Rightarrow \overline{x} \in \overline{x}\overline{\curlywedge}\overline{z}.$$

Proof. We only prove (i), since (ii) is proved dually. From Propositions 19 and 18 we have: (a) $\overline{y} \in \overline{x}\overline{\curlyvee}\overline{y} \Rightarrow \overline{x} \in \overline{x}\overline{\curlywedge}\overline{y} \Rightarrow \overline{x} = \overline{x \wedge y}$ and (b) $\overline{z} \in \overline{y}\overline{\curlyvee}\overline{z} \Rightarrow \overline{z} = \overline{y \vee z}$. Now $y \vee z = (x \wedge y) \vee (y \vee z) \in (x \wedge y) \sqcup (y \vee z)$. Using this and (a) and (b) we have $\overline{z} = \overline{y \vee z} \in \overline{(x \wedge y) \sqcup (y \vee z)} = \overline{x \sqcup z} = \overline{x}\overline{\curlyvee}\overline{z}$. $\square$

When R is a s-congruence, Property $T7'$ implies Properties $T6', T8'$, as can be seen from the next proposition.

Proposition 22 *Let R be a s-congruence on H. Then $T7' \Rightarrow T6', T8'$.*

Proof. We will first prove the following: if $T7'$ holds (i.e. for all $x, y \in H$ we have: $\overline{x}, \overline{y} \in \overline{x}\overline{\curlyvee}\overline{y} \Rightarrow \overline{x} = \overline{y}$) then $\forall x \in H$ we have that $\overline{x}$ is w-convex. To show this, take any $x \in H$ and any $y, z \in H$ such that: (a) $x \leq z \leq y$ and (b) $\overline{x} = \overline{y}$. We have: $z = x \vee z \in x \sqcup z \Rightarrow \overline{z} \in \overline{x \sqcup z} = \overline{y \sqcup z}$. Similarly, $y = y \vee z \in y \sqcup z \Rightarrow \overline{y} \in \overline{y \sqcup z}$. Hence (by $T7'$) $\overline{y} = \overline{z}$ and so $\overline{x}$ is w-convex. Since w-convexity of R implies $T6'$ by Proposition 19 and $T8'$ by Proposition 21, the proof is complete. $\square$

We are now ready to prove that: R is w-convex iff $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ is a superlattice.

Proposition 23 *Let R be a s-congruence on H. Then R is w-convex iff $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ is a superlattice.*

Proof. (i) Assume that R is w-convex. Then from Propositions 19, 20, 21 respectively, it follows that properties $T6', T7', T8'$ hold. Furthermore, since R is an s-congruence, $(T1 - T4)$ hold by Proposition 12. It follows that $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ satisfies $(T1 - T4, T6' - T8')$ and hence is a superlattice.

(ii) Assume $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ is a superlattice. Take any $x, y, z \in H$ such that: (a) $\overline{x} = \overline{y}$ and (b) $x \leq z \leq y$. Now $z = x \vee z \in x \sqcup z \Rightarrow \overline{z} \in \overline{x \sqcup z} = \overline{y \sqcup z}$; similarly $y = z \vee y \in z \sqcup y \Rightarrow \overline{y} \in \overline{z \sqcup y} = \overline{y \sqcup z}$. Since $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ is a superlattice, it follows that $T7'$ holds; from $T7'$ and $\overline{z} \in \overline{y \sqcup z}$, $\overline{y} \in \overline{z \sqcup y}$ it follows that $\overline{y} = \overline{z}$. Hence R is w-convex. $\square$

Before proceeding, let us provide some additional notation and prove an auxiliary proposition.

Notation. For all $A, B \subseteq H$ we define

$$A \vee B \doteq \{x \vee y : x \in A, y \in B\}, \quad A \wedge B \doteq \{x \wedge y : x \in A, y \in B\}.$$

Proposition 24 *Let R be an equivalence relation on H. If for some $z \in H$ we have that $\overline{z}$ is a sub-$\vee$-semi-lattice of H, then $\overline{z} \vee \overline{z} = \overline{z}$.*

Proof. Clearly $\overline{z} \subseteq \overline{z} \vee \overline{z}$. Now, say $u \in \overline{z} \vee \overline{z}$; then exist $x, y \in \overline{z}$ such that $u = x \vee y$. Since $\overline{z}$ is a sub-$\vee$-semi-lattice, it follows that $u = x \vee y \in \overline{z}$. So $\overline{z} \vee \overline{z} \subseteq \overline{z}$. $\square$

Let us now continue the study of properties of s-congruences. We have seen that when R is a s-congruence on H, then w-convexity of R is equivalent to $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ being a superlattice. The next proposition shows that a number of other properties are equivalent to the above two.

