Lagrangian structures in geometrized jet framework

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Dedicated to Prof. Dr. Radu Miron,
member of the Romanian Academy, at his 75th anniversary

Abstract

The first section discusses the existence of canonic nonlinear connections in first order jet spaces $J^1(T,M)$ endowed with a Lagrange structure and are derived the Euler-Lagrange equations for the Kronecker case $\tilde{g}(t,x,y)_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t,x)\otimes g_{ij}(t,x,y)$. In Section 2, emerging from the Cartan linear N-connection, are presented the general Einstein equations with sources, which are further specified for the ARL (almost Riemann Lagrangian) jet case, and for the Riemannian jet linearized weak gravitational metric case as well. Are derived as well the deflection-generated associated electromagnetic tensors, and are stated the corresponding Maxwell equations with sources for the general geometrized jet case. Section 3 describes the paths and the Lorentz curves of the generalized Lagrangian model, emphasizing the ARL case.

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1 Lagrangians and nonlinear connections on $J^1(T,M)$

Consider $\xi = (E = J^1(T,M), \pi, T\times M)$, the first order jet bundle of mappings $\varphi : T \to M$, where T and M are $\mathcal{C}^\infty$ real differentiable manifolds with $\dim T = m$, $\dim M = n$ respectively. The local coordinates in E will be denoted by

$$(t^\alpha, x^i, y^A)_{(\alpha,i,A)\in I_*} \equiv (y^\mu)_{\mu\in I},$$

where we consider the sets of indices

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$$I = I_h \cup I_v, \quad I_h = I_{h_1} \cup I_{h_2}, \quad I_v = \overline{m+n+1, m+n+mn}$$
$$I_{h_1} = \overline{1,m}, \quad I_{h_2} = \overline{m+1, m+n}, \quad I_* = I_{h_1} \times I_{h_2} \times I_v.$$

Throughout the paper, the indices will implicitly take values as follows:

$$\alpha, \beta, \ldots \in I_{h_1}; \quad i, j, \ldots \in I_{h_2}; \quad A, B, \ldots \in I_v; \quad \lambda, \mu, \ldots \in I.$$

When appropriate, for any index $A = m + n + n(i - m - 1) + \alpha$, we shall identify $A \equiv \binom{i}{\alpha}$ and $y^A \equiv x^{\binom{i}{\alpha}} = \frac{\partial x^i}{\partial t^\alpha}$.

Primarily, E can be endowed with a Lagrangian (also called [11] the extended Lagrangian of electrodynamics), having the form

$$L(t, x, y) = \tilde{g}_{AB}(t, x, y) y^A y^B + U_A(t, x) y^A + \Phi(t, x), \tag{1}$$

with $\tilde{g}_{AB}$ nondegenerate tensor field, $U_A(t, x)$ a 1-form on E and $\Phi(t, x)$ a scalar function on $T \times M$.

The associated Euler-Lagrange equations produce a spray which under certain restrictive conditions, provides further a *non-linear connection* $N = \{N_\mu^A\}_{\mu \in I_h, A \in I_v}$ on E which produces a splitting [12, 7]

$$TE = HE \oplus VE, \tag{2}$$

where $VE = Ker\ \pi_*$. As well, N determines the local adapted basis of $\mathcal{X}(E)$

$$\mathcal{B} = \{\delta_\alpha, \delta_i, \delta_A\}_{(\alpha, i, A) \in I_*} \equiv \{\delta_\mu\}_{\mu \in I}, \tag{3}$$

where we denote briefly $\partial_\alpha = \frac{\partial}{\partial t^\alpha}, \partial_i = \frac{\partial}{\partial x^i}$ and

$$\delta_\alpha = \partial_\alpha - N_\alpha^A \delta_A, \quad \delta_i = \partial_i - N_i^A \delta_A, \quad \delta_A = \dot{\partial}_A = \frac{\partial}{\partial y^A}. \tag{4}$$

The dual basis of $\mathcal{B}$ in (3) writes then

$$\mathcal{B}^* = \{\delta^\alpha, \delta^i, \delta^A\}_{(\alpha, i, A) \in I_*} \equiv \{\delta^\mu\}_{\mu \in I},$$

where

$$\delta^\alpha = dt^\alpha, \quad \delta^i = dx^i, \quad \delta^A \equiv \delta y^A = dy^A + N_\alpha^A dt^\alpha + N_i^A dx^i. \tag{5}$$

An open problem is related to the existence of Lagrangian-produced non-linear connections in the general Kronecker case ([10]).

To this goal, we state the following result.

Theorem 1. *The Euler-Lagrange equations for the Kronecker case*

$$\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t, x) g_{ij}(t, x, y), \tag{6}$$

where $h_{\alpha\beta}$ and g_{ij} non-degenerate tensor fields, are

$$
\begin{aligned}
E^i(L) = \quad & h^{\alpha\beta}\left[y^i_{\alpha\beta} + y^l_{\alpha\gamma} g^{ik} y^{\binom{j}{\beta}} \cdot \partial_{\binom{\{k}{\gamma}} g_{jl\}} + \tfrac{1}{2} y^l_{\delta\gamma} \cdot y^{\binom{j}{\alpha}} y^{\binom{m}{\beta}} g^{ik} \partial^2_{\binom{l}{\delta}\binom{k}{\gamma}} g_{mj}\right] + \\
& + \tfrac{1}{2} g^{ik}\left(h^{\beta\gamma} y^{\binom{m}{\beta}} y^{\binom{j}{\gamma}} y^{\binom{l}{\alpha}} \partial^2_{l\binom{k}{\alpha}} g_{mj} - h^{\beta\delta} h^{\gamma\varepsilon} y^i_\beta y^{\binom{j}{\gamma}} y^{\binom{l}{\alpha}} \cdot \partial_l h_{\delta\varepsilon} \cdot \partial_{\binom{k}{\alpha}} g_{ij}\right) + \\
& + h^{\alpha\beta} y^{\binom{j}{\alpha}} y^{\binom{k}{\beta}} \left|{i \atop jk}\right| - g_{kj} y^{\binom{j}{\gamma}} h^{\alpha\beta} \left|{\gamma \atop \alpha\beta}\right| + \tfrac{1}{2} h^{\beta\gamma} y^{\binom{l}{\beta}} \left|{\delta \atop \alpha\delta}\right| g^{ik} \partial_{\binom{k}{\alpha}} g_{lj} - \\
& - h^{\alpha\beta} y^{\binom{j}{\beta}} g^{ik} \partial_\alpha g_{kj} + \tfrac{1}{2} h^{\beta\gamma} y^{\binom{l}{\beta}} y^{\binom{j}{\gamma}} g^{ik} \partial^2_{\alpha\binom{k}{\alpha}} g_{jl} - \\
& - \tfrac{1}{2} h^{\beta\delta} h^{\gamma\varepsilon} y^{\binom{l}{\beta}} y^{\binom{j}{\gamma}} \cdot \partial_\alpha h_{\delta\varepsilon} \cdot g^{ik} \partial_{\binom{k}{\alpha}} g_{lj} - \tfrac{1}{2} g^{ik} y^{\binom{l}{\alpha}} \partial_{[k} U_{\binom{l]}{\alpha}} + \\
& + \tfrac{1}{2} g^{ik}\left(U_{\binom{k}{\alpha}} \left|{\varepsilon \atop \alpha\varepsilon}\right| + \partial_\alpha U_{\binom{k}{\alpha}} - \partial_k \Phi\right) + T^i(L) = 0, \quad \forall i \in I_{h_2},
\end{aligned}
\tag{7}
$$

