SPECIAL RIEMANNIAN METRICS ON COMPACT MANIFOLDS

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Abstract

Let M be a compact manifold. One very importaant problem is to determine, if on M there is a metric with positive sectional curvature. We study this problem by the means of the spectrum of the Laplace operator.

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1. Introduction

Let M be a compact orientable Riemannian manifold of dimension n. The existence of a Riemannian metric g on M with some properties is an open problem. Especially to determine Riemannian metrics with positive sectional curvature is a difficult problem. It is an open problem to find out if the manifold $S^k \times S^\lambda$, $k, \lambda \geq 2$ can carry a Riemannian metric with positive sectional curvature. In order to face this problem we use the notion of positively k-pinched manifold. Therefore this problem can be stated as follows: *If a manifold is given, is there a Riemannian metric positively k-pinched, where k is small as we like ?*

The aim of the present paper is to study this problem. The whole paper contains four sections. The first section is the introduction. Some basic elements for Riemannian manifolds are given in the second section. The third section contains some results concerning exterior forms on a Riemannian manifold. The basic results of this paper are contained in the fourth section.

2. Some basic elements on riemannian manifolds

Let M be a compact orientable manifold of dimension n. It is known that the Riemannian metrics $R(M)$ on M form a Banach space. If $g \in R(M)$, then (M, g) is a compact

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orientable Riemannian manifold. We assume that $n = 2m$, that is $\dim M = 2m$. Let $D^q(M, D^0)$ and $D_q(M, D^0)$ be the modules of contravariant and covariant tensor fields of order q on M over the algebra $D^0(M)$ of real functions on M. These two modules are isomorphic. The isomorphism in a local level can be expressed as follows:

$$f : D^q(M, D^0) \to D_q(M, D^0) \tag{1}$$

$$f : \alpha(\alpha^{i_1 \dots i_q}) \to f(\alpha) = (\alpha_{j_1 - j_q} = g_{i_1 j_1} \dots g_{i_q j_q} \alpha^{i_1 \dots i_q}) \tag{2}$$

where (g_{ij}) are the local components of g with respect to the local coordinate system $(x_1, \dots, x_n)$ of the local chart (U, φ). This local isomorphism permits to substitute the notion *covariant* by *contravariant* and vice versa. From now on we use the notion of tensor field of order q.

If α and β are two tensor fields of order q on the manifold M. The local product of α and β is defined by:

$$(\alpha, \beta) = \frac{1}{q!} \alpha^{i_1 \dots i_q} \beta_{i_1 \dots i_q} \tag{3}$$

and the local norm of α is defined by:

$$|\alpha|^2 = (\alpha, \alpha) = \frac{1}{q!} \alpha^{i_1 \dots i_q} \alpha_{i_1 \dots i_q}.$$

Let α be the volume element of the manifold. The global product of two tensors α and β of order q and the global norm of α are given by:

$$< \alpha, \beta >= \int_M (\alpha, \beta) \omega, \ ||\alpha||^2 = \int_M |\alpha|^2 \omega. \tag{4}$$

If α is an exterior q-form, then we have ([5, p. 3]):

$$\frac{1}{2} \Delta(|\alpha|^2) = (\alpha, \Delta\alpha) - |\nabla\alpha|^2 + \frac{1}{2(q-1)!} Q_q(\alpha), \tag{5}$$

where

$$|\nabla\alpha|^2 = \frac{1}{q!} \nabla^k \alpha^{i_1 \dots i_q} \nabla_k \alpha_{i_1 \dots i_q} \text{ and} \tag{6}$$

$$Q_q(\alpha) = (q-1) R_{kl,mn} \alpha^{kli_3 \dots i_q} \alpha^{mn}_{i_3 \dots i_q} - 2 R_{kl} \alpha^{ki_2 \dots i_q} \alpha^l_{i_2 \dots i_q}. \tag{7}$$

The following formula is valid [4, p. 187]:

$$< \alpha, \Delta\alpha >= ||\delta\alpha||^2 + ||d\alpha||^2, \tag{8}$$

where $< \alpha, \Delta\alpha >$ is the global product of the exterior q-forms α, $\Delta\alpha$ and $||d\alpha||^2$, $||\delta\alpha||^2$ are the global norms, of $d\alpha$ and $\delta\alpha$ respectively.

3. Some results on exterior forms on a Riemannian manifold

Let P be a point of the manifold. If $X, Y \in T_P(M)$, where $T_P(M)$ is the tangent space of M in P, then we denote by

$$< X, Y > \text{ and } |X| \tag{9}$$

the scalar product of the vectors X, Y and the norm of the vector $|X|$ respectively. It is known that the inner product $<>$ on $T_P(M)$ is induced by the Riemannian metric g on M. The curvature tensor R in the point P, by means of two given vectors Y, Z defines an endomorphism on $T_P(M)$ as follows:

$$R(Y,Z) : T_P(M) \to T_P(M),\ R(Y,Z) : X \to R(Y,Z)X. \tag{10}$$

The Riemannian tensor field R_1 in the point P defines a quadrilinear mapping

$$(R_1) : T_P(M) \times T_P(M) \times T_P(M) \times T_P(M) \to \mathbb{R}, \tag{11}$$

$$(R_1) : (X,Y,Z,W) \to (R_1)_P(X,Y,Z,W) =< R(X,Y)Z, W > . \tag{12}$$

Let λ be a plane of the tangent space $T_P(M)$ which is spanned by two linearly independent vectors X, $Y \in T_P(M)$. The sectional curvature of the plane λ is given by:

$$\sigma(\lambda) = \sigma(X,Y) = -\frac{\langle R(X,Y)X, Y\rangle}{\langle X, Y\rangle^2 - |X|^2 |Y|^2}. \tag{13}$$

PROBLEM 2.1 Is there a Riemannian metric g on M, that is $g \in R(M)$, with some special properties?

