ON THE CLASSIFICATION OF THE FIVE-DIMENSIONAL LIE SUPERALGEBRAS

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Abstract

The purpose of this contribution is to initiate a classification of Lie superalgebras (LS) of dimension **5**, over the field $\mathbb{R}$ of real numbers.

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1 Introduction - Preliminaries

In this section we give some definitions which are very usefull for the sequel.

Definition 1.1 A *Lie superalgebra* $L = L_0 \oplus L_1$ is a superalgebra over a base field $K = \mathbb{R}$ or C, with a bilinear bracket operation $[.\,,.]$ satisfying the following axioms:

(i) $[x,y] = -(-1)^{\alpha\beta}[y,x]$, *(graded skew-symmetry)*

for all $x \in L_\alpha$, $y \in L_\beta$, $\alpha, \beta \in \{\bar{0}, \bar{1}\} = \mathrm{Z}_2$.

(ii) $(-1)^{\alpha\gamma}[[x,y],z] + (-1)^{\alpha\beta}[[y,z],x] + (-1)^{\beta\gamma}[[z,x],y] = 0,$

(graded Jacobi identity) for all $x \in L_\alpha$, $y \in L_\beta$, $z \in L_\gamma$, $\alpha, \beta, \gamma \in \mathrm{Z}_2$.

Definition 1.2 L_0 is called the *even* part of L and is a Lie algebra. L_1 is called the *odd* part of L and is a L_0-module.

Definition 1.3 We say that $L = L_0 \oplus L_1$ and $L' = L'_0 \oplus L'_1$ are *equivalent*, if there are isomorphisms $L_0 \leftrightarrow L'_0$ and $L_1 \leftrightarrow L'_1$, which preserve the bracket multiplication.

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We can ask the question: given a Lie algebra L_0 and a L_0-module M, how many LS $L = L_0 \oplus L_1$ can we construct where L_1 and M are isomorphic as L_0-modules? Answering this question is the basis for the classification scheme.

Definition 1.4 A LS L is *trivial*, if $[L_1, L_1] = \{0\}$; otherwise, L is *non-trivial*.

Definition 1.5 We say that L is a (m, n)-*Lie superalgebra* and has dimension $m+n$, if $\dim L_0 = m$ and $\dim L_1 = n$. Here we consider $m + n = 5$, in particular only the $(1, 4)$ and $(4, 1)$ cases.

Finally, we make a usefull remark.

Remark 1.6 From graded skew-symmetry and graded Jacobi identity, we obtain the following relations, which hold for all $a, b, c \in L_0$ and $\alpha, \beta, \gamma \in L_1$:

$$\begin{aligned}
[a, b] &= -[b, a], && (1.1)\\
[a, \alpha] &= -[\alpha, a], && (1.2)\\
[\alpha, \beta] &= [\beta, \alpha], && (1.3)\\
[[a, b], c] &+ [[b, c], a] + [[c, a], b] = 0, && (1.4)\\
[[a, b], \alpha] &+ [[b, \alpha], a] + [[\alpha, a], b] = 0, && (1.5)\\
[[a, \alpha], \beta] &+ [[\alpha, \beta], a] - [[\beta, a], \alpha] = 0, && (1.6)\\
[[\alpha, \beta], \gamma] &+ [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = 0. && (1.7)
\end{aligned}$$

2 The Method

In this section we present the method that we applied, in order to get the results. First, for the

I. (4,1)-dimensional Lie superalgebras

Let $L = L_0 \oplus L_1$ be a $(4, 1)$-LS over $\mathbb{R}$ and $\{a_1, a_2, a_3, a_4 \; ; \; \alpha\}$ a set of generators. In particular a_1, a_2, a_3, a_4 are the generators of the even part L_0 and α is the generator of the odd part L_1. The one-dimensional representation $\rho : L_0 \to L_1$ is defined by:

$$\rho(a_i)(\alpha) \equiv [a_i, \alpha] = \lambda_i \alpha, \quad i = 1, 2, 3, 4.$$

Then, we either have one of the following two cases:
(1) $\lambda_i = 0$, for each $i = 1, 2, 3, 4$, or
(2) there exists at least one $\lambda_i \neq 0$, for example λ_4.