with

$$
\begin{aligned}
T^i(L) = \quad & \tfrac{1}{2} h^{\alpha\delta} h^{\beta\varepsilon} g_{lm} y^{\binom{l}{\alpha}} y^{\binom{m}{\beta}} g^{ik} \partial_k h_{\delta\varepsilon} - h^{\alpha\delta} h^{\beta\varepsilon} g_{kj} y^{\binom{l}{\alpha}} y^{\binom{j}{\beta}} g^{ik} \partial_l h_{\delta\varepsilon} + \\
& + \tfrac{1}{4} g^{ik} h^{\delta\varepsilon} y^{\binom{l}{\alpha}} \partial_l h_{\delta\varepsilon}\left(2h^{\alpha\beta} g_{kj} y^{\binom{j}{\beta}} + h^{\beta\gamma} y^{\binom{l}{\beta}} y^{\binom{m}{\gamma}} \partial_{\binom{k}{\alpha}} g_{lm} + U_{\binom{k}{\alpha}}\right) - \\
& - \tfrac{L}{4} g^{ik} h^{\alpha\beta} \partial_k h_{\alpha\beta},
\end{aligned}
\tag{8}
$$

where we denote $\tau_{[i\ldots j]} = \tau_{i\ldots j} - \tau_{j\ldots i}$ *and* $\tau_{\{i\ldots j\}} = \tau_{i\ldots j} + \tau_{j\ldots i}$. *Proof.* By tedious straightforward computation, by applying the Hilbert-Palatini variational principle, and the relations

$$
\begin{aligned}
\tfrac{\partial h^{\alpha\beta}}{\partial w} &= -h^{\alpha\delta} h^{\beta\varepsilon} \partial_w h_{\delta\varepsilon}, \ \tfrac{\partial h^{\alpha\beta}}{\partial h_{\delta\varepsilon}} = -h^{\alpha\delta} h^{\beta\varepsilon}, \\
\tfrac{\partial\sqrt{|h|}}{\partial w} &= \tfrac{\sqrt{|h|}}{2} h^{\alpha\beta} \partial_w h_{\alpha\beta}, \ \ h^{\delta\varepsilon} \partial_\alpha h_{\delta\varepsilon} = 2\left|{\beta \atop \alpha\beta}\right|,
\end{aligned}
$$

where we have denoted $|h| = \det(h_{\alpha\beta})_{\alpha\beta=\overline{1,m}}$, one obtains the Euler-Lagrange equations (7) attached to the Lagrangian density $\mathcal{L} = L\sqrt{|h|}$. □

Particular cases.

I. In case that $U_{\binom{k}{\alpha}}$ depends only on the jet-coordinates, e.g., for the Caratheodory type Lagrangian ([8]), where $U_{\binom{k}{\alpha}} = \partial_{\binom{k}{\alpha}} \varphi(y)$ is a gradient, for the autonomous Lagrangian case with $m = 1$ and $h_{11} = 1$, one gets easily the results on TM obtained in [8].

II. In the case that $h_{\alpha\beta}$ is a metric tensor on T and

$$\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t) g_{ij}(t, x, y),$$

the theorem leads to the particular results including mainly the calculation of the jet spray in terms of L for $h-$ reducible Lagrangians which were derived in [12].

III. The AFL (almost Finsler Lagrange) jet subcase, in which $g_{ij}(t, x, y)$ is $0-$homogeneous in y, leads in the $m = 1$ autonomous case to the AFL case considered in [8].

The three subsequent cases are important subcases of the AFL jet case.

IV. In the ARL (almost Riemann Lagrange) jet case, where

$$\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t) g_{ij}(t,x), \tag{9}$$

if $h_{\alpha\beta}$ is a metric tensor on T, the theorem produces the spray and the nonlinear connection from [12], [10]. Namely, the Lagrangian L in (1) produces the canonical nonlinear connection $N = (N^{\binom{i}{\alpha}}_{\beta}, N^{\binom{i}{\alpha}}_{j})$ of coefficients

$$N^{\binom{i}{\alpha}}_{\beta} = -\left|{}^{\gamma}_{\alpha\beta}\right| y^{\binom{i}{\gamma}}, \quad N^{\binom{i}{\alpha}}_{j} = \left|{}^{i}_{jk}\right| y^{\binom{k}{\alpha}} + \frac{1}{4} g^{ik}(2\partial_\alpha g_{jk} + h_{\alpha\beta} U_{\binom{k}{\beta} j}), \tag{10}$$

where we denoted the h_2-curl of U by $U_{\binom{k}{\beta} j} = \delta_{[j} U_{\binom{k]}{\beta}}$. The Lagrangian L is in this case a Kronecker $h^{\alpha\beta} - h-$regular Lagrangian and produces by

$$\tilde{g}_{AB} = \frac{1}{2} \dot{\partial}^2_{AB} L, \tag{11}$$

the vertical metric tensor field $\tilde{g}_{AB}$.

