HOMOGENEOUS DOMAIN C^4 AND LIE ALGEBRAS

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Abstract

The aim of this brief expository note is to describe the connection between normal j-algebras and convex cones, who are necessary for the structure of a Siegel domain.

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1 Introduction

One of the basic problems in the study of homogeneous bounded domains which are not symmetric.

For a lot of years there was an idea, that every homogeneous bounded domains is symmetric.This idea is one of the basic problems of E.Cartan.

In [4], Pyatetskij-Shapiroj proved the following result:

There is only one non symmetric homogeneous bounded domain in C^4*. But he didn't give the explicit form of this homogenous bounded domain which is non symmetric in* C^4 $\left(\textit{some } C^5\right)$.

In [6] prof. G.Tsagas and the first author gave the explicit form of a normal j-algebra in dimensions four and five.

The study of non symmetric homogeneous bounded domains in C^4 and C^5 amounts to the study of the above j-algebras.

Another way of study of the above nonsymmetric homogeneous bounded domains is in that Siegel domains.

The aim of this brief expository note is to describe the connection between normal j-algebras and convex cones, who are necessary for the structure of a Siegel domain.

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2 The construction of the solvable Lie group and its Lie algebra

Let

$$A = \sum_{1 \le j,i \le 5} A_{ij} \tag{2.1}$$

be a T-algebra of rank five provided with an involative anti-automorphism *, and let g be a solvable Lie algebra included in A.

$$g = \sum_{1 \le j,i \le 5} A_{ij} = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & x_{22} & 0 & 0 & 0 \\ 0 & 0 & x_{33} & 0 & 0 \\ 0 & 0 & 0 & x_{44} & 0 \\ 0 & 0 & 0 & 0 & x_{55} \end{bmatrix}, \quad \begin{matrix} x_{ij} \in I\!R^* \\ i = 1, \ldots, 5, \\ j = 2, \ldots, 5 \end{matrix} \tag{2.2}$$

From this construction of g we conclude that the enformophism $J_0 : \ J_0 = (b_{kl}) \in I\!R^*$, $1 \le k, l \le 8$ which must satisfy the relations

$$J_0 : g \to g, J_0 : X \to J_0(X), J_0^2 = -I \tag{2.3}$$

$$[X, Y] + J_0 [J_0(X), Y] + J_0 [X, J_0(Y)] = [J_0(X), J_0(Y)] \tag{2.4}$$

is

$$J_0 = \begin{bmatrix} k & 0 & 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & \chi & 0 & 0 & 0 & \nu & 0 & 0 \\ 0 & 0 & \varphi & 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \tau & 0 & 0 & 0 & \sigma \\ -\frac{1+k^2}{\mu} & 0 & 0 & 0 & -k & 0 & 0 & 0 \\ 0 & -\frac{1+\chi^2}{\nu} & 0 & 0 & 0 & -\chi & 0 & 0 \\ 0 & 0 & -\frac{1+\varphi^2}{\rho} & 0 & 0 & 0 & -\varphi & 0 \\ 0 & 0 & 0 & -\frac{1+\tau^2}{\sigma} & 0 & 0 & 0 & -\tau \end{bmatrix}, \tag{2.5}$$

where μ ,ν, ρ ,$\sigma \in I\!R^*$. We consider a linear form w on g with the properties:

$$w : g \to g, \ w : X \to \omega(X), \ \omega [J_0(X), J_0(Y)] = \omega ([X, Y]) \tag{2.6}$$

and

$$w ([J_0(X), Y]) > 0. \tag{2.7}$$

We take

$$\omega (x) = \langle x_0, x \rangle (2 - k), \tag{2.8}$$

where $\langle \cdot, \cdot \rangle$ the usual inner product on g and $x_0 = (k_1, k_2, k_3, k_4.........., k_8)$ is a fixed vector.

In order that ω satisfies the conditions (2.6) , (2.7) we must have : $k_1\mu \succ 0$, $k_2\mu \succ 0$, $k_3\mu \succ 0$, $k_4\mu \succ 0$.

Now we determinate the solvable Lie group S which corresponds to the solvable Lie algebra g.

