SUFFICIENT CRITERIA FOR A SUPERDIFFERENTIABLE SUPERCURVE TO BE A MINIMUM ON RIEMANNIAN SUPERMANIFOLDS

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Abstract

Let L define a regular problem in the calculus of variations on supermanifolds. Sufficient criteria for a piecewise superdifferentiable supercurve C in sense of Rogers to be a strict weak local minimum on Riemannian supermanifolds is given.

AMS Subject Classification: 58A50.
Key words: Legendre supertransformations, super-Hamiltonian, supervector space, super-Lagrangian, strict weak local minimum, superbundle, supermanifolds, sub-supermanifolds.

1 Introduction

Let V be a supervector space [3], V^* be the dual supervector space [6], M be a supermanifold in the sense of Rogers [8] and $T(M)$ be the tangent superspace or superbundle [6]over M. Let us consider only algebras over the reals. For each positive integer L, B_L [8] will denote the Grassmann algebra over the reals with generators 1^L ,β_1^L,..., β_L^L and relations

$$1^L \cdot \beta_i^L = \beta_i^L \cdot 1^L = \beta_i^L \quad i = 1, ..., L,$$

$$\beta_i^L \cdot \beta_j^L = -\beta_j^L \cdot \beta_i^L \quad i, j = 1, ..., L.$$

B_L is a graded algebra [9] and can be written as a direct sum

$$B_L = (B_L)_0 \oplus (B_L)_1 \quad [8],$$

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2002,* 40-48

where $(B_L)_0$ and $(B_L)_1$ are the even and odd part of (B_L) respectively. We consider the (m,n)-dimensional supereuclidean space as ${B_L}^{m,n} = {(B_L)_0}^m \oplus {(B_L)_1}^n$ [8] with $L > n$. Let M_L denote (due to Kostant [7]) the set of finite sequences of positive integers
$\mu = (\mu_1, ..., \mu_k)$ with $1 \leq \mu_1 < ... < \mu_k \leq L$. M_L includes the sequence with no elements, denoted ϕ. As it follows in [6] for each μ in M_L,

$$\beta_\mu^{(L)} = \beta_{\mu_1}^{(L)} \cdots \beta_{\mu_k}^{(L)}, \quad k = 1, ..., L$$

and

$$\beta_\phi^{(L)} = 1^{(L)}$$

a typical element b of B_L may be expressed as

$$b = \sum_{\mu \in M_L} b^\mu \cdot \beta_\mu^{(L)},$$

where the coefficients b^μ are real numbers. With the norm on B_L defined by

$$||b|| := \sum_{\mu \in M_L} |b^\mu|,$$

B_L is a Banach algebra [8] .

We consider the body map (in DeWitt's terminology [6])

$$\varepsilon_L : B_L \to \mathbf{R}$$

given by

$$\varepsilon_L(b) = b^\phi.$$

For the first time, I have introduced on [3] ${B_L}^{m+n} = {B_L}^{m,n} \oplus {B_L}^{n,m}$ with $n = 2r$, the scalar product

$$\langle v, w \rangle = x^1 \cdot y^1 + \cdots + x^m \cdot y^m + \theta^1 \cdot \theta'^{r+1} + \cdots + \theta^r \cdot \theta'^n - \theta^{r+1} \cdot \theta'^1 - \cdots - \theta^n \cdot \theta'^r$$

$(\forall)$ $v = (x^1, \ldots, x^m, \theta^1, \ldots, \theta^n), w = (y^1, \ldots, y^m, \theta'^1, \ldots, \theta'^n) \in {B_L}^{m+n}$.

Definition 1.[8] A function $f : {B_L}^{m,n} \to B_L$ is called a *superdifferentiable function* if there exist $f_\mu \in C^\infty(, \mathbf{R}^m, \mathbf{R})$ such that:

$$f(x, \theta) = \sum_{\mu \in M_n} f_\mu(x) \theta^\mu,$$

where $M_n = \{(\mu_1, \ldots, \mu_n)/1 \leq \mu_1 < \ldots < \mu_n \leq n\}$ [6].