V. More particular, in the ARLS (almost Riemann Lagrange separated) jet case in which g_{ij} is a metric tensor field on M, i.e.,

$$\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t) g_{ij}(x), \tag{12}$$

if $h_{\alpha\beta}$ is a metric tensor on T, the theorem leads to the spray and non-linear connection from [11]. The two nondegenerate metric tensors g and h and the potentials U_A, determine the nonlinear connection $N = (N^{\binom{i}{\alpha}}_{\beta}, N^{\binom{i}{\alpha}}_{j})$ of coefficients

$$N^{\binom{i}{\alpha}}_{\beta} = -\left|{}^{\gamma}_{\alpha\beta}\right| y^{\binom{i}{\gamma}}, \quad N^{\binom{i}{\alpha}}_{j} = \left|{}^{i}_{jk}\right| y^{\binom{k}{\alpha}} + \frac{1}{4} g^{ik} \cdot h_{\alpha\beta} U_{\binom{k}{\beta} j}. \tag{13}$$

VI. The ALML (almost locally Minkowski Lagrange) jet case, where in a subatlas $\mathcal{A}$ of E one has

$$\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t,x) g_{ij}(y), \tag{14}$$

necessarily requires that $T \times M$ be $\mathcal{A}-$locally affine. This extends the case $J^1(\mathbb{R}, M) \equiv TM$ considered in [8].

2 Einstein equations on $J^1(T, M)$. Special cases.

Assuming fixed on E a non-linear connection $N = \{N^A_\alpha, N^A_i\}$, we may consider a linear connection $\nabla = \{L^\lambda_{\mu\nu}\}_{\lambda,\mu,\nu \in I}$ in E, whose coefficients relative to the adapted basis (3) are provided by

$$\delta^\lambda(\nabla_{\delta_\nu} \delta_\mu) = L^\lambda_{\mu\nu}, \ \ \forall \lambda, \mu, \nu \in I; \tag{15}$$

these determine $3^3 = 27$ distinct subsets, according to the three sets of indices. The torsion $\mathcal{T}$ and the curvature $\mathcal{R}$ of ∇ have the adapted components defined by the relations

$$\delta^\lambda(\mathcal{T}(\delta_\nu, \delta_\mu)) = T^\lambda_{\mu\nu}, \ \ \delta^\lambda(\mathcal{R}(\delta_\nu, \delta_\mu)\delta_\rho) = R_\rho{}^\lambda{}_{\mu\nu}, \quad \forall\, \lambda, \mu, \nu, \rho \in I.$$

The subsets of nontrivial coefficients of ∇ can be reduced subsequently by considering the following particular classes of connections:

1. As first particular case, we consider the set of connections $\Gamma(N)$ (called "N-connections") whose covariant derivative preserves the sections $\mathcal{S}(HE)$ and $\mathcal{S}(VE)$. Their coefficients form $3 \cdot 2^2 + 3 = 15$ generally nonvanishing subsets related to the three index classes, due to the relations

$$L^\lambda_{\mu\nu} = 0, \ \ \forall\, (\lambda, \mu) \in (I_h \times I_v) \cup (I_v \times I_h). \tag{16}$$

Note that if E carries a metric structure, the Levi-Civita (metric and torsionless) connection is not generally a member of $\Gamma(N)$ [7]. It can be easily shown that the associated torsion and curvature of a connection $\nabla \in \Gamma(N)$ satisfy

$$\begin{aligned} &T^\omega_{BC} = 0, && \forall\, \omega \in I_h, B, C \in I_v, \\ &R_\rho{}^\lambda{}_{\mu\nu} = 0, && \forall\, \mu, \nu \in I, (\lambda, \rho) \in (I_h \times I_v) \cup (I_v \times I_h), \end{aligned} \tag{17}$$

and hence the torsion subsets reduce from 3^3 to 25 and the curvature ones from 3^4 to $5 \cdot 3^2$.

2. As further subcase, we consider the *special N-connections* $\Gamma_*(N)$, whose covariant derivatives preserve the distributions $Span(\delta_\alpha)_{\alpha \in I_{h_1}}$ and $Span(\delta_i)_{i \in I_{h_2}}$. They have just 9 sets of generally nonvanishing coefficients and besides (16), they satisfy as well

$$L^\lambda_{\mu\nu} = 0, \ \ \forall\, (\lambda, \mu) \in (I_{h_1} \times I_{h_2}) \cup (I_{h_2} \times I_{h_1}), \tag{18}$$

As consequence, we have e.g.,

$$R_\rho{}^\lambda{}_{\mu\nu} = 0, \quad \forall\, \mu, \nu \in I, (\lambda, \rho) \in (I_{h_1} \times I_{h_2}) \cup (I_{h_2} \times I_{h_1}).$$

and the number of torsion and curvature sets of $\Gamma_*(N)$ reduce to 12 and 18 respectively [12], [10].

3. Among the connections $\Gamma_*(N)$ we evidentiate the so-called "Γ-linear h-normal connections" $\Gamma_n(N)$ [10], which depend on the four essential sets of components

$$\nabla \equiv \{L^\alpha_{\beta\gamma}, L^i_{j\gamma}, L^i_{jk}, L^i_{jA}\}, \tag{19}$$

and have the other 5 sets provided by

$$\begin{aligned} &L^A_{B\gamma} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}\gamma} = \delta^\beta_\alpha L^i_{j\gamma} - \delta^i_j \left|{}^{\beta}_{\alpha\gamma}\right|, \ \ L^A_{Bk} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}k} = \delta^\beta_\alpha \left|{}^{i}_{jk}\right|, \\ &L^A_{BC} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}C} = \delta^\beta_\alpha L^i_{jC}, \ \ L^\alpha_{\beta j} = 0, \quad L^\alpha_{\beta C} = 0. \end{aligned}$$

Here the number of torsion and curvature sets reduces to 9 and respectively to 7.