Proposition 25 *Let R be a s-congruence on H. Then the following are equivalent.*

1. *For all* $x \in H$, $\overline{x}$ *is a sub-*$\vee$*-semi-lattice of* H.
2. *For all* $x \in H$, $\overline{x}$ *is a sublattice of* H.
3. *For all* $x \in H$, $\overline{x}$ *is w-convex.*
4. *For all* $x \in H$, $\overline{x}$ *is s-convex.*
5. $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ *is a superlattice.*

Proof. We will show that (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1); also that (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5).

(1) $\Rightarrow$ (3): Suppose that for all $x \in H$, $\overline{x}$ is a sub-$\vee$-semi-lattice of H. Take any $A \in H/R$ and any $x, y, z \in H$ such that: (a) $\overline{x} = \overline{y} = A$ and (b) $x \leq z \leq y$. From (a) we have: $\overline{x \sqcup z} = \overline{y \sqcup z}$. From (b) it follows that $z = x \vee z \in x \sqcup z \Rightarrow \overline{z} \in \overline{x \sqcup z} = \overline{y \sqcup z}$. Hence exists some $u \in y \sqcup z$ such that: $\overline{u} = \overline{z}$. But $u \in y \sqcup z \Rightarrow u \vee y = u \vee z = y \vee z = y \Rightarrow \overline{u \vee z} = \overline{y} = \overline{x}$. Also, $u \vee z \in \overline{u} \vee \overline{z} = \overline{z} \vee \overline{z} = \overline{z}$ (since $\overline{z}$ is a sub-$\vee$-semi-lattice). Hence $\overline{z} = \overline{u \vee z} = \overline{x} = A$. So for all $x \in H$, $\overline{x}$ is w-convex.

(3) $\Rightarrow$ (2): Suppose that for all $x \in H$, $\overline{x}$ is w-convex. Take any $A \in H/R$ and any x, y such that $\overline{x} = \overline{y} = A$. Then $\overline{x \sqcup x} = \overline{x \sqcup y}$. Now $x \vee y \in x \sqcup y \Rightarrow \overline{x \vee y} \in \overline{x \sqcup y} = \overline{x \sqcup x}$. Hence exists some z such that: (a) $z \in x \sqcup x$ and (b) $\overline{z} = \overline{x \vee y}$. We have $z \in x \sqcup x \Rightarrow x \vee z = x \vee x = x \Rightarrow z \leq x$. So we have $z \leq x \leq x \vee y$ and $\overline{z} = \overline{x \vee y}$; by w-convexity of R it follows that $x \vee y \in \overline{x} = A$. We show dually that $x \wedge y \in A$ and so we have established that for all $x \in H$, $\overline{x}$ is a sublattice of H.

(2) $\Rightarrow$ (1): This is obvious.

(4) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (4): Assume that for all $x \in H$ we have that $\overline{x}$ is a sublattice. Then, we have already established that, for all $x \in H$, $\overline{x}$ is also w-convex. Since for all $x \in H$, $\overline{x}$ is a w-convex sublattice, it is also s-convex.

(3) $\Leftrightarrow$ (5): This was proved in Proposition 23. □

4 A New Order Relationship

We have already noted that: if $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ is a superlattice (equivalently, if R is a s-convex s-congruence on H) the order relationship $\precsim$ can be defined on H/R in terms of either $\overline{\curlyvee}$ or $\overline{\curlywedge}$. We now introduce two additional relationships: $\preceq$ and $\sqsubseteq$. Then we show that $\precsim$, $\preceq$ and $\sqsubseteq$ are equivalent when $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ is a superlattice.

Definition 26 Let R be an equivalence on H. For all $A, B \in H/R$ we write $A \preceq B$ iff: $\forall x \in A, \forall y \in B$ we have $x \wedge y \in A$ and $x \vee y \in B$.

Proposition 27 *If* R *is a s-convex equivalence on* H, *then* $\preceq$ *is an order relationship on* H/R.