Let M be a Hausdorff topological space. [7] (a) An (m,n) chart on M over B_L is a pair (U, ψ) with U an open set of M and ψ a homeomorphism of U onto an open subset of ${B_L}^{m,n}$. (b) An (m,n) superdifferentiable structure on M over B_L is a collection $\{(U_\alpha, \psi_\alpha) | \alpha \in \Lambda\}$ of (m,n) charts on M such that $(I) M = \cup_{\alpha \in \Lambda} U_\alpha$ and (II) for each pair α , β in Λ the mapping $\psi_\beta \circ {\psi_\alpha}^{-1}$ is a superdifferentiable function of

$\psi_\alpha(U_\alpha \cap U_\beta)$ onto $\psi_\beta(U_\alpha \cap U_\beta)$, (III) the collection $\{(U_\alpha, \psi_\alpha)|\alpha \in \Lambda\}$ is a maximal collection of open charts for which (I) and (II) hold.

Definition 2. [8] An (m,n)-*dimensional superdifferentiable supermanifold* over B_L is a Hausdorff topological space M with an (m,n) superdifferentiable structure over B_L.

Definition 3. [6] A subset M' of a supermanifold M of dimension (m,n) is called a *sub-supermanifold* of dimension (m',n'), $m \geq m'$, $n \geq n'$, if M' is contained in the union of a set $\{(U,\psi)\}$ of charts each of which has the property that, for all $(x,\theta) \in U \cap M'$,

$$\psi(x,\theta) = (x^1, ..., x^{m'}, a^{m'+1}, ..., a^m, \theta^1, ..., \theta^{n'}, \eta^{n'+1}, ..., \eta^n),$$

where $(a^{m'+1}, ..., a^m, \eta^{n'+1}, ..., \eta^n)$ is a fixed element of ${B_L}^{m-m',n-n'}$ depending on the chart in question.

The pairs $\{(U',\psi')\}$, where $U' = U \cap M'$ and $\psi'(x,\theta) = (x^1, ..., x^{m'}, \theta^1, ..., \theta^{n'})$ constitute an *atlas* for M' .

Example 1. Let us consider ${B_L}^{m,n}$ the (m,n)-dimensional supereuclidean space. ${B_L}^{m,n}$ is an (m,n)-dimensional superdifferentiable supermanifold over B_L from the *Definition 2.*. We consider the following subset of ${B_L}^{m,n}$, ${S_L}^{m-1,n-2} = \{(x,\theta) \in {B_L}^{m,n}/{(x^1)}^2+...+{(x^m)}^2+2\cdot\theta^1)\cdot\theta^{r+1})+...+2\cdot\theta^r)\cdot\theta^n) = 1+2\cdot\beta_1)\cdot\beta_{r+1})+...+\beta_r)\cdot\beta_n)\}$ and we conclude that ${S_L}^{m-1,n-2}$ is an $(m-1,n-2)$-dimensional sub-supermanifold of ${B_L}^{m,n}$.

Definition 4. [4], [8] The function $C : [a,b] \to M$ is called a *superdifferentiable supercurve* [3] if the functions $x^i \circ C$ $(\forall)$ $1 \leq i \leq m$ and $\theta^\alpha \circ C$ $(\forall)$ $1 \leq \alpha \leq n$ are superdifferentiable [8], the functions $\varepsilon_L \circ x^i \circ C$ $(\forall)$ $1 \leq i \leq m$ and $\varepsilon_L \circ \theta^\alpha \circ C$ $(\forall)$ $1 \leq \alpha \leq n$ are differentiable in $\mathbf{R}$ and (x^i, θ^α) are the coordinates of a point $p \in M$.