A. (4,1)-TRIVIAL LS

In case **(1)**, any $(4, 1)$-trivial LS is an ordinary Lie algebra of dimension four (see J Patera, R T Sharp, P Winternitz and H Zassenhaus, (1976)).

In case **(2)**, the change of basis in L_0, defined by:

$$
\begin{aligned}
a'_i &= a_i - \frac{\lambda_i}{\lambda_4} a_4, \quad i = 1,2,3 \\
a'_4 &= a_4,
\end{aligned}
$$

gives

$$
\begin{aligned}
&[a'_i, \alpha] = 0, \quad \text{for} \quad i = 1,2,3 \\
&[a'_4, \alpha] = \lambda_4 \alpha, \quad \lambda_4 \neq 0.
\end{aligned}
$$

Thus, we observe:
1) λ_4 can be reduced to unity by scaling a_4.
2) In the sequel we use again a_i instead of a'_i. It is easy to see that $h \equiv \ker(\rho)$ is a three-dimensional ideal of L_0, generated by a_1, a_2, a_3, with a_4 acting on h as an external derivation. This means that the following relation holds

$$
[a_4, [a_i, a_j]] = [[a_4, a_i], a_j] + [a_i, [a_4, a_j]], \quad i, j = 1,2,3, \; i \neq j.
$$

3) We can also see that the quotient Lie algebra $L_0/\ker(\rho)$ is isomorphic to the one-dimensional Abelian Lie algebra. Thus, we have:

$$
\mathbf{L_0} = \mathbf{h} \; + \; \mathbb{R}\, \mathbf{a_4}.
$$

Finally, using all different forms of the 3-dimensional Lie algebra h over the reals and the graded Jacobi identity, we find **19** inequivalent trivial $(4,1)$-dimensional LS.

B. (4,1)- NON TRIVIAL LS

From the graded Jacobi identity, we have:

$$
\begin{aligned}
[a_i, [\alpha, \alpha]] &= 2[[a_i, \alpha], \alpha], \quad i = 1,2,3,4 && (2.1) \\
[\alpha, [\alpha, \alpha]] &= 0. && (2.2)
\end{aligned}
$$

In case **(1)**, where $\lambda_i = 0 \; \forall i = 1,2,3,4$, we obtain that L_0 is a 4-dimensional Lie algebra with $[L_0, L_1] = 0$. From (2.1) we get $[a_i, [\alpha, \alpha]] = 0 \; \forall \; i$. So, $[\alpha, \alpha]$ belongs to the center $Z(L_0)$ of L_0.
In this case, we find **5** different non-trivial $(4,1)$-dimensional LS.

In case **(2)**, this means that there is at least one $\lambda_i \neq 0$, we conclude that $[\alpha, \alpha]$ belongs to the center $Z(h)$ of the ideal h, since the relation (2.1) gives $[a_i, [\alpha, \alpha]] = 0 \; \forall i = 1,2,3$.
In this case, we find **11** different non-trivial $(4,1)$-dimensional LS.

II. (1,4)-dimensional Lie superalgebras

Let $L = L_0 \oplus L_1$ be a (4, 1)-LS over $\mathbb{R}$ and $\{a\ ;\ \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ a set of generators. In particular, a is the generator of L_0 and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the generators of L_1.

A. (1,4)- TRIVIAL LS

The action of a on L_1 is completely determined by a 4×4 matrix over $\mathbb{R}$.

Thus, there exist **15** inequivalent families of $(1, 4)$-trivial LS, arising from the inequivalent Jordan forms or the rational canonical form over $\mathbb{R}$, of 4×4 matrix .