Proof. We will show that for all $A, B, C \in H/R$ we have: (i) $A \preceq A$, (ii) $A \preceq B$ and $B \preceq A \Rightarrow A = B$ and (iii) $A \preceq B$ and $B \preceq C \Rightarrow A \preceq C$.

(i) Take any $x, y \in A$. Since A is s-convex, $x \vee y \in A$ and $x \wedge y \in A$.

(ii) Take any $x \in A$ and any $y \in B$. Now $A \preceq B \Rightarrow x \vee y \in B$; and $B \preceq A \Rightarrow x \vee y \in A$. Since A, B are classes and $A \cap B \neq \emptyset$, it follows that $A = B$.

(iii) Take any $x \in A$, any $y \in B$ and any $z \in C$. Now $x \wedge y \in A$, $x \vee y \in B$, $y \wedge z \in B$, $y \vee z \in C$. Since $x \wedge y \in A$, $y \wedge z \in B$ and $A \preceq B$, it follows that $x \wedge y \wedge z \in A$. Also, by assumption $x \in A$. Since $x \wedge y \wedge z \leq x \wedge z \leq x$, by s-convexity it follows that $x \wedge z \in A$. We can prove similarly that $x \vee z \in C$. Hence $A \preceq C$. □

Proposition 28 *Let R be an equivalence on H. If $\preceq$ is an order relationship on H/R, then for all $A \in H/R$ we have that A is a sublattice.*

Proof. Take any $A \in H/R$ and any $x, y \in A$. Since $A \preceq A$, it follows that $x \wedge y, x \vee y \in A$. □

Definition 29 Let R be an equivalence on H. For all $A, B \in H/R$ we write $A \sqsubseteq B$ iff:

$$\textit{(i) } \forall x \in A \quad \exists y \in B \textit{ such that } x \leq y; \quad \textit{(ii) } \forall y \in B \quad \exists x \in A \textit{ such that } x \leq y.$$

Proposition 30 *Let R be an equivalence on H. For all $A, B, C \in H/R$ we have:*

$$\textit{(i) } A \sqsubseteq A; \quad \textit{(ii) if } A \sqsubseteq B \textit{ and } B \sqsubseteq C, \textit{ then } A \sqsubseteq C.$$

Proof. (i) For all $x \in A$ we have $x \leq x$, so $A \sqsubseteq A$.

(ii) Take any $x \in A$; then exists $y \in B$ such that $x \leq y$. For this y exists $z \in C$ such that $y \leq z$. Hence for every $x \in A$ exists some $z \in C$ such that $x \leq z$. We can similarly prove that for all $z \in C$ exists some $x \in A$ such that $x \leq z$. □

Proposition 31 *Let R be a w-convex equivalence on H and $A, B \in H/R$. Then*

$$(A \neq B \textit{ and } A \sqsubseteq B) \Rightarrow \nexists (x, y) \in A \times B \textit{ such that } y < x.$$

Proof. Assume that exists some $x \in A, y \in B$ such that $y < x$. There also exists some $y_1 \in B$ such that $x \leq y_1$. Hence we have $y < x \leq y_1 \Rightarrow x \in B$ (by convexity) which implies that $A \cap B \neq \emptyset$. But A, B were assumed to be distinct classes, so we have reached a contradiction. □

We will now prove that: if $(H/R, \curlyvee, \curlywedge)$ is a superlattice then $\precsim, \preceq, \sqsubseteq$ are equivalent. To this end we first need an auxiliary proposition.

Proposition 32 *Let R be an equivalence on H. If R is s-convex, then for all $A \in H/R$ we have: (i) $A \vee A = A$, (ii) $A \wedge A = A$.*

Proof. We will only prove (i), since (ii) is proved similarly. Clearly $A \subseteq A \vee A$. Now take any $x \in A \vee A$, then there exist $y, z \in A$ such that $x = y \vee z$. But, $y \vee z \in A$ by s-convexity of A. So $A \vee A \subseteq A$. □

Proposition 33 *Let R be a s-convex s-congruence on H. Then for all $A, B, C \in H/R$ the following are equivalent: (i) $A \precsim B$, (ii) $A \preceq B$, (iii) $A \sqsubseteq B$.*