2 Legendre Supertransformation, Super-Hamiltonian, Super-Lagrangian

Definition 5. [4] Let L be a superdifferentiable function on $T(M) \times B_L$ and we make distinction between this superdifferentiable function L and the positive integer L. Then L defines a superdifferentiable map $L' : T(M) \times B_L \to T^*(M) \times B_L$ called the *Legendre supertransformation*, given in local coordinates by $x^i \circ L' = x^i$ $(\forall)$ $1 \leq i \leq m$,

$\theta^\alpha \circ L' = \theta^\alpha$ $(\forall)$ $1 \leq \alpha \leq n$,

$y^i \circ L' = (\partial L/\partial x^i)$ $(\forall)$ $1 \leq i \leq m$

$\delta^\alpha \circ L' = (\partial L/\partial \theta^\alpha)$ $(\forall)$ $1 \leq \alpha \leq n$ and $t \circ L' = t$.

Definition 6. [4] If the Legendre supertransformation is an immersion [5] of $T(M) \times B_L$ into $T^*(M) \times B_L$, then the function L will be called a *regular super-Lagrangian*.

Definition 7. [4] If the Legendre supertransformation is an immersion, the map ${L'}^{-1}$ comes locally in a similar way from a function H on $T^*(M) \times B_L$ called the *super-Hamiltonian*:

$$H(y, \delta) = \langle {L'}^{-1}(y, \delta), (y, \delta) \rangle - L \circ \mathcal{L}^{-1}(y, \delta).$$

The function $E = H \circ L'$ is globally well defined on $T(M) \times B_L$.

Theorem 1. [4] *If M is a Riemannian supermanifold* [5] *and $L(v, t) = (1/2) \cdot \langle v, v \rangle$ then L is a regular super-Lagrangian and L' coincides on each tangent superspace* [5] *with the map of $T_q(M) \to T_q^*(M)$ given by the scalar product* [3]. *Futhermore, in this case, $E = H \circ L' = L$.*

Proof. See [4]. □

Definition 8. [4] Let $C : [a, b] \to M$ be a superdifferentiable supercurve on M. Then C determines a supercurve, $\tilde{C}$, on $T(M) \times B_L$ defined by

$$\tilde{C}(t) = (C'(t), t)$$

for each $t \in [a, b]$. Therefore, we can consider the integral

$$I(C) = \int_a^b L(\tilde{C}(t))dt.$$

Let C_j and C_j^1 be the restrictions of C and C^1 respectively to the interval $[s_j, s_{j+1}]$, where $a = s_0 < ... < s_r = b$ and $W \subset M$ C_j and C_j^1 be superdifferentiable supercurves of $(s_j + \varepsilon, s_{j+1} - \varepsilon)$ into W.

Definition 9. [4] A supercurve C is called *(strict) weak local minimum* if there are W and $\varepsilon > 0$ such that $\varepsilon_L(I(C)) \leq \varepsilon_L(I(C^1))$ or $\varepsilon_L(I(C)) < \varepsilon_L(I(C^1))$ for all piecewise superdifferentiable supercurves satisfying

$$C^1(a) = C(a) \quad \textit{and} \quad C^1(b) = C(b).$$

Proposition 1. [4] *Let C be a weak local minimum of L. Then at every point t, where C is superdifferentiable the tangent supervector $Y_q = C'(t)$ satisfies*

$$Y_q \rfloor d\omega_q = 0$$

for $\theta^\alpha(t) = t \cdot (\overline{\delta}^\alpha(t) + \overline{\delta}^{\alpha+r}(t))$ and $\theta^{\alpha+r}(t) = t \cdot (\overline{\delta}^{\alpha+r}(t) - \overline{\delta}^\alpha(t))$ for all $\alpha \in \{1, ..., r\}$ and $(y, \overline{\delta})$ are coordinates on $({B_L}^{m+n})^*$, where $e_i \rfloor e^{j_1} \wedge \ldots \wedge e^{j_r} =$

$$\begin{cases} 0 & if \quad i \neq j_k \quad for \quad any \quad k \\ (-1)^{(i)+k-1} \cdot e^{j_1} \wedge \ldots \wedge e^{j_{k-1}} \wedge e^{j_{k+1}} \wedge \ldots \wedge e^{j_r} & if \quad i = j_k \end{cases}$$

and (i) is 0 if $e_i \in {B_L}^{m,n}$ and is 1 if $e_i \in {B_L}^{n,m}$ and where $(e_i)_{i=1,\ldots,m+n}$ is a basis of ${B_L}^{m+n}$ and $(e^j)_{j=1,\ldots,m+n}$ is a basis of $({B_L}^{m+n})^*$.