In the general case when $h_{\alpha\beta}(t)$ and $g_{ij}(t,x,y)$ in the Lagrangian L in (1) are non-degenerate, we endow E with a semi-Riemannian metric

$$G = \underbrace{h_{\alpha\beta}(t)dt^\alpha \otimes dt^\beta}_{h} + \underbrace{g_{ij}(t,x,y)dx^i \otimes dx^j}_{g} + \underbrace{\tilde{g}_{AB}(t,x,y)\delta y^A \otimes \delta y^B}_{\tilde{g}}, \tag{20}$$

with $\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t)g_{ij}(t,x,y)$. Then within $\Gamma_n(N)$ we evidentiate *the Cartan connection* ([12], [10]) which is metrical and satisfies the conditions

$$L^i_{j\gamma} = \frac{g_{ik}}{2}\partial_\gamma g_{jk}, \quad L^i_{[jk]} = 0, \quad L^i_{[j\binom{k]}{\alpha}} = 0.$$

This exhibits generally (for $m > 1$) just 8 torsion sets and 7 curvature sets, and 5 torsion sets provided that g is y-independent. Its essential four sets of coefficients (19) specify in this case to

$$\begin{aligned} &L^\alpha_{\beta\gamma} = \left|\begin{smallmatrix}\alpha\\ \beta\gamma\end{smallmatrix}\right|, \ \ L^i_{j\gamma} = \tfrac{1}{2}g^{ik}\delta_\gamma g_{kj}, \ \ L^i_{jk} = \left|\begin{smallmatrix}i\\ jk\end{smallmatrix}\right|, \\ &L^i_{jA} \equiv L^i_{j\binom{k}{\gamma}} = \tfrac{1}{2}g^{il}(\delta_{\binom{\{k}{\gamma}}g_{jl\}} - \delta_{\binom{l}{\gamma}}g_{jk}). \end{aligned} \tag{21}$$

Its essential torsion coefficients are given by [10]

$$\begin{cases} T_{\gamma}{}^{\binom{i}{\alpha}}{}_{\binom{j}{\beta}} = \partial_{\binom{j}{\beta}}N_\gamma^{\binom{i}{\alpha}} - \delta^\beta_\alpha L^i_{j\gamma} + \delta^i_j L^\beta_{\alpha\gamma} \\ T_{k}{}^{\binom{i}{\alpha}}{}_{\binom{j}{\beta}} = \partial_{\binom{j}{\beta}}N_k^{\binom{i}{\alpha}} - \delta^\beta_\alpha L^i_{jk} \\ T_{\binom{j}{\beta}}{}^{\binom{i}{\alpha}}{}_{\binom{k}{\gamma}} = \delta^\beta_i L^i_{j\binom{k}{\gamma}} - \delta^\gamma_i L^i_{k\binom{j}{\beta}} \\ T_\beta{}^i{}_j = -L^i_{j\beta}, \ \ T^i_{jA} = L^i_{jA} \\ T_\beta{}^A{}_\gamma = \delta_{[\gamma}N^A_{\beta]}, \ T_\beta{}^A{}_j = \delta_{[j}N^A_{\beta]}, \ T_i{}^A{}_j = \delta_{[j}N^A_{i]}, \end{cases} \tag{22}$$

and explicitly provided for the ARL case in [12, Theorem 4.4]. The nontrivial non-holonomy coefficients $\omega^\lambda_{\mu\nu}$ are provided by the relations

$$\begin{aligned} [\delta_\mu, \delta_\nu] &= \omega^A_{\mu\nu}\delta_A \equiv T^A_{\mu\nu}\delta_A, \qquad \forall \mu,\nu \in I_h, \\ [\delta_\mu, \delta_B] &= \omega^A_{\mu B}\delta_A \equiv \partial_B N^A_\mu \delta_A, \quad \forall \mu \in I_h, \end{aligned}$$

and explicitly provided for the ARL case in [10, Theorem 2.3]. As well, the nontrivial curvature N-tensor fields are

$$
\begin{cases}
R_{\beta\ \gamma\delta}^{\ \alpha} = \partial_{[\delta} L^{\alpha}_{\beta\gamma]} + L^{\varepsilon}_{\beta[\gamma} L^{\alpha}_{\varepsilon\delta]} \\
R_{j\ km}^{\ i} = \delta_{[k} L^{i}_{jm]} + L^{\beta}_{j[m} L^{i}_{\beta k]} + L^{i}_{jA} T^{A}_{mk} \\
R_{j\ \gamma\mu}^{\ i} = \delta_{[\mu} L^{i}_{j\gamma]} + L^{\varepsilon}_{j[\gamma} L^{i}_{\varepsilon\mu]} + L^{i}_{jA} T^{A}_{\gamma\mu},\ \forall\mu \in I_h \\
R_{j\ \mu A}^{\ i} = \partial_A L^{i}_{j\mu} - L^{i}_{jA|\mu} + L^{i}_{jB} T^{B}_{\mu A},\ \forall\mu \in I_h \\
R_{j\ CD}^{\ i} = \partial_{[D} L^{i}_{jC]} + L^{k}_{j[C} L^{i}_{kD]},
\end{cases}
\tag{23}
$$

and

$$
\begin{cases}
R_{\binom{j}{\beta}\ \gamma\delta}^{\ \binom{i}{\alpha}} = \delta^{\beta}_{\alpha} R_{j\ \gamma\delta}^{\ i} + \delta^{i}_{j} R_{\alpha\ \gamma\delta}^{\ \beta} \\
R_{\binom{j}{\beta}\ \mu k}^{\ \binom{i}{\alpha}} = \delta^{\beta}_{\alpha} R_{j\ \mu k}^{\ i},\ \ \forall\mu \in I_h \\
R_{\binom{j}{\beta}\ \mu A}^{\ \binom{i}{\alpha}} = \delta^{\beta}_{\alpha} R_{j\ \mu A}^{\ i}, \forall\mu \in I,
\end{cases}
$$

where we denoted by $|\alpha$, $|i$ and $|A$ the covariant derivations given by ∇_{δ_μ}, for $\mu \in I_{h_1}, I_{h_2}$ and I_v respectively.

The nontrivial associated Ricci N-tensor fields are

$$
\begin{aligned}
&R_{\alpha\beta} = R^{\gamma}_{\alpha\beta\gamma},\ R_{i\alpha} = R^{k}_{i\alpha k},\ R_{ij} = R^{k}_{ijk},\ R_{iA} = -R^{j}_{ijA}, \\
&R_{\binom{i}{\alpha}\beta} = R^{k}_{i\beta\binom{k}{\alpha}},\ R_{\binom{i}{\alpha}j} = R^{k}_{ij\binom{k}{\alpha}},\ R_{\binom{i}{\alpha}\binom{j}{\beta}} = R^{k}_{i\binom{j}{\beta}\binom{k}{\alpha}},
\end{aligned}
\tag{24}
$$

and the scalars of curvature

$$
R_h = h^{\alpha\beta} R_{\alpha\beta}, \quad R_g = g^{ij} R_{ij}, \quad R_v = \tilde{g}^{AB} R_{AB}. \tag{25}
$$

