NOETHER'S LAWS OF CONSERVATION IN TIME DEPENDENT LAGRANGE SPACES

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Abstract

Some formulae for the energy E_L for the non-autonomous (or time dependent) Lagrangians are given. In section 4 the Noether Theorem for such kind of Lagrangians is proved. An explicit expression of the infinitesimal Noether symmetries is also given.

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Key words: Lagrange space, time dependent Lagrangian, energy, Noether symmetries.

1 Introduction

This paper is a continuation of the paper "A New Geometrization of Time Dependent Lagrangian" by C. Frigioiu and M. Kirkovits [4], which will be presented at Bolyai-Gauss-Lobachevsky International Conference Tîrgu Mureş, in July, 3-6, 2002.

The geometry of time dependent Lagrangians and the rheonomic Lagrange spaces was studied by M. Anastasiei and H. Kawaguchi [2]. In present paper, we prove the Euler-Lagrange equations which are fundamental in the geometry of rheonomic spaces $RL^n = (M, L(t,x,y))$. So, by using the Euler-Lagrange equations, we prove two important results (3.16) and (3.17) about the energy E_L.

Theorem (3.2) shows us that the energy E_L is conserved along an extremal curve of $L(t,x,y)$. Using the variational principle for the integral $I(c)$ and $I'(c)$ from (4.18) and (4.19) we obtain the infinitesimal symmetries given by Noether Theorem 4.1.

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2 The Manifold $(R \times TM, \pi, M)$

Let M be a smooth C^∞ manifold of finite dimension n and (TM, π, M) be its tangent bundle. We denote by $(x^i), i, j, k = 1, 2, ..., n$ the local coordinates on M and by (x^i, y^i) the local coordinates on TM.

The tangent bundle

$$E = (R \times TM, \pi, M) \tag{2.1}$$

has the total space $E = R \times TM$ which is a $n+1-$ real manifold. In a domain of a local chart $(a, b) \times U^\star$, the points $(t, x, y) \in E$ have the local coordinates (t, x^i, y^i).

The canonical projection $\pi : E \to M$ is defined by:

$$\pi(t, x, y) = x, \quad \forall (t, x, y) \in E. \tag{2.2}$$

A change of local coordinates on E has the following form:

$$\begin{gathered} \tilde{t} = t, \\ \tilde{x}^i = \tilde{x}^i(x^1, x^2, ..., x^n), \quad det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0, \\ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j. \end{gathered} \tag{2.3}$$

The natural basis of tangent space $T_u E$ at the point $u \in (a, b) \times U^\star$ is given by

$$\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right). \tag{2.4}$$

The coordinates transformation (2.3) determines the transformations of the natural basis as follows

$$\begin{gathered} \frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}}, \\ \frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{y}^j}{\partial x^i} \frac{\partial}{\partial \tilde{y}^j}, \\ \frac{\partial}{\partial y^i} = \frac{\partial \tilde{y}^j}{\partial y^i} \frac{\partial}{\partial \tilde{y}^j}, \end{gathered} \tag{2.5}$$

where

$$\frac{\partial \tilde{y}^j}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i}; \quad \frac{\partial \tilde{y}^j}{\partial x^i} = \frac{\partial^2 \tilde{x}^j}{\partial x^i \partial x^h} y^h.$$

.

We know that TM admits a natural tangent structure $J : \chi(TM) \to \chi(TM)$, give by:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}; J\left(\frac{\partial}{\partial y^i}\right) = 0 \quad for \quad i, j, k = 1, 2, ..., n. \tag{2.6}$$

In order to prolong this structure on E we must define $J\left(\frac{\partial}{\partial t}\right)$. We shall keep the notation $J:\chi(TM)\to\chi(TM)$, for the tangent structure on $E=R\times TM$, too. Thus, we have:

$$J\left(\frac{\partial}{\partial t}\right)=0; J\left(\frac{\partial}{\partial x^i}\right)=\frac{\partial}{\partial y^i}; J\left(\frac{\partial}{\partial y^i}\right)=0 \quad for \quad i,j,k=1,2,...,n. \tag{2.7}$$

By a direct calculation, we find that $J\circ J=0$ and the Nijenhuis tensor N_J vanishes.