From the above we conclude that the solvable Lie group S of g is defined by:

$$S = \begin{bmatrix} 1 & \frac{x}{y}\left(e^{y}-1\right) & \frac{\alpha}{\tau}\left(e^{\tau}-1\right) & \frac{\gamma}{\delta}\left(e^{\delta}-1\right) & \frac{k}{\lambda}\left(e^{\lambda}-1\right) \\ 0 & e^{y} & 0 & 0 & 0 \\ 0 & 0 & e^{\tau} & 0 & 0 \\ 0 & 0 & 0 & e^{\delta} & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda} \end{bmatrix} \tag{2.9}$$

The inner products on the solvable Lie algebra g is defined by

$$\begin{aligned} \langle x, y\rangle &= w\left(\left[J_0\left(x\right), y\right]\right), \\ \langle x, y\rangle &= w\left(\left[J_0\left(x, y\right)\right]\right), \end{aligned}$$

where w is given by (2.8). This inner product determines the Kahler metric on G, which is essentially the Bergmann metric on it.

3 Some basic constructions

Let $D \subseteq R^n$ be a convex domain, non invariant under any affine transformations on $\mathbb{R}^n$. If the group $A\left(D\right)$ acts transitivelly in D, then the domain D is said to be homogeneous. From an homogeneous convex domain in D in $\mathbb{R}^n$, we define an homogeneous convex cone $V = V(D) \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ as follows [7]

$$D(V) = \left\{(tx, t) \in \mathbb{R}^n \times \mathbb{R} :\; x \in D,\, t > 0\right\}, \tag{3.1}$$

which is called convex cone on the domain D. Let $G\left(Y\right)$ be the group of all linear automorphisms of V and g_v be the cunonical $G\left(Y\right)$-invariant Riemannian metric on V.

Then the natural embedding [7] is equivariant with respect to the groups $A\left(D\right)$ and $G\left(Y\right)$. Therefore, the Riemannian metric on D: $\mathrm{g}_D = x \cdot g_v$ induced from (V, g_0) by X is $A\left(D\right)-$invariant.

The Riemannian metric g_D is called the canonical metric on D.

We note that the canonical metric g_D is given from the characteristic function φ_V of V as follows:

Let us put $\varphi_D = \varphi_V * X$. Then

$$g_D = \sum \frac{\partial^2 \log \varphi_D}{\partial x^i \partial x^j} dx^i dx^j. \tag{3.2}$$

After all the above we define the relation which gives the connection between the solvable algebra of one J-algebra g and one convex cone of a Siegel domain genus II.

4 The solvable Lie algebras and convex cones of dimension 8

Let

$$A = \sum_{1 \leq i,j \leq 5} A_{ij} \tag{4.1}$$

be a T-algebra of rank 5 provided with an involutive anti-automorphism *.Generally, the elements of A_{ij} will be demoted as a_{ij}, b_{ij}, ... and an arbituary element of A will be written as a matrix $a = (a_{ij})$ where a_{ij} is the A_{ij}-component of a.

We define the subsets g, $T(A)$, $V(A)$ and X as follows:

a)

$$g = \left\{ \lambda = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & x_{22} & 0 & 0 & 0 \\ 0 & 0 & x_{33} & 0 & 0 \\ 0 & 0 & 0 & x_{44} & 0 \\ 0 & 0 & 0 & 0 & x_{55} \end{pmatrix} \in A : x_{ij} \in I\!R^*,\ 1 \leq j \leq i \leq 5 \right\}, \tag{4.2}$$

which is a solvable Lie algebra, dimension 8 over $I\!R$.

b)

$$T(A) = \{\lambda = (\lambda_{ij}) \in A : \lambda_{ii} > 0,\ 2 \leq i \leq 5,\ \lambda_{ij} = 0,\ 1 \leq i,j \leq 5\} \tag{4.3}$$

is a subset of g with the diagonal elements positive.

c)

$$V(A) = \left\{ \lambda + \lambda^* = \begin{pmatrix} 0 & x & y & z & \Lambda \\ x & k_{22} & 0 & 0 & 0 \\ y & 0 & k_{33} & 0 & 0 \\ z & 0 & 0 & k_{44} & 0 \\ \Lambda & 0 & 0 & 0 & k_{55} \end{pmatrix},\ \begin{array}{c} k_{22} = x_{22} + x_{242} \\ k_{33} = x_{33} + x_{33} \\ k_{44} = x_{44} + x_{44} \\ k_{55} = x_{55} + x_{55} \end{array} \right\}, \tag{4.4}$$

$x = x_{12}$, $y = x_{13}$, $z = x_{14}$, $\Lambda = x_{15}$.

d)

$$V(A) \subset X = \{x \in A,\ x^* = x\} \tag{4.5}$$

is the set of symmetric (Hermitian) matrices.