Then, denoting $R = R_h + R_g + R_v$, the Einstein equations with sources write

$$
\begin{cases}
R_{\alpha\beta} - \frac{1}{2} R h_{\alpha\beta} = \kappa \mathcal{T}_{\alpha\beta} \\
R_{ij} - \frac{1}{2} R g_{ij} = \kappa \mathcal{T}_{ij} \\
R_{AB} - \frac{1}{2} R g_{AB} = \kappa \mathcal{T}_{AB},
\end{cases}
\qquad
\begin{cases}
0 = \mathcal{T}_{\alpha i}, & 0 = \mathcal{T}_{\alpha A}, \\
R_{i\alpha} = \kappa \mathcal{T}_{i\alpha}, & R_{A\alpha} = \kappa \mathcal{T}_{A\alpha} \\
R_{iA} = \kappa \mathcal{T}_{iA}, & R_{Ai} = \kappa \mathcal{T}_{Ai},
\end{cases}
\tag{26}
$$

where $\mathcal{T} = \mathcal{T}_{\mu\nu} \delta^{\mu} \otimes \delta^{\nu} \in \mathcal{T}^{0}_{2}(E)$ is the *energy-momentum tensor field* and κ is the cosmological constant. They satisfy the *conservation laws*

$$
E^{\mu}_{\ \nu|\mu} = \kappa \mathcal{T}^{\mu}_{\ \nu|\mu},\ \ \forall\mu \in I = I_{h_1} \cup I_{h_2} \cup I_v,
$$

where $E_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R G_{\mu\nu}$ is the Einstein N-tensor field and the indices are raised by means of the metric G in (20) on E.

Regarding the energy-momentum tensor field $\mathcal{T} = \{T_{\mu\nu}\}$, we distinguish several remarkable cases [2] which extend the ones discussed in [16]:

I. The case of *electromagnetic field source*, when

$$T_{\mu\nu} = F_{\mu\rho}F^{\rho}_{\mu} - \frac{1}{4}G_{\mu\nu}F^{\rho\pi}F_{\rho\pi}.$$

Here the electromagnetic 2-form

$$F = F_{A\mu}\delta y^{A} \wedge \delta y^{\mu}, \tag{27}$$

has the nontrivial components

$$\begin{cases} F_{A\beta} & \equiv F_{\binom{i}{\alpha}\beta} = \frac{1}{2}\left(h^{\alpha\gamma}g_{ik}y^{\binom{k}{[\gamma}}\right)_{|\beta]} \\ F_{Aj} & \equiv F_{\binom{i}{\alpha}j} = \frac{1}{2}\, d_{\binom{[i}{\alpha}j]} = \frac{1}{2}\, y_{\binom{[i}{\alpha}|j]} = \frac{1}{2}\left(y^{\binom{k}{\gamma}}h^{\alpha\gamma}g_{k[i}\right)_{|j]}, \\ F_{AB} & \equiv F_{\binom{i}{\alpha}\binom{j}{\beta}} = \frac{1}{2}\tilde{g}_{\binom{[i}{\alpha}C}y^{C}{}_{|\binom{j]}{\beta}}. \end{cases} \tag{28}$$

Essentially, F is produced by the *deflection tensor fields*

$$d^{A}_{\mu} = \delta^{A}\nabla_{\delta_{\mu}}\mathcal{C}, \ \ \mu \in I, \ A \in I_{v},$$

generated by the Liouville field $\mathcal{C} = y^{A}\delta_{A}$. Note that the raising/lowering of the indices was performed using the metric G, producing the associated N-tensor field

$$\tilde{F} = F_{A}{}^{\mu}\delta_{\mu} \otimes \delta^{A}, \qquad F^{\alpha}_{A} = h^{\alpha\beta}F_{A\beta}, \ F^{i}_{A} = g^{ij}F_{Aj}, \ F^{C}_{A} = g^{CD}F_{AD}. \tag{29}$$

The *energy-momentum* tensor fields have in this case the essential coefficients given by

$$\begin{cases} \mathcal{T}_{\alpha\beta} = F_{A\alpha}F_{B\beta}\tilde{g}^{AB} - \frac{1}{4}h_{\alpha\beta}F_{*} \\ \mathcal{T}_{ij} = F_{Ai}F_{Bj}\tilde{g}^{AB} - \frac{1}{4}g_{ij}F_{*} \\ \mathcal{T}_{AB} = F_{AC}F_{BD}\tilde{g}^{CD} - \frac{1}{4}g_{AB}F_{*} \end{cases} \qquad \begin{cases} \mathcal{T}_{\alpha i} = F_{C\alpha}F_{Di}\tilde{g}^{CD} \\ \mathcal{T}_{\alpha A} = F_{C\alpha}F_{DA}\tilde{g}^{CD} \\ \mathcal{T}_{iA} = F_{Ci}F_{DA}\tilde{g}^{CD} \end{cases}$$

where $F_{*} = F_{AC}F_{BD}\tilde{g}^{AB}\tilde{g}^{CD}$.

Moreover, we have the following

Theorem 2. *The 2-form F is subject to the two sets of* the Maxwell extended equations with sources

$$F_{\binom{i}{\alpha}k|\gamma} = \tfrac{1}{2}[d_{\binom{i}{\alpha}\beta|k} + d_{\binom{i}{\alpha}m}T^{m}_{\beta k} + d_{\binom{i}{\alpha}C}T^{C}_{\beta k} - (T^{j}_{\gamma i|k} + L^{j}_{kC}T^{C}_{\gamma i})y_{\binom{j}{\alpha}}]$$

$$F_{\binom{i}{\alpha}\binom{k}{\beta}|\gamma} = \tfrac{1}{2}[d_{\binom{i}{\alpha}\beta|\binom{k}{\gamma}} + d_{\binom{i}{\alpha}m}T^{m}_{\beta\binom{k}{\gamma}} + d_{\binom{i}{\alpha}C}T^{C}_{\beta k} - (\partial_{\binom{k}{\gamma}}T^{j}_{\gamma i} + L^{j}_{kC}T^{C}_{\gamma\binom{i}{\beta}})y_{\binom{j}{\alpha}}]$$

$$\underset{ijk}{S}F_{\binom{i}{\alpha}j|k} = -\tfrac{1}{2}\underset{ijk}{S}(L^{m}_{iC}y_{\binom{m}{\alpha}} + d_{\binom{i}{\alpha}C})T^{C}_{jk}$$

$$\underset{ijk}{S}F_{\binom{i}{\alpha}\{j|\binom{k\}}{\gamma}} = 0, \ \ \underset{ijk}{S}F_{\binom{i}{\alpha}\binom{j}{\beta}|\binom{k}{\gamma}} = 0;$$

$$g^{BC}F_{B\alpha|C} = -4\pi J_\alpha$$
$$g^{BC}F_{Bi|C} = -4\pi J_i$$
$$g^{BC}F_{AB|C} + g^{ij}F_{Ai|j} + h^{\alpha\beta}F_{A\alpha|\beta} = 4\pi J_A,$$

where $J = J_\mu \delta^\mu \in \mathcal{X}^*(E)$ *is the adapted electric current, and where we denoted by* S *the cyclic summation of the corresponding indices below.*