B. (1,4)- NON TRIVIAL LS

From the graded Jacobi identity, we have:

$$[[\alpha_i, \alpha_i], \alpha_i] = 0, \quad i = 1, 2, 3, 4. \tag{2.3}$$

Let $[\alpha_i, \alpha_j] = S_{ij}\ a, \quad i, j = 1, 2, 3, 4.$ and $S_{ij} \in \mathbb{R}$.
Then the matrix $S = (S_{ij})$ is a 4×4 real symmetric matrix and by a linear transformation takes a diagonal form $S = diag\ (s_1, s_2, s_3, s_4)$. We distinguish the following cases:
(1) If $s_i = 0$ for all $i = 1, 2, 3, 4,$ then the LS is trivial.
(2) If there is at least one $s_i \neq 0$. Let $[\alpha_1, \alpha_1] = s_1 a,\ s_1 \neq 0$. Then $[a, \alpha_1] = 0$.
Using again the graded Jacobi identity, we have

$$[a, \alpha_i] = 0, \quad i = 2, 3, 4.$$

Therefore, when $s_2 s_3 s_4 = 0$ the LS decomposes. So, in order to obtain an indecomposable LS we must have $s_i \neq 0$ for all i.
(3) If $s_i \neq 0$ for all i, then by scalling each α_i we can take $s_i = 1$ or $-1\ \forall\ i = 1, 2, 3, 4$. This leads to 3 inequivalent forms of the matrix $S = diag\ (s_1, s_2, s_3, s_4)$.
So, we finally find **3** different families of $(1, 4)$-non trivial LS.

3 Tabulations

Here we tabulate into families of equivalence classes the real indecomposable LS of dimension four, in particular the $(1, 4)$ and $(4, 1)$ cases, which are not Lie Algebras. For typographical convenience we use the generators a, b, c, d instead of a_1, a_2, a_3, a_4 and $\alpha, \beta, \gamma, \delta$ instead of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

(4,1)-TRIVIAL

	Relations	**Comments**
1	$[d,a]=pa,\,[d,b]=qb,\,[d,c]=rc;\,[d,\alpha]=\alpha$	$p,q,r\neq 0$
2	$[a,b]=c;\,[d,a]=pa,\,[d,b]=qb,\,[d,c]=(p+q)c;\,[d,\alpha]=\alpha$	$p,q\neq 0$
3	$[a,b]=b,\,[a,c]=\mu c;\,[d,c]=rc;\,[d,\alpha]=\alpha$	$r\neq 0,\ \mu\neq 0,1$
4	$[a,b]=b,\,[a,c]=b+c;\,[d,b]=qb,\,[d,c]=qc;\,[d,\alpha]=\alpha$	$q\neq 0$
5	$[a,b]=c;\,[d,b]=qb,\,[d,c]=qc;\,[d,\alpha]=\alpha$	$q\neq 0$
6	$[a,b]=\kappa b-c,\,[a,c]=b+\kappa c;\,[d,b]=qb,\,[d,c]=qc;\,[d,\alpha]=\alpha$	$q,\kappa\neq 0$
7	$[a,b]=-c,\,[a,c]=b;\,[d,b]=qb,\,[d,c]=qc;\,[d,\alpha]=\alpha$	$q\neq 0$
8	$[d,a]=pa,\,[d,b]=a+pb,\,[d,c]=b+pc;\,[d,\alpha]=\alpha$	$p\neq 0$
9	$[a,c]=a,\,[b,c]=\lambda a+b;\,[d,b]=a,\,[d,c]=b;\,[d,\alpha]=\alpha$	$\lambda\neq 0$
10	$[a,c]=a,\,[b,c]=b;\,[d,b]=a,\,[d,c]=b;\,[d,\alpha]=\alpha$	
11	$[b,c]=\kappa a;\,[d,b]=a,\,[d,c]=b;\,[d,\alpha]=\alpha$	$\kappa\neq 0$
12	$[d,a]=pa,\,[d,b]=a+pb,\,[d,c]=qc;\,[d,\alpha]=\alpha$	$p,q\neq 0$
13	$[a,b]=c;\,[d,a]=pa,\,[d,b]=a+pb,\,[d,c]=2pc;\,[d,\alpha]=\alpha$	$p\neq 0$
14	$[b,a]=a+c,\,[b,c]=\mu c;\,[d,b]=a;\,[d,\alpha]=\alpha$	$\mu\neq 0,\ \lvert\mu\rvert\leq 1$
15	$[b,a]=\kappa a-\lambda c,\,[b,c]=\lambda a+\kappa c;\,[d,b]=a;\,[d,\alpha]=\alpha$	$\lambda\neq 0$
16	$[d,a]=pa,\,[d,b]=qb-rc,\,[d,c]=rb+qc;\,[d,\alpha]=\alpha$	$p,r\neq 0$
17	$[b,c]=a;\,[d,a]=2qa,\,[d,b]=qb-rc,\,[d,c]=rb+qc;\,[d,\alpha]=\alpha$	$r\neq 0$
18	$[a,b]=b,\,[a,c]=c;\,[d,b]=-rc,\,[d,c]=rb;\,[d,\alpha]=\alpha$	$r\neq 0$
19	$[b,c]=a;\,[d,b]=-rc,\,[d,c]=rb;\,[d,\alpha]=\alpha$	$r\neq 0$