Proof. We will prove (i) $\Leftrightarrow$ (ii), (ii) $\Leftrightarrow$ (iii).
(i) $\Rightarrow$ (ii). Assume that $A \precsim B$. Take any $x \in A$, $y \in B$. Equivalently $\overline{y} \in \overline{x} \curlyvee \overline{y}$ and from Proposition 18 we have that $\overline{y} = \overline{x \vee y}$, i.e. that $x \vee y \in \overline{y}$. Similarly we prove that $x \wedge y \in \overline{x}$. Hence $\overline{x} \preceq \overline{y}$, i.e. $A \preceq B$.
(ii) $\Rightarrow$ (i). Assume that $A \preceq B$. Take any $x \in A$, $y \in B$. Then $x \wedge y \in A = \overline{x} \Rightarrow \overline{x \wedge y} = \overline{x}$. This, by Proposition 18, implies $\overline{x} \in \overline{x} \curlywedge \overline{y}$ and so $\overline{x} \precsim \overline{y}$, i.e. $A \precsim B$.
(ii) $\Rightarrow$ (iii). Assume that $A \preceq B$. Take any $x \in A$, $y \in B$. Then $x \wedge y \in A$; so for any $y \in B$ exists $x_1 = x \wedge y \in A$ such that $x_1 \leq y$. Similarly we can show that for any $x \in A$ exists $y_1 = x \vee y \in B$ such that $x \leq y_1$. Hence $A \sqsubseteq B$.
(iii) $\Rightarrow$ (ii). Assume that $A \sqsubseteq B$. Take any $x \in A$, $y \in B$. We will consider three cases for the relationship between x and y and we will show that in very case $x \wedge y \in A, x \vee y \in B$.

(a) If $x \leq y$, then $x \wedge y = x \in A$, $x \vee y = y \in B$.

(b) Assume $y < x$. Since $A \sqsubseteq B$, we must have $A = B$; because if $A \neq B$ then, by Proposition 31, we cannot have $y < x$. Furthermore A is a sublattice, so $x \wedge y \in A$ and $x \vee y \in A = B$.

(c) Assume $x||y$. Since $A \sqsubseteq B$, exists some $x_1 \in A$ such that $x_1 \leq y$. So $x \wedge x_1 \leq x \wedge y \leq x$ and $x \wedge x_1, x \in A$; then by convexity $x \wedge y \in A$. Similarly it can be shown that $x \vee y \in B$.

So we have shown that for all $x \in A$, $y \in B$ we have $x \wedge y \in A, x \vee y \in B$, i.e. that $A \preceq B$. □
Remark. Hence when R is a s-convex s-congruence, the $\sqsubseteq$ relationship is an order.

In the case of main interest to us, namely when $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$ is a superlattice, the three relationships $\precsim$, $\preceq$, $\sqsubseteq$ are equivalent and they can be denoted by a single symbol. In such a case we will use the symbol $\preceq$ to denote this *order* on the superlattice $(H/R, \overline{\curlyvee}, \overline{\curlywedge})$. Let us now establish further properties of the $\preceq$ order.

Proposition 34 *Let R be an s-convex s-congruence on H. Then for all $x, y \in H$ we have:*

$$x \leq y \Rightarrow \overline{x} \preceq \overline{y}.$$

Proof. We have $x \leq y \Rightarrow x = x \wedge y \Rightarrow \overline{x} = \overline{x \wedge y}$. But then $\overline{x} \preceq \overline{y}$ by Proposition 18. □

Corollary 35 *Let R be an s-convex s-congruence on H. Then for all $x, y, z \in H$ we have:*

$$(i)\ x \leq y \Rightarrow \overline{x \vee z} \preceq \overline{y \vee z} \quad (ii)\ x \leq y \Rightarrow \overline{x \wedge z} \preceq \overline{y \wedge z}.$$

Proof. This follows immediately from Proposition 34. □

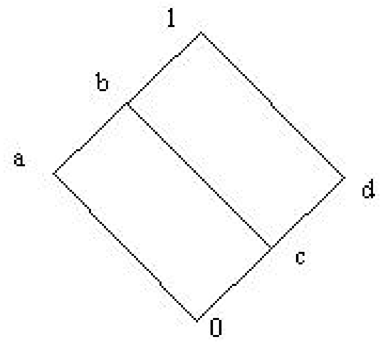
Proposition 36 *Let R be an s-convex s-congruence on H. Then R is a l-congruence on H, i.e. for all $x, y \in H$ we have:*

$$(i)\ \overline{x} = \overline{y} \Rightarrow \overline{x \vee z} = \overline{y \vee z}, \quad (ii)\ \overline{x} = \overline{y} \Rightarrow \overline{x \wedge z} = \overline{y \wedge z}.$$