II. In the case of a *perfect fluid* with the extended velocity vector field $\mathcal{V} = \mathcal{V}^\mu \delta_\mu \in \mathcal{X}(E)$, the energy-momentum N-tensor field is given by

$$\mathcal{T}_{\mu\nu} = (P+\rho)\mathcal{V}_\mu \mathcal{V}_\nu + pG_{\mu\nu},$$

where ρ is the *mass-energy density*, p is the *pressure* and, in the 4-dimensional Lorentz-metric case $\mathcal{V}$ satisfies the condition $\mathcal{V}^i\mathcal{V}_i = -1$. In particular, for $p = 0$, is obtained the case of *cosmologic dust* (presureless fluid).

III. In the case of source given by *the Klein-Gordon field* Φ, we have

$$\mathcal{T}_{\mu\nu} = \Phi_{|\mu}\Phi_{|\nu} - \frac{1}{2}(G^{\pi\rho}\Phi_{|\pi}\Phi_{|\rho} + m^2\Phi^2)G_{\mu\nu},$$

where m is the mass, and the field Φ satisfies the condition $G^{\mu\nu}\Phi_{|\mu|\nu} = m^2\Phi$ [16].

IV. In the case of *the radiation field*, we get

$$\mathcal{T}_{\mu\nu} = \Phi^2 K_\mu K_\nu,$$

where the N-vector field $K = K^\mu\delta_\mu \in \mathcal{X}(E)$ obeys in the Lorentz metric case the condition $G_{\mu\nu}K^\mu K^\nu = 0$.

Particular cases.

I. In the ARLS case, the vertical metric $N-$ tensor field $\tilde{g}$ is produced by the Lagrangian via (11), and the nontrivial coefficients of the Cartan connection are [11]

$$L^\alpha_{\beta\gamma} = \left|{}^{\alpha}_{\beta\gamma}\right|, \quad L^i_{jk} = \left|{}^{i}_{jk}\right|.$$

They have the non-trivial torsion $N-$fields

$$T^{\binom{m}{\gamma}}_{\alpha\beta} = -R^\delta_{\gamma\alpha\beta}x^m_\delta, \;\; T^{\binom{m}{\gamma}}_{ij} = R^m_{ij\gamma} + \tfrac{h_{\gamma\beta}g^{mk}}{4}U_{\binom{k}{\beta}\{i|j\}},$$
$$T^{\binom{m}{\gamma}}_{\alpha j} = -\tfrac{h_{\gamma\beta}g^{mk}}{4}[L^\beta_{\alpha\delta}U_{\binom{k}{\delta}j} + \partial_\alpha U_{\binom{k}{\beta}j}],$$

where $R^\alpha_{\beta\gamma\delta}$ and R^i_{jkl} are the nontrivial curvature tensors of $h_{\alpha\beta}$ and g_{ij} respectively; the Einstein equations contain the classical ones on $(T\times M, h+g)$,

$$\begin{cases} R_{\alpha\beta} - \frac{R_h+R_g}{2}h_{\alpha\beta} = \kappa\mathcal{T}_{\alpha\beta} \\ R_{ij} - \frac{R_h+R_g}{2}g_{ij} = \kappa\mathcal{T}_{ij} \\ -\frac{R_h+R_g}{2}h^{\alpha\beta}g_{ij} = \kappa\mathcal{T}_{\binom{i}{\alpha}\binom{j}{\beta}}. \end{cases}$$

II. In the ARLS case with $m = 1,\ n = 4$ and $h_{11} = 1$, one recaptures as particular case, the pseudo-Riemannian weak gravitational model endowed with the metric

$$g_{ij}(x) = \eta_{ij} + \varepsilon_{ij}(x), \tag{30}$$

where the weakness of the gravitational field g_{ij} is expressed by its decomposition into the flat Minkowski metric $n_{ij} = diag(-1, 1, 1, 1)$ and a small perturbation $\varepsilon_{ij}(x)$, a symmetric tensor field with $|\varepsilon_{ij}(x)| << 1$. We note that the perturbation $\varepsilon_{ij}(x)$ has to satisfy certain supplementary gauge conditions ([3]). In the linearized approach, the indices are raised via n_{ij}, e.g., $\varepsilon^{rs} = n^{ri}n^{sj}\varepsilon_{ij}$. This point of view permits to develop the linearized version of a given generalized model of General Relativity, in which the symmetric tensor field corresponds to a weak pseudo-Riemannian gravitational field [3]. In the linearized case, one has $\left|{}^{i}_{jk}\right| \cong \varepsilon^{i}_{jk} \equiv n^{is}|s;jk|$ and the *Einstein equations* get the typical form corresponding to weak gravitational waves

$$R_{ij} - \frac{1}{2}Rn_{ij} \equiv \frac{1}{2}(\Box\varepsilon_{ij} + \partial^2_{ij}\varepsilon - \partial^2_{\{js}\varepsilon^{s}_{i\}}) - n_{ij}R = \kappa T_{ij},$$

where $\varepsilon = n^{ij}\varepsilon_{ij}$, R_{ij} is the Ricci tensor of g_{ij}, R is the scalar curvature and "$\Box$" denotes the d'Alambertian

$$\Box = -\partial^2_{00} + \partial^2_{11} + \partial^2_{22} + \partial^2_{33} \equiv -\partial^2_{tt} + \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz}.$$

3 Paths and Lorentz equations

Let $c : J = [a, b] \subset \mathbb{R} \to E$ be a smooth curve, whose image lies in a chart $\tilde{U} \subset E$,

$$c(s) = (t^{\alpha}(s), x^{i}(s), y^{A}(s)) \equiv (y^{\mu}(s)), \forall t \in J,$$

and let ∇ be a linear $N-$connection on E.