Then it is known [8] that there exists an homogeneous convex cone in the real vector space X and $T\,(A)$ is a connected Lie group which acts on V simply transitively as a linear transformation by:

$$\lambda = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & x_{22} & 0 & 0 & 0 \\ 0 & 0 & x_{33} & 0 & 0 \\ 0 & 0 & 0 & x_{44} & 0 \\ 0 & 0 & 0 & 0 & x_{55} \end{pmatrix},$$

$$S+S^*=\begin{pmatrix} 0 & S_{12} & S_{13} & S_{14} & S_{15} \\ S_{12} & S_{22}+S_{22} & 0 & 0 & 0 \\ S_{13} & 0 & S_{33}+S_{33} & 0 & 0 \\ S_{14} & 0 & 0 & S_{44}+S_{44} & 0 \\ S_{15} & 0 & 0 & 0 & S_{55}+S_{55} \end{pmatrix} \to$$

$$(\lambda+S)+(\lambda+S)^*=$$

$$\begin{bmatrix} 0 & x_{12}+S_{12} & x_{13}+S_{13} & x_{14}+S_{14} & x_{15}+S_{15} \\ x_{12}+S_{12} & x_{22}+S_{22} & 0 & 0 & 0 \\ x_{13}+S_{13} & 0 & x_{33}+S_{33} & 0 & 0 \\ x_{14}+S_{14} & 0 & 0 & x_{44}+S_{44} & 0 \\ x_{15}+S_{15} & 0 & 0 & 0 & x_{55}+S_{55} \end{bmatrix} \tag{4.6}$$

that is

$$(\lambda, S\cdot S^*) \in T(A)\times V(A) \to (\lambda\cdot S)\cdot(\lambda\cdot S^*) \in V. \tag{4.7}$$

Conversely, every homogeneous convex cone is realized in this form up to a linear equivalence.

Moreover, the element $e=(e_{ij})$, $e_{ij}=(\delta_{ij})$ (the Kronecker delta). is the unit element of T and also e is contained in V.

Hence, the tanget space $T_e(V)\cdot fV$ at the point e may by naturally identified with the ambient space X and also with the Lie algebra λ of T. On the other hand, the solvable Lie algebra λ may be identified with the subset $\sum_{1\le j<j\le 5} A_{ij}$ of A provided with the bracket $[a,b]=ab-ba$.

A canonical linear isomorphism between λ and X is given by [5]:

$$\xi : a \in \lambda = \sum_{1\le j<j\le 5} A_{ij} \to a+a^* \in X = T_e(V). \tag{4.8}$$

Theorem 4.9 [**5**] *The connection between solvable Lie algebras and convex cones of dimension* 8 *was described by a canonical isomorphism* (4.8)*. The canonical Riemannian metric g_V at the point e has an inner product $\langle\cdot,\cdot\rangle$ on λ defined by*

$$\langle a,b\rangle = g_{V(e)}\left(\xi\left(a\right),\xi\left(b\right)\right). \tag{4.9}$$

for every $a,b\in\lambda_V$. The inner product $\langle\cdot,\cdot\rangle$ has the following expression

$$\langle a,b\rangle = g_P\left(\xi\left(a\right),\xi\left(b\right)\right). \tag{4.10}$$

References

[1] E. Cartan, *Sur les domaines bornees homogénes de l'espace de variable complex*, Abn. Math. Sem. Hamburg Univ, 11 (1936) 116-162.

[2] G. Dimou, *New representations of a special nonsymmetric homogeneous domain in C^n, $n=8$*, Mathematica Balkanica NS, 9 (1995) 43-46.

[3] A. Ledyev and M. Obata, *Affine and Riemannian S-manifolds*, J.Diff. Geometry, 2 (1968) 451-459.

[4] I Pyatetsky and J. Shapiro, *Automorphic functions and the geometry of classical domains*, Gordon and Breach, New York, 1969.

[5] T. Tadashi, *Symmetric homogeneous convex domains*, Nagoya Math. Jour., 93 (1994) 1-17.

[6] Gr. Tsagas and G. Dimou, *New representation of non-symmetric homogeneous bounded domains in C^4 and C^5*, Inter. Jour. Mat., Univ. of Florida, (1992) 741-752.

[7] E. B. Vinberg, *The theory of convex homogeneous cones*, Trans. Moscow Mat. Soc., 12 (1963) 340-430.

[8] E. B. Vinberg, *The structure of the group of authomorphisms of homogeneous convex cones*, Trans. Moscow Mat. Soc., 13 (1965) 63-93.

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