Proof. We only prove (i), since (ii) is proved similarly. $\overline{x} = \overline{y} \Rightarrow \overline{x \sqcup z} = \overline{y \sqcup z}$. Since $x \vee z \in x \sqcup z$, exists some $u \in y \sqcup z$ such that $\overline{x \vee z} = \overline{u}$. Since $u \in y \sqcup z$ it follows that $u \leq y \vee z$ and then (by Proposition 34) $\overline{x \vee z} = \overline{u} \preceq \overline{y \vee z}$. Similarly we prove $\overline{y \vee z} \preceq \overline{x \vee z}$. Hence $\overline{y \vee z} = \overline{x \vee z}$. □

Remark. The converse is **not** true, i.e. an l-congruence (which is necessarily s-convex) is not necessarily an s-congruence. This can be seen in the next example.

Example 37 *Take the lattice of Figure 1 and let $R = \{A, B\}$, $A = \{0, x_1\}$, $B = \{x_2, 1\}$. It is easy to check that R is a l-congruence, but it is* not *a s-congruence. For instance* $0 \sqcap x_1 = \{0\}$, $x_1 \sqcap x_1 = \{x_1, x_2, 1\}$, $\overline{0 \sqcap x_1} = \{\overline{0}\} = \{A\}$, $\overline{x_1 \sqcap x_1} = \{\overline{x_1}, \overline{x_2}, \overline{1}\}$ $= \{A, B\}$*; so* $\overline{0} = \overline{x_1}$, *but* $\overline{0 \sqcap x_1} \neq \overline{x_1 \sqcap x_1}$.



Example 38 *In fact, the above example is also related to the following proposition.*

Proposition 39 *Let $(H, \vee, \wedge)$ be a chain and R be a s-congruence on H. If there is some $x \in H$ such that $card(\overline{x}) \geq 2$, then $H/R = \{H\}$.*

Proof. Given x such that $card(\overline{x}) \geq 2$, choose some $y \neq x$ such that $\overline{x} = \overline{y}$. Since $(H, \vee, \wedge)$ is a chain, we will have either $x < y$ or $y < x$. Without loss of generality, take $x < y$. Then it is easy to check that $\overline{x \sqcup x} = \overline{x \sqcup y} = \overline{y}$. Define $A = \{z : z \leq x\}$; then it is easy to check that $A = x \sqcup x$. Now take any $z \in A = x \sqcup x$; then $\overline{z} \in \overline{x \sqcup x}$ $= \overline{y}$, i.e. $z \in \overline{z} = \overline{y}$. Hence $A \subseteq \overline{y}$. Defining $B = \{z : x \leq z\}$, one can show similarly that $B \subseteq \overline{y}$. Hence $H = A \cup B \subseteq \overline{y} \subseteq H$ and so the only class of H/R is $\overline{y} = H$. □

Remark. Hence, if $(H, \vee, \wedge)$ is a chain, the only s-congruences on H are $R_1 = \{H\}$ and $R_2 = \{\{x\}\}_{x \in H}$. The connection to Example 37 is now obvious.

Proposition 40 *Let R be an s-convex s-congruence on H. Then for all $x, y \in H$ we have:*

$$(i)\ \overline{x} \preceq \overline{y} \Rightarrow \overline{x \vee z} \preceq \overline{y \vee z}, \quad (ii)\ \overline{x} \preceq \overline{y} \Rightarrow \overline{x \wedge z} \preceq \overline{y \wedge z}.$$

Proof. We only prove (i), since (ii) is proved similarly. Assuming $\overline{x} \preceq \overline{y}$, it follows that exists some $y_1 \in \overline{y}$ such that $x \leq y_1$. Then (from Corollary 35) we have $\overline{x \vee z} \preceq \overline{y_1 \vee z}$. But, from Corollary 36 we have $\overline{y_1 \vee z} = \overline{y \vee z}$ and the proof is complete. □

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