Definitions. a) The field $\mathcal{V} = \frac{\delta y^{\mu}}{\mathrm{d}s}$ defined on c will be called *covariant velocity* field of the curve c. The components of $\mathcal{V}$ are explicitely given by

$$\{\mathcal{V}^{\mu}\}_{\mu\in I} \equiv \left(\dot{t}^{\alpha}, \dot{x}^{i}, \frac{\delta y^{a}}{\mathrm{d}s} = \dot{y}^{A} + N^{A}_{\beta}\dot{t}^{\beta} + N^{A}_{j}\dot{x}^{j}\right)_{(\alpha,i,A)\in I_*},$$

where we denote by dot the s-derivation. We have also denoted by $\mathcal{F} = \mathcal{F}^{\mu}\delta_{\mu}$, where

$$\mathcal{F}^{\mu} = \frac{\nabla\mathcal{V}^{\mu}}{ds} \overset{not}{=} \frac{\delta\mathcal{V}^{\mu}}{ds} + L^{\mu}_{\nu\rho}\mathcal{V}^{\nu}\mathcal{V}^{\rho},$$

the *covariant force* on c, which provides the motion of the test-body along c.

b) We shall say that c is a *stationary curve* with respect to ∇ iff $\mathcal{F} = 0$ along the curve.

c) The curve c is called $h-$*curve*, if $\pi_v(\mathcal{V}) = 0$, and $v-$*curve*, if $\pi_h(\mathcal{V}) = 0$, where by π_h and π_v we denoted respectively the $h-$ and $v-$projectors of the canonic splitting induced by N. If a $h-/v-$curve satisfies also the extra condition $\mathcal{F}=0$, then it is called $h-/v-$*path*, respectively.

Analytically, these curves are described by the following

Theorem 3. *Let $c : J \subset \mathbb{R} \to E$ be a curve. Then the following hold true:*

a) c is a $h-$curve iff

$$\mathcal{V}^A = 0 \;\Leftrightarrow\; \frac{\delta y^a}{ds} = 0 \;\Leftrightarrow\; \dot{y}^a + N^A_\alpha \dot{t}^\alpha + N^A_j \dot{x}^j = 0. \tag{31}$$

b) c is a $v-$curve iff

$$\begin{aligned}\mathcal{V}^\mu = 0,\ \forall \mu \in I_h \quad &\Leftrightarrow\; \frac{\delta y^\mu}{ds} = 0, \forall \mu \in I_{h_1} \cup I_{h_2} \;\Leftrightarrow \\ &\Leftrightarrow\; c(s) = (t_0, x_0, y(s)), s \in J.\end{aligned} \tag{32}$$

c) c is an $h-$path ("stationary $h-$curve or "horizontal geodesic") iff besides (31) it satisfies

$$\frac{d\mathcal{V}^\mu}{ds} + L^\mu_{\nu\rho}\mathcal{V}^\nu\mathcal{V}^\rho = 0,\ \forall \mu \in I_h. \tag{33}$$

Note that the implicit sum in the right term involves just horizontal index types.

d) c is a $v-$path ("stationary $v-$curve or "vertical geodesic") iff besides (32) it satisfies

$$\frac{\delta\mathcal{V}^A}{ds} + L^A_{BC}\mathcal{V}^B\mathcal{V}^C = 0,\ \forall A \in I_v. \tag{34}$$

The implicit sum in the right term involves just vertical index types.

We consider the electromagnetic tensor fields in (28) and (29), the metric G in (20), a fixed nonlinear connection N, and the Cartan connection attached to G having the coefficients (21). Then *the Lorentz equations* attached to G, N and ∇ have the generic shape

$$G_{\nu\rho}\frac{\nabla\mathcal{V}^\rho}{ds} = \tilde{F}_{A\nu}\mathcal{V}^A \quad \Leftrightarrow \quad \frac{\nabla\mathcal{V}^\mu}{ds} = \mathcal{F}_A{}^\mu\mathcal{V}^A, \tag{35}$$

where $\mathcal{V} = \mathcal{V}^\mu\delta_\mu$ is *the covariant velocity* along the considered extended path of the moving test-particle. In detail, the Lorentz equations have the form

$$\begin{aligned}\ddot{t}^\alpha \quad &+L^\alpha_{\beta C}\dot{t}^\beta\mathcal{V}^C + L^\alpha_{jC}\dot{x}^j\mathcal{V}^C + L^\alpha_{\beta\gamma}\dot{t}^\beta\dot{t}^\gamma + \\ &+L^\alpha_{j\gamma}\dot{x}^j\dot{t}^\gamma + L^\alpha_{\beta k}\dot{t}^\beta\dot{x}^k + L^\alpha_{jk}\dot{x}^j\dot{x}^k = F^\alpha_B\mathcal{V}^B\end{aligned} \tag{36}$$

$$\begin{aligned}\ddot{x}^i \quad &+L^i_{\beta C}\dot{t}^\beta\mathcal{V}^C + L^i_{jC}\dot{x}^j\mathcal{V}^C + L^i_{\beta\gamma}\dot{t}^\beta\dot{t}^\gamma + \\ &+L^i_{j\gamma}\dot{x}^j\dot{t}^\gamma + L^i_{\beta k}\dot{t}^\beta\dot{x}^k + L^i_{jk}\dot{x}^j\dot{x}^k = F^i_B\mathcal{V}^B\end{aligned} \tag{37}$$

$$\begin{aligned}\dot{\mathcal{V}}^A + N^A_\alpha\dot{t}^\alpha \quad &+N^A_i\dot{x}^i + L^A_{C\beta}\mathcal{V}^C\dot{t}^\beta + \\ &+L^A_{Cj}\mathcal{V}^C\dot{x}^j + L^A_{BC}\mathcal{V}^B\mathcal{V}^C = F^A_B\mathcal{V}^B,\end{aligned} \tag{38}$$

where $\mathcal{V}^A = \dot{y}^A + N^A_\beta \dot{t}^\beta + N^A_i \dot{x}^i,\ A \in I_v$. As well, one may consider *the Lorentz h-paths*, characterized by the relations

$$\mathcal{V}^A = 0,\ A \in I_v \quad \Leftrightarrow \quad \frac{\delta y^A}{ds} = 0,\ A \in I_v.$$

We note that, since the right side of (36)-(38) are identically vanishing, these curves *coincide with the usual h-paths of* (E, N, ∇).