(4,1)- NON TRIVIAL

	Relations	**Comments**
1	$[d,a]=2a,\,[d,b]=qb,\,[d,c]=rc;\,[d,\alpha]=\alpha;\,[\alpha,\alpha]=a$	$q,r\neq 2$
2	$[a,b]=c;\,[d,a]=pa,\,[d,b]=(2-p)b,$ $[d,c]=2c;\,[d,\alpha]=\alpha;\,[\alpha,\alpha]=c$	$p\neq 0,q\neq 2$
3	$[a,b]=c;\,[d,b]=2b,\,[d,c]=2c;\,[d,\alpha]=\alpha;\,[\alpha,\alpha]=c$	
4	$[d,a]=2a,\,[d,b]=a+2b,\,[d,c]=b+2c;\,[d,\alpha]=\alpha;\,[\alpha,\alpha]=a$	
5	$[d,a]=2a,\,[d,b]=a+2b,\,[d,c]=qc;\,[d,\alpha]=\alpha;\,[\alpha,\alpha]=c$	$q\neq 0,2$
6	$[d,a]=pa,\,[d,b]=a+pb,\,[d,c]=2c;\,[d,\alpha]=\alpha;\,[\alpha,\alpha]=c$	$p\neq 0,2$
7	$[a,b]=c;\,[d,a]=a,\,[d,b]=a+b,\,[d,c]=2c;\,[d,\alpha]=\alpha;\,[\alpha,\alpha]=c$	
8	$[a,b]=c;\,[d,a]=a,\,[d,b]=a+b,\,[d,c]=2c;\,[d,\alpha]=\alpha;\,[\alpha,\alpha]=-c$	
9	$[d,a]=2a,\,[d,b]=qb-rc,\,[d,c]=rb+qc;\,[d,\alpha]=\alpha;\,[\alpha,\alpha]=a$	$r\neq 0$
10	$[b,c]=a;\,[d,a]=2a,\,[d,b]=b-rc,$ $[d,c]=rb+c;\,[d,\alpha]=\alpha;\,[\alpha,\alpha]=a$	$r\neq 0$
11	$[b,c]=a;\,[d,a]=2a,\,[d,b]=b-rc,$ $[d,c]=rb+c;\,[d,\alpha]=\alpha;\,[\alpha,\alpha]=-a$	$r\neq 0$
12	$[d,a]=2a,\,[d,c]=b;\,[\alpha,\alpha]=b$	
13	$[d,b]=a,\,[d,c]=b;\,[\alpha,\alpha]=a$	
14	$[a,b]=c;\,[d,a]=a,\,[d,b]=-b;\,[\alpha,\alpha]=c$	
15	$[a,b]=c;\,[d,a]=-b,\,[d,b]=a;\,[\alpha,\alpha]=c$	
16	$[a,b]=c;\,[d,a]=-b,\,[d,b]=a;\,[\alpha,\alpha]=-c$	