Proof. See [4]. □

Definition 10. [4] A piecewise superdifferentiable supercurve C on M such that $\tilde{C}$ is an integral supercurve of X is called an *extremal*.

Theorem 2. [4] *Let L define a regular problem in the calculus of variations on supermanifolds. A necessary condition that a piecewise superdifferentiable supercurve C in sense of Rogers be a weak local minimum for L is that C be superdifferentiable and $\tilde{C}$ be an integral supercurve of X, where X is defined by* (3) $X\rfloor d\sigma = 0$,$\omega = dL$ $\langle X, dt\rangle = 1$,*the superform $\sigma = L'^{*}\omega$ is well defined on $T(M) \times B_L$ and L' is an immersion of $T(M) \times B_L$ into $T^{*}(M) \times B_L$.*

Proof. See [4]. □

Let U be an open subset of M, let I an open subset of B_L, let π denote the projection of $T(M) \times B_L$ onto M and let ψ be a map of $U \times I \to B_L$ such that $(\pi \times id) \circ \psi(x, \theta, t) = (x, \theta, t)$ for $(x, \theta, t) \in U \times I$. For each fixed t, $\psi(\cdot, t)$ gives a supervector field on U; thus ψ represents a " system of superdifferential superequations " on U. Suppose, furthermore that (4) $L \circ \psi = 0$ and $\varepsilon_{\circ} L > 0$ in $(\pi \times id)^{-1}(U \times I) - \psi(U \times I)$.

Lemma 1. [5] *Let U,I, L, and ψ as above, $[a, b] \subset I$ and $C : [a, b] \to M$ be a solution supercurve of ψ lying in U, i.e., let a) $C([a, b]) \subset U$ and b) $\tilde{C}(t) = \psi(C(t), t)$. . Then C is minimum supercurve for L. In fact, if D is any piecewise superdifferentiable supercurve $D : [a, b] \to U$ with $D(a) = C(a)$ then* (5) $\varepsilon_L(I(C)) < \varepsilon_L(I(D))$ *unless $C = D$.*

Proof. See [5]. □

Lemma 2. [5] *Suppose we replace in* Lemma 1.*, the second part of condition* (4) *by $\varepsilon_L \circ L > 0$ in $W - \psi(U \times I)$, where W is some neighborhood of $\psi(U \times I)$ in $(\pi \times id)^{-1}(U \times I)$. Then we conclude* (5) *for all $D : [a, b] \to U$ with $D(a) = C(a)$ such that $\tilde{D}(t) \in W$ (when $\tilde{D}$ is defined).*

For proof, the argument is the same as in *Lemma1*.

If L is the superenergy of a Riemann supermanifold, the only map ψ giving (4) assings to each $(x, \theta) \in U$ the zero tangent supervector at (x, θ) . The integral supercurves C then become the constant map which surely satisfy (5).

Let S be a function defined on $U \times I$. Define the function $\tilde{S}$ on $\pi^{-1}(U \times I)$ by (6) $\tilde{S}(v, t) = v(S) + (\partial S/\partial t)$ or in local coordinates

$$(7) \qquad \tilde{S}(x^1, ..., x^m, \theta^1, ..., \theta^n, \dot{x}^1, ..., \dot{x}^m, \dot{\theta}^1, ..., \dot{\theta}^n, t) = \textstyle\sum_{i=1}^{m} (\partial S/\partial x^i) \cdot \dot{x}^i +$$

$$+ \sum_{\alpha=1}^{r} (\partial S/\partial \theta^{\alpha}) \cdot \dot{\theta}^{\alpha+r} - \sum_{\alpha=1}^{r} (\partial S/\partial \theta^{\alpha+r}) \cdot \dot{\theta}^{\alpha} + (\partial S/\partial t),$$

where $n = 2r$. Then (8)

$$\int_a^b L(\tilde{S}(\tilde{D}(t))dt = \varepsilon_L(S(D(b), b)) - \varepsilon_L(S(D(a), a)).$$

for any superdifferentiable supercurve $D : [a, b] \to U$.