In some cases the problem is very difficult and it is still open.

We assume that the Riemannian manifold (M, g) is positively k-pinched, that means $\sigma(\lambda)$ satisfies the inequalities:

$$0 < k \leq \sigma(\lambda) \leq 1,\ (\forall)\lambda \in T_P(M),\ (\forall)P \in M. \tag{14}$$

The components of the Riemannian curvature in any point of this manifold satisfies the inequalities [2, p. 74-93]:

$$|R_{ijih}| \leq \frac{1}{2}(1-k),\ R_{ihjl} < \frac{1}{3}(1-k). \tag{15}$$

Let $(x_1, ..., x_n)$ be a normal coordinate system around the point P, which is taken as the origin of the system. It is known that there is an orthonormal base:

$$X_1, X_2, ..., X_n \tag{16}$$

of $T_P(M)$, such that its dual base:

$$X_1^*, X_2^*, ..., X_n^* \tag{17}$$

has the property that a given exterior 2-form α can be written as follows:

$$\alpha = \alpha_{12} X_1^* \wedge X_2^* + \alpha_{34} X_3^* \wedge X_4^* + \cdots + \alpha_{2m-1,2m} X_{2m-1}^* \wedge X_{2m}^*. \tag{18}$$

We form the exterior $2m$-form β defined by

$$\beta = \frac{1}{m!} \alpha \wedge \alpha \wedge \ldots \wedge \alpha. \tag{19}$$

The relation (19), by means of (18), takes the form:

$$\beta = \alpha_{12}\alpha_{34} \ldots \alpha_{2m-12m} X_1^* \wedge X_2^* \wedge \ldots \wedge X_{2m}^*. \tag{20}$$

From the relations (18) and (20) we have:

$$|\alpha|^2 = \alpha_{12}^2 + \alpha_{34}^2 + \cdots + \alpha_{2m-1,2m}^2, \; |\beta|^2 = \alpha_{12}^2 \alpha_{34}^2 ... \alpha_{2m-1,2m}^2. \tag{21}$$

If we calculate $Q_2(\alpha)$ at the point P from the formula (7) and take under consideration the inequalities (14) and (15) and the equalities (21), then we obtain:

$$\frac{1}{2} Q_2(\alpha) \leq -4(m-1)k|\alpha|^2 + \gamma(1-k)/3, \tag{22}$$

where

$$\gamma = \alpha_{12}\alpha_{34} + \alpha_{12}\alpha_{56} + ... + \alpha_{12}\alpha_{2n-1,2n} + \alpha_{34}\alpha_{56} + ... + \alpha_{2n-3,2n}\alpha_{2n-1,2n}. \tag{23}$$

It has been proved [9, p. 306]:

$$||\delta\beta||^2 \leq \frac{(2m-1)(m-2)}{m^{m-2}} |\nabla\alpha|^2 |\alpha|^{2m-2}, \tag{24}$$

which by integration implies:

$$\|\alpha\beta\|^2 \leq \frac{(2m-1)(m-2)}{m^{m-2}} \int \alpha 2\alpha 2m - 2\omega. \tag{25}$$

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4 . Main results

We assume that on the Riemannian manifold (M, g) exists the exterior $2m$-form β belonging to a zero class, that means its integral on the manifold is zero, in other words

$$\int_M \beta = 0. \tag{26}$$

Since the manifold M is of dimension $2m$, if we apply the $*$ operator on the form β, then we have the function

$$f = *\beta. \tag{27}$$

From the relation $\delta = -*d*$ of the operators δ, $*$, d and the relations (8) and (27) we obtain:

$$\|\delta\beta\|^2 = \|df\|^2 =< f, \Delta f > . \tag{28}$$

If we integrate the equation (26) we have:

$$\int\limits_M \beta = \int\limits_M f\omega = 0. \tag{29}$$

Let $\{\lambda_i\}$ be the spectrum of the Laplace operator Δ on the the functions on the manifold M. If $\{f_i\}$ are the eigenfunctions of Δ, then we obtain:

$$\Delta f_i = \lambda_i f_i. \tag{30}$$

It is known that the function f can be written:

$$f = \mu_* + \sum \mu_i f_i. \tag{31}$$

The relations (29), (30) and (31) imply:

$$\mu_* = \int\limits_M \beta = \int\limits_M f\omega = 0. \tag{32}$$

The relation (31) by means of (32) takes the form:

$$f = \sum \mu_i f_i. \tag{33}$$

The spectrum of the