(1,4)-TRIVIAL

	Relations	Comments
1	$[a,\alpha]=\alpha, [a,\beta]=q\beta, [a,\gamma]=r\gamma, [a,\delta]=s\delta$	$0 \leq \lvert s\rvert \leq \lvert r\rvert \leq \lvert q\rvert \leq 1$
2	$[a,\alpha]=p\alpha+\beta, [a,\beta]=p\beta+\gamma, [a,\gamma]=p\gamma+\delta, [a,\delta]=p\delta$	$p \neq 0$
3	$[a,\alpha]=\beta, [a,\beta]=\gamma, [a,\gamma]=\delta$	
4	$[a,\alpha]=p\alpha, [a,\beta]=p\beta+\gamma, [a,\gamma]=p\gamma+\delta, [a,\delta]=p\delta$	$p \neq 0$
5	$[a,\alpha]=p\alpha+\beta, [a,\beta]=p\beta, [a,\gamma]=p\gamma+\delta, [a,\delta]=p\delta$	$p \neq 0$
6	$[a,\alpha]=\beta, [a,\gamma]=\delta$	
7	$[a,\alpha]=p\alpha, [a,\beta]=p\beta, [a,\gamma]=p\gamma+\delta, [a,\delta]=p\delta$	$p \neq 0$
8	$[a,\alpha]=p\alpha, [a,\beta]=q\beta+\gamma, [a,\gamma]=q\gamma+\delta, [a,\delta]=q\delta$	$p,q \neq 0,\ p \neq q$
9	$[a,\alpha]=p\alpha, [a,\beta]=\gamma, [a,\gamma]=\delta$	$p \neq 0$
10	$[a,\alpha]=p\alpha, [a,\beta]=q\beta, [a,\gamma]=q\gamma+\delta, [a,\delta]=q\delta$	$p,q \neq 0,\ p \neq q$
11	$[a,\alpha]=p\alpha+\beta, [a,\beta]=p\beta, [a,\gamma]=q\gamma+\delta, [a,\delta]=q\delta$	$p \neq q$
12	$[a,\alpha]=p\alpha, [a,\beta]=p\beta, [a,\gamma]=q\gamma+\delta, [a,\delta]=q\delta$	$p \neq 0,\ p \neq q$
13	$[a,\alpha]=p\alpha+q\beta, [a,\beta]=-q\alpha+p\beta, [a,\gamma]= r\gamma+s\delta, [a,\delta]=-s\gamma+r\delta$	$q,s \neq 0$
14	$[a,\alpha]=p\alpha, [a,\beta]=q\beta, [a,\gamma]= r\gamma+s\delta, [a,\delta]=-s\gamma+r\delta$	$s \neq 0$, $\lvert p\rvert \geq \lvert q\rvert > 0$
15	$[a,\alpha]=p\alpha+\beta, [a,\beta]=p\beta, [a,\gamma]= r\gamma+s\delta, [a,\delta]=-s\gamma+r\delta$	$s \neq 0$

(1,4)-NON TRIVIAL

	Relations	Comments
1	$[\alpha,\alpha]=a, [\beta,\beta]=a, [\gamma,\gamma]=a, [\delta,\delta]=a$	
2	$[\alpha,\alpha]=a, [\beta,\beta]=a, [\gamma,\gamma]=a, [\delta,\delta]=-a$	
3	$[\alpha,\alpha]=a, [\beta,\beta]=a, [\gamma,\gamma]=-a, [\delta,\delta]=-a$	

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