Lemma 3. [5] *Let $\psi : U \times I \to T(M) \times B_L$ be such that $(\pi \times id) \circ \psi = id$, and C satisfies a) and b) from* Lemma 1*. Suppose that there is a function S defined on $U \times I$ such that* (9) $(L - \tilde{S}) \circ \psi = 0$ *and* (10) $\varepsilon_L(L - \tilde{S})(p) > 0$ *for $p \in (\pi \times id)^{-1}(U \times I) - \psi(U \times I)$. Then for any supercurve $D : [a, b] \to U$ (where*

$[a,b]=I$), with $D(a)=C(a)$ we have

$$(11)\qquad \varepsilon_L(I(C)) < \varepsilon_L(I(D)) - \varepsilon_L(S(D(b),b)) + \varepsilon_L(S(D(a),a))$$

unless $D=C$. If (10) *holds only for $p \in W - \psi(U \times I)$, where W is a neighborhood of $\psi(U\times I)$ in $(\pi \times id)^{-1}(U\times I)$, then* (11) *is true for those D satisfying $\tilde{D}(t) \in W$ for $t \in [a,b]$.*

The proof is the same as that of *Lemma1.* using (8).

Definition 11. [5] We will say that L is *positive definite* if, for all points, p, of $T(M)\times B_L$ the matrix $A=$

$$\begin{pmatrix} \varepsilon_L(((L_{\dot{x}^i\dot{x}^j})(p))) & \varepsilon_L(((L_{\dot{x}^i\dot{\theta}^\beta})(p))) \\ \varepsilon_L(((L_{\dot{\theta}^\alpha\dot{x}^j})(p))) & \varepsilon_L(((L_{\dot{\theta}^\alpha\dot{x}^\beta})(p))) \end{pmatrix}$$

is positive definite, where $(L_{\dot{x}^i\dot{x}^j})(p) = (\partial^2 L/\partial\dot{x}^i\partial\dot{x}^j)(p)$, $(L_{\dot{x}^i\dot{\theta}^\beta})(p) = (\partial^2 L/\partial\dot{x}^i\partial\dot{\theta}^\beta)(p)$, $(L_{\dot{\theta}^\alpha\dot{x}^j})(p) = (\partial^2 L/\partial\dot{\theta}^\alpha\partial\dot{x}^j)(p)$ and $(L_{\dot{\theta}^\alpha\dot{\theta}^\beta})(p) = (\partial^2 L/\partial\dot{\theta}^\alpha\partial\dot{\theta}^\beta)(p)$.

Theorem 3. [5] *If L is positive definite, then condition* (9) *and* (12) $\partial[(L-\tilde{S})\circ\psi]\partial\dot{x}^i = 0$ *and* (13) $\partial[(L-\tilde{S})\circ\psi]\partial\dot{\theta}^\alpha = 0$ *implies that* (10) *hold for all $p \in W$ for some neighborhood W of $\psi(U\times I)$. If L is the superenergy of a Riemann supermetric then* (12) *implies* (10)*.[Condition* (12)*, although expressed in terms of local coordinates.It says that if v is supervector tangent to $T_{(x,\theta)}(M)$ (more precisely to $(\pi\times id)^{-1}(x,\theta,t)$, at $\psi(x,\theta,t)$, then $v(L-\tilde{S})=0$.]*

Proof. See [5]. □

3 Hamilton-Jacobi Superequation on Supermanifolds and Sufficient Criteria for a Superdifferentialble Supercurve to be a Minimum on Riemannian Supermanifolds