On the manifold E, there exists a vertical distribution V, generated by $n+1$ local vector fields $\left(\frac{\partial}{\partial t},\frac{\partial}{\partial y^1},\frac{\partial}{\partial y^2},...,\frac{\partial}{\partial y^n}\right)$,

$$V:u\in E\to V_u\subset T_uE \tag{2.8}$$

and

$$V_u=V_{0,n}\oplus V_{n,u} \quad \forall u\in E,$$

where the linear space $V_{0,n}$ is generated by the vector field $\frac{\partial}{\partial t}\mid_u$ and it is an 1-dimensional linear subspace of the tangent space T_uE. Also, the n-dimensional linear space $V_{n,u}$ generated by the fields $\left(\frac{\partial}{\partial t},\frac{\partial}{\partial y^1},\frac{\partial}{\partial y^2},...,\frac{\partial}{\partial y^n}\right)\mid_u$ is a linear subspace of T_uE.

A non-linear connection in E is a distribution:

$$N:u\in E\to N_u\subset T_uE, \tag{2.9}$$

which is supplementary to the vertical distribution V:

$$T_uE=N_u\oplus V_u, \quad \forall u=(t,x,y)\in E. \tag{2.10}$$

The local basis adapted to the descomposition (2.10) is $\left(\frac{\partial}{\partial t},\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i}\right)$, where

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N_i^0(t,x,y)\frac{\partial}{\partial t}-N_i^j(t,x,y)\frac{\partial}{\partial y^j}.$$

The real functions $(N_i^0(t,x,y),N_i^j(t,x,y))$ are locally defined on E and subject to the following transformation rule under (2.1):

$$\frac{\partial\tilde{x}^j}{\partial x^i}\tilde{N}_j^0=N_i^0, \tag{2.11}$$

$$\tilde{N}_m^j\frac{\partial\tilde{x}^m}{\partial x^i}=\frac{\partial\tilde{x}^j}{\partial x^m}N_i^m-\frac{\partial\tilde{y}^j}{\partial x^i}.$$

The coordinate transformation (2.1) determines the transformation of the local basis to the decomposition (2.10) as follows:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}},$$
$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j}, \tag{2.12}$$
$$\frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{y}^j}.$$

3 Time Dependent Lagrangians

Definition 3.1 *A time dependent Lagrangian is a scalar function L from $R \times TM$ to R defined by $(t, x, y) \to L(t, x, y)$.*

Definition 3.2 *A time dependent Lagrangian L is called differentiable if L is of the class C^∞ on the manifold $R\times\tilde{T}M$ and L is continous in the points $(t, x, o) \in R\times TM$.*

Let us consider on the manifold M a smooth parametrized curve $c : t \in [0, 1] \to ((x^i(t)) \in U \subset M$. Its extension $\tilde{c} : [0, 1] \to (a, b) \times \pi^{-1}(U) \subset E$ is represented by:

$$\tilde{c}(t) = \left(t, x^i(t), \frac{dx^i}{dt}(t)\right),$$

where $t \in [0, 1]$ is the time.

The integral of action of the time dependent Lagrangian L along c is defined by:

$$I(c) = \int_0^1 L\left(t, x(t), \frac{dx}{dt}(t)\right) dt. \tag{3.13}$$

On the open set U we consider the curves:

$$c_\varepsilon : t \in [0, 1] \to ((x^i) + \varepsilon V^i(t)) \in M,$$

where ε is a real number, sufficiently small in absolute value so that $Im(c_\varepsilon) \subset U$,$V^i(x(t))$ denoted by $V^i(t)$ being a regular vector field on U, restricted to the curve c. We assume that the curves c_ε have the same end points $c(0)$ and $c(1)$, with the curve c and at these points they have the same tangents. Therefore, the vector fields $V^i(t)$ satisfy the conditions

$$V^i(0) = V^i(1), \frac{dV^i}{dt}(0) = \frac{dV^i}{dt}(1) = 0.$$

The extension of c to $R \times T\tilde{M}$ is $\tilde{c}_\varepsilon$ given by:

$$c_\varepsilon : t \in [0, 1] \to \left(t, (x^i) + \varepsilon V^i(t), \frac{dx^i}{dt} + \varepsilon\frac{dV^i}{dt})\right) \in (a, b) \times \pi^{-1}(U).$$