As for the *Lorentz v-paths*, these have fixed base-point, i.e.,

$$\mathcal{V}^\mu = 0,\ \mu \in I_h \quad \Leftrightarrow \quad (t, x) = (t_0, x_0) \in T \times M,$$

and hence the associated equations rewrite

$$\begin{cases} F^\alpha_B \mathcal{V}^B = 0, \quad F^i_B \mathcal{V}^B = 0 \\ F^A_B \mathcal{V}^B = \dot{\mathcal{V}}^A + L^A_{BC} \mathcal{V}^B \mathcal{V}^C. \end{cases}$$

For the ARLS case with the nonlinear connection (10) induced by the Lagrangian (1), the electromagnetic tensors are

$$F^\alpha_A \equiv F^\alpha_{\binom{i}{\beta}\gamma} = 0,\ \ F^i_A = g^{ij}\tilde{F}_{Aj} = -\frac{1}{4} g^{ij} U_{Aj},\ \ F^B_A = 0,$$

the nonvanshing Cartan connection coefficients are

$$L^\alpha_{\beta\gamma} = \left|{}^{\ \alpha}_{\beta\gamma}\right|, \quad L^i_{jk} = \left|{}^{\ i}_{jk}\right|, \ \ L^A_{B\gamma} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}\gamma} = -\delta^i_j \left|{}^{\ \beta}_{\alpha\gamma}\right|, \ \ L^A_{Bk} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}k} = -\delta^\beta_\alpha \left|{}^{\ i}_{jk}\right|,$$

and the Lorentz equations (36)-(38) reduce to

$$\begin{cases} \ddot{t}^\alpha + \left|{}^{\ \alpha}_{\beta\gamma}\right| \dot{t}^\beta \dot{t}^\gamma = 0 \\ \ddot{x}^i + \left|{}^{\ i}_{jk}\right| \dot{x}^j \dot{x}^k = -\frac{1}{4} g^{ij} U_{Aj} \mathcal{V}^A \\ \dot{\mathcal{V}}^A = 0. \end{cases}$$

Note that in this case (g dependent on x only), the Berwald connection [12] has the same coefficients as the Cartan connection, and hence the associated Lorentz curves, h- and v-paths are described by the same equations. The Lorentz h-paths obey the extra equations

$$\dot{y}^A + N^A_\beta \dot{t}^\beta + N^A_j \dot{x}^j = 0,$$

which write explicitely

$$\dot{y}^{\binom{i}{\alpha}} - \left|{}^{\ \gamma}_{\alpha\beta}\right| y^{\binom{i}{\gamma}} \dot{t}^\beta + \left(\left|{}^{\ i}_{jk}\right| y^{\binom{k}{\alpha}} + \frac{1}{4} g^{ik} h_{\alpha\beta} U_{\binom{k}{\beta}j} \right) \dot{x}^j = 0.$$

The Lorentz v-paths for the Cartan connection satisfy just the extra condition $-\mathcal{V}^A = 2\dot{\mathcal{V}}^A$, having as solutions the curves $(t_0, x_0, y^A = k^A_1 e^{-s/2} + k^A_2),\ s \in \mathbb{R}$, with

$k_{1,2} \in \mathbb{R}^{mn}$, semilines within the fibers of E, the linear geodesics of the flat fiber - in accordance with the particular case $J^1(\mathbb{R}, M) \equiv TM$ studied in [8].

In this typical particular case, (for $m = 1$ and $s = t^1 = t$), we can use *the Finsler-Lagrange tangent space notations* from [7]. Shifting the indices left by one unit ($I_{h_2} = \overline{1, n},\ I_v = \overline{n+1, 2n}$), we have $y^A \equiv y^{\binom{i}{1}} \overset{not}{=} y^i$, and set locally $h_{11} = 1$. For the Lagrangian (1) we consider its particular form

$$L(x, y) = mc\,\gamma_{ij}(x) y^i y^j + \frac{2e}{m} U_i(x) y^i + \Phi(x), \tag{39}$$

where γ_{ij} is a pseudo-Riemannian metric and $U = U_i dx^i$ is a 1-form on M. The fundamental tensor derived from L via (11) is then

$$\tilde{g}_{\binom{i}{1}\binom{j}{1}}(t, x, y) = g_{ij}(x) = mc\,\gamma_{ij}(x).$$

The non-linear connection induced by L has the components

$$N_1^A = 0, N_j^{\binom{i}{1}} = \left|{}^{\,i}_{jk}\right| y^k + g^{ik} U_{\binom{k}{1} j},\ i = \overline{1, n},\ A = \overline{n+1, 2n},$$

with $U_{\binom{k}{1}} = \frac{e}{m} A_k$. In this case, the Cartan (21) and Berwald canonic connections have just null and Christoffel (re-indexed) components. Choosing for ∇ the Cartan connection, the Lorentz generalized equations (37) confine to the known ones of Lagrange spaces [7] and coincide with the equations of the Lagrangian spray $G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k + \frac{e}{2m^2 c}\gamma^{ij} A_{[j;k]} y^k$. They have the equivalent form [6, p. 171]

$$\ddot{x}^i + 2G^i(x, y) = 0,\ \ y^i = \frac{dx^i}{ds}, \tag{40}$$

where "$;k$" expresses the canonic covariant derivative on (M, γ_{ij}). We note that in the absence of the electromagnetic force F_{μ_A}, the equations (35) become the equations of stationary curves of the connection ∇. In the particular case $m = 1, s = t^1, h_{11} = 1$, we also note that in the absence of U, for ∇ the Cartan connection, the equations (35) become *the equations of geodesics* of the manifold M. We remark that in the case $m = 1, h_{11} = 1$, the equations above lead to the characterizations of the corresponding curves in [8], and the equations of $h - paths$ become the Lorentz equations.

4 Conclusions.

We have discussed the existence of canonic nonlinear connections in first order jet spaces $J^1(T, M)$ endowed with a Lagrange structure and have derived the explicit Euler-Lagrange equations for the general Kronecker case $\tilde{g}(t, x, y)_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t, x) \otimes g_{ij}(t, x, y)$. Then, for the Cartan linear N-connection, are presented the general Einstein equations with sources, further specified for both the ARL (almost Riemann Lagrangian) jet case, and for the Riemannian jet linearized weak gravitational metric case. As well, are derived the deflection-generated associated electromagnetic tensors, and are stated the corresponding Maxwell equations with sources for the general geometrized jet case. The paths and the Lorentz curves of the Lagrangian model are analytically characterized, emphasizing the ARL special case.

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