In the case of positive definite super-Lagrangians we are looking for maps ψ for which there is an S satisfying (9) and (12). It is now convenient to apply the Legendre supertransformation to the whole setup and pass to $T^*(M)\times B_L$. If ψ is a map of $U\times I \to T(M)\times B_L$ then $\phi_1 = L'\circ\psi$ maps $U\times I \to T^*(M)\times B_L$. Let $x^1,...,x^m,\theta^1,...,\theta^n,\dot{x}^1,...,\dot{x}^m,\dot{\theta}^1,...,\dot{\theta}^n,t$ be the local coordinates on $T(M)\times B_L$ and $x^1,...,x^m,\theta^1,...,\theta^n,y^1,...,y^m,\delta^1,...,\delta^n$ be the corresponding local coordinates on $T^*(M)\times B_L$ .Let $\dot{x}^i(x,\theta,t) = \dot{x}^i\circ\psi(x,\theta,t)$, $\dot{\theta}^\alpha(x,\theta,t) = \dot{\theta}^\alpha\circ\psi(x,\theta,t)$, $y^i(x,\theta,t) = y^i\circ\phi_1(x,\theta,t)$ and $\delta^\alpha(x,\theta,t) = \delta^\alpha\circ\phi_1(x,\theta,t)$ $y^i(x,\theta,t) = (\partial L/\partial\dot{x}^i)(\psi(x,\theta,t))$ and $\delta^\alpha(x,\theta,t) = (\partial L/\partial\dot{\theta}^\alpha)(\psi(x,\theta,t))$. Then $(\partial(L-\tilde{S})/\partial\dot{x}^i)\circ\psi = (\partial L/\partial x^i)\circ\psi - (\partial S/\partial x^i)$ and $(\partial(L-\tilde{S})/\partial\dot{\theta}^\alpha)\circ\psi = (\partial L/\partial\theta^\alpha)\circ\psi - (\partial S/\partial\theta^\alpha)$ by (7). Thus (12) becomes (14) $y^i(x,\theta,t) = (\partial S/\partial x^i)$ and (13) becomes (15) $\delta^\alpha(x,\theta,t) = (\partial S/\partial\theta^\alpha)$. Also $L-\tilde{S} = L - \sum_{i=1}^m (\partial S/\partial x^i)\cdot\dot{x}^i - \sum_{\alpha=1}^r (\partial S/\partial\theta^\alpha)\cdot\dot{\theta}^{\alpha+r} + \sum_{\alpha=1}^r (\partial S/\partial\theta^{\alpha+r})\cdot\dot{\theta}^\alpha - (\partial S/\partial t) = L - \sum_{i=1}^m y^i\cdot\dot{x}^i - \sum_{\alpha=1}^r \delta^\alpha\cdot\dot{\theta}^{\alpha+r} + \sum_{\alpha=1}^r \delta^{\alpha+r}\cdot\dot{\theta}^\alpha - (\partial S/\partial t)$.

But $L - \sum_{i=1}^{m} y^i \cdot \dot{x}^i - \sum_{\alpha=1}^{r} \delta^\alpha \cdot \dot{\theta}^{\alpha+r} + \sum_{\alpha=1}^{r} \delta^{\alpha+r} \cdot \dot{\theta}^\alpha = -H$ so that (9) becomes (16) $(\partial S/\partial t) + H \circ \phi_1 = 0$. We can regard (14), (15) and (16) as a single partial superdifferential superequation (17) $(\partial S/\partial t) + H(x^1, ..., x^m, \theta^1, ..., \theta^n, (\partial S/\partial x^1), ..., (\partial S/\partial x^m), (\partial S/\partial \theta^1), .. \quad .., (\partial S/\partial \theta^n), t) = 0$. This equation is called the Hamilton-Jacobi superequation. We observe that (14), (15) and (16) can be written more succinctly as (18) $\phi_1^* \omega = dS$ .In fact, $\phi_1^* \omega = \sum_{i=1}^{m} y^i(x, \theta, t) dx^i + \sum_{\alpha=1}^{n} \delta^\alpha d\theta^\alpha - (H \circ \psi) dt$ and $dS = \sum_{i=1}^{m} (\partial S/\partial x^i) dx^i + \sum_{\alpha=1}^{n} (\partial S/\partial \theta^\alpha) d\theta^\alpha - (\partial S/\partial t) dt$. Comparing the coefficients of $dx^i, d\theta^\alpha$ and dt gives (14), (15) and (16). We are therefore looking for maps ϕ_1 for which there is a function S such that $\phi_1^* \omega = dS$ .There will certainly be such an S if $\phi_1^* d\omega = 0$ and $H^1(U \times I) = 0$. In other words we have:

Lemma 4. *Let L be positive definite . $H^1(U \times I) = 0$ and $\phi_1 : U \times I \to (\pi \times id)^{-1}(U \times I)$ satisfy $(\pi \times id)\phi_1 = id$ and*

$$(19) \qquad \phi_1^* d\omega = 0.$$

Then if S is defined by (18) *and $\psi = {L'}^{-1} \circ \phi_1$, then ψ and S satisfy* (9) *and* (10) *for all p in some neighborhood of $\psi(U \times I)$. If L is the superenergy of a Riemannian supermetric, then* (10) *holds for all $p \in (\pi \times id)^{-1}(U \times I)$.*

In our quest for a map ϕ_1 satisfying (18), we will split the problem up into two parts. One part is to find (20) a sub-supermanifold N of $T^*(M) \times B_L$ satisfying $i^*(\omega) = dS^*$, where i is the injection of $N \to T^*(M) \times B_L$ and S^* is a suitable function on N and other part is to find (21) a sub-supermanifold N of $T^*(M) \times B_L$ satisfying $i^*(\omega) = dS^*$ such that $\pi \times id$ restricted to N is a diffeomorphism [6] of N onto $U \times I$ (for suitable $U \subset M$ and $I \subset B_L$).

We are interested in $(m+1, n+2)$-dimensional sub-supermanifold N with the properties above.

Lemma 5. *Let N_0 be an (m,n)-dimensional sub-supermanifold of $T^*(M) \times B_L$. Suppose that the Euler superfield, $Y = {L'}_* X$, is nowhere tangent to N_0 and ${i_0}^*(d\omega) = 0$, where i_0 is the injection of $N_0 \to T^*(M) \times B_L$. Let N be the sub-supermanifold swept out, locally by N under flow α , generated by Y . Then $i^*(d\omega) = 0$ if i is the injection of $N \to T^*(M) \times B_L$.*

Proof. What we have to show is that for every pair of supervectors Z_1 and Z_2 tangent to N at $p \in N$, $\langle Z_1 \wedge Z_2 | d\omega_p \rangle = 0$. Now the tangent superspace $T_p(N)$ is spanned by ${\alpha_t}^*(T_{\alpha_{-t}(p)}(N_0))$ and $Y(p)$, where $\alpha_{-t}(p) \in N_0$. If Z_1 and Z_2 both lie in ${\alpha_t}^*(T_{\alpha_{-t}(p)}(N_0))$ then, since ${\alpha_t}^* d\omega = d\omega$, $\langle Z_1 \wedge Z_2 | d\omega_p \rangle = \langle {\alpha_t}^* Z_1 \wedge {\alpha_t}^* Z_2 | d\omega_p \rangle = \langle Z_1 \wedge Z_2 | d\omega_{\alpha_{-t}(p)} \rangle = 0$, where $Z_1, Z_2 \in T_{\alpha_{-t}(p)}(N_0)$. By liniarity, the only remaining case is $\langle Y \wedge Z_1 | d\omega_p \rangle = \langle Z_1 | Y(p) \rfloor d\omega \rangle = 0$, by (3). This proves *Lemma5*.. □