The integral of action of the differentiable Lagrangian $L(t,x,y)$ on the curves c_ε is:

$$I(c_\varepsilon) = \int_0^1 L\left(t, x+\varepsilon V, \frac{dx}{dt} + \varepsilon \frac{dV}{dt}\right) dt.$$

A necessary condition for the functional $I(c)$ to be an extremal value for $I(c_\varepsilon)$ is

$$\frac{dI(c_\varepsilon)}{d\varepsilon} |_{\varepsilon=0} = 0.$$

A direct calculus leads us to:

$$\frac{dI(c_\varepsilon)}{d\varepsilon} |_{\varepsilon=0} = \int_0^1 E_i(L) V^i dt,$$

where $E_i(L) := \dfrac{\partial L}{\partial x^i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial y^i}\right)$ and $y^i = \dfrac{dx^i}{dt}$.

Theorem 3.1 *In order that the integral of action $I(c)$ was an extremal value for the functionals $I(c_\varepsilon)$, it is necessary that the following Euler-Lagrange equations hold:*

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) = 0;\ y^i = \frac{dx^i}{dt}. \tag{3.14}$$

The equations (3.14) are fundamental in the geometry of time dependent Lagrangian L.

We may also consider the function

$$E_L = y^i \frac{\partial L}{\partial y^i} - L(t,x,y). \tag{3.15}$$

This function is called the energy of L.We proved in [4] the following theorem:

Theorem 3.2 *a) Along any curve c from M, the following formula hold:*

$$\frac{dE_L}{dt} = -\frac{dx^i}{dt} E_i(L) - \frac{\partial L}{\partial x^i}. \tag{3.16}$$

.

b) The variation of the energy E_L along of an extremal curve of L is given by:

$$\frac{dE_L}{dt} = -\frac{\partial L}{\partial t}. \tag{3.17}$$

4 A Noether Theorem

Let us consider the integrals of action (3.13) for the time dependent Lagrangians $L(t,x,y)$ and $L(t,x,y)+\frac{dF(t,x)}{dt}$, where $F(t,x)$ is a differentiable Lagrangian:

$$I(c)=\int_0^1 L\left(t.x(t),\frac{dx}{dt}(t)\right)dt \tag{4.18}$$

and

$$I'(c)=\int_0^1 \left[L\left(t.x(t),\frac{dx}{dt}(t)\right)+\frac{dF(t,x)}{dt}\right]dt. \tag{4.19}$$

Lemma 4.1 *The integrals of actions $I(c)$ and $I'(c)$ have the same extremal curves, for any differentiable Lagrangian $F(t,x)$.*

Proof.

$$E_i(L):=\frac{\partial L}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right)$$

and

$$E_i\left(L+\frac{dF}{dt}\right):=E_i(L)+E_i\left(\frac{dF}{dt}\right).$$

But, we can prove that $E_i\left(\frac{dF}{dt}\right)=0$. So, we obtain $E_i(L)=E_i\left(L+\frac{dF}{dt}\right)$ for any differentiable Lagrangian $F(x,t)$. That imply $I(c)$ and $I'(c)$ have the same extremal curves. □

Definition 4.1 *A symmetry of the differentiable time dependent Lagrangian L is a C^∞- diffeomorphism $\varphi: R\times TM\to R\times TM$, which preserves the variational principle of the integral of action from (4.18) and (4.19).*

Generally, there do not exist such diffeomorphims. But locally the diffeomorphism φ does exist.

So, we may consider the notion of local symmetry of the Lagrangian $L(t,x,y)$, taking φ as a local diffeomorphism.

We start with an infinitesimal transformation φ on $(a,b)\times M$ of the form:

$$\begin{cases} x'^i = x^i+\varepsilon V^i(t,x) \\ t' = t+\varepsilon\tau(t,x)\end{cases} \quad for \quad i=1,2,...,n, \tag{4.20}$$

where ε is a real number, sufficiently small as absolute value, so that the points (x^i,t) and (x'^i,t) belong to the same local chart $(a,b)\times U\subset I\!R\times M$.

In the following considerations, the terms of order greater than 1 in ε will be neglected.

The inverse of the diffeomorphism (4.20) is :

$$\begin{cases} x^i = x'^i-\varepsilon V^i(t,x) \\ t = t'-\varepsilon\tau(t,x)\end{cases} \quad for \quad i=1,2,...,n. \tag{4.21}$$

The vector field $V^i(t, x(t)) = V^i(t)$ on $(a, b) \times U$ has the property:

$$V^i(0) = V^i(1) = 0. \tag{4.22}$$

Therefore, looking at (4.18) and (4.19), the infinitesimal transformation (4.20) is a symmetry for the time dependent Lagrangian $L(t, x, y)$, if for any C^∞- function, $F(t, x)$ satisfies the following equation:

$$L\left(t', x', \frac{dx'}{dt}\right) dt' = \left\{L\left(t, x, \frac{dx}{dt}\right) + \frac{dF(t, x)}{dt}\right\}. \tag{4.23}$$

From (4.20) and (4.21) we deduce:

$$\begin{cases} \frac{dt'}{dt} &= 1 + \varepsilon\frac{d\tau}{dt} \Rightarrow \frac{dt}{dt'} = 1 - \varepsilon\frac{d\tau}{dt} \\ \frac{dx'^i}{dt'} &= \frac{dx^i}{dt} + \varepsilon\left(\frac{dV^i}{dt} - \frac{dx^i}{dt}\frac{d\tau}{dt}\right) \end{cases}. \tag{4.24}$$

The equality (4.23) by virtue of (4.24) and neglecting the terms in $\varepsilon^2, \varepsilon^3$... leads to:

$$\varepsilon\left[\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x^i}V^i + \frac{\partial L}{\partial y^i}\left(\frac{dV^i}{dt} - \frac{dx^i}{dt}\frac{d\tau}{dt}\right)\right] + \varepsilon L\frac{d\tau}{dt} = \frac{dF(t, x)}{dt}, \tag{4.25}$$

$$y^i = \frac{dx^i}{dt},$$

where we set εF instead of F and we obtain:

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x^i}V^i + \frac{\partial L}{\partial y^i}\left(\frac{dV^i}{dt} - \frac{dx^i}{dt}\frac{d\tau}{dt}\right) + L\frac{d\tau}{dt} = \frac{dF(t, x)}{dt}, \tag{4.26}$$

$$y^i = \frac{dx^i}{dt}.$$

The equation (4.27) can be written under the form:

$$\left(\frac{\partial L}{\partial t} + \frac{dE_L}{dt}\right)\tau + E_i(L)V^i + \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}V^i - E_L\tau\right) = \frac{dF(t, x)}{dt}. \tag{4.27}$$

So, we have the following Noether theorem:

Theorem 4.1 *For any infinitesimal symmetry (4.20) of the time dependent Lagrangian $L(t, x, y)$ and for any function $F(t, x)$ the following function:*

$$\Im(L, F) \stackrel{def}{=} V^i\frac{\partial L}{\partial y^i} - \tau E_L - F(t, x) \tag{4.28}$$

is conserved on the solution curves of the Euler-Lagrange equations

$$E_i(L) = 0; \quad y^i = \frac{dx^i}{dt}.$$

Proof. The equations

$$E_i(L) = 0; y^i = \frac{dx^i}{dt}$$

and (4.27) imply the conclusion of the theorem. □

Conclusions: In this paper we prove the following results:

a) the Euler-Lagrange equations for time dependent Lagrangians. These equations are fundamental in rheonomic Lagrange geometry;

b) two new formulae for the energy E_L of the time dependent Lagrangians $L(t, x, y)$ are estabilished

c) a demonstration of the Noether theorem is explicitly given and the infinitesimal Noether symmetries are determinated.

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