We should remark that since $\langle Y | dt \rangle = 1$, any simultaneous sub-supermanifold, i.e., any supermanifold on which $t = const$, is nowhere tangent to Y. As an example of an N_0 satisfying the hypothesis of *Lemma5*. let $x^1, ..., x^m, \theta^1, ..., \theta^n$ be a coordinate system in a neighborhood U about a point p and let $v^* = \sum_{i=1}^{m} {y_0}^i d{x_p}^i + \sum_{\beta=1}^{n} {\delta_0}^\beta d{\theta_p}^\beta \in {T_p}^*(M)$. Let N_0 be the sub-supermanifold of ${T_p}^*(M)) \times B_L$, consisting, in local coordinates, of all points of the form $(x^1, ..., x^m, {y_0}^1, ..., {y_0}^m, \theta^1, ..., \theta^n, {\delta_0}^1, ..., {\delta_0}^n, t_0)$, i.e., defined by the equation $y^i = {y_0}^i$, $\delta^\beta = {\delta_0}^\beta$ and

$t = t_0$. Then $i_0{}^*(d\omega) = i_0{}^*(\sum_{i=1}^m dy^i \wedge dx^i + \sum_{\beta=1}^n \delta^\beta \wedge d\theta^\beta - dH \wedge dt) = 0$ since $i_0{}^*(dy^i) = i_0{}^*(d\delta^\beta) = i_0{}^*(dt) = 0$. If we choose U to be diffeomorphic [6] to ${B_L}^{m,n}$ then $N = N_0 \times I$ is diffeomorphic to ${B_L}^{m+1,n+2}$, therefore $H^1(N) = 0$. By *Lemma5.*, $i^*(d\omega) = 0$ or $i^*(\omega) = dS^*$, where S^* is some function on N. Furthermore, $\pi \times id$ is a diffeomorphism of some neighborhood of $(q, t_0) = i^{-1} \circ \pi \times id^{-1}(p, t_0)$ onto some neighborhood of (p, t_0) in M. To prove this, it suffices to check $\pi \times id_* \circ i_*$ is a non-singular map of $T_{(q,t_0)}(N) = T_{(q}(N_0) + T_{t_0}(B_L)$ onto $T_{(p,t_0)}(M \times B_L)$. But π is a diffeomorphism of N_0 onto U and $i_*(\partial/\partial t)_{(q,t_0)} = X(i(q, t_0))$ thus $\pi \times id_* i_*(\partial/\partial t)_{(q,t_0)}$ has a non-zero component relative to $\partial/\partial t$. This implies that $\pi \times id_* i_*$ is non-singular. Thus N satisfies the conditions (20) and (21) . Therefore, if we define $\phi_1 = i \circ \eta$, where η is the inverse of $\pi \times id \circ i : N \to M$, and $S = \phi_{1*} S_*$, then ϕ_1 and S satisfy the conditions of *Lemma4.*. Since v^* is arbitrary, we have:

Theorem 4. *Let L be a positive definite super-Lagrangian and C an extremal of L with $C(t_0) = p$. Then there exists a sufficiently small neighborhood U about p_0, an interval I about t_0 and a function S defined on $U \times I$ such that any supercurve B in sufficiently small C^1 neighborhood of C, with $B(t) \in U$ for all $t \in I$, and*

(22) $B(t_0) = p$

satisfies

(23) $\varepsilon_L(I(C)) < \varepsilon_L(I(B)) + \varepsilon_L(S(C(t), t)) - \varepsilon_L(S(B(t), t)).$

In particular, C is a strict weak local minimum on any interval $[t_0, t]$, where $t \in I$. If L is the superenergy of a Riemannian supermanifold [6], *then* (22) *holds for all B such that $B(t) \in U$ for all $t \in I$, so that C is a strict strong local minimum for ϕ_1 on $[t_0, t]$.*

In fact, since v^* in the discussion preceding the theorem is arbitrary, we can take $v^* = \bar{C}(t_0)$. If we then define $\psi = {L'}^{-1}\phi_1$, it follows from the previous discussion that the hypothesses of *Lemma3.* are satisfied, implying *Theorem4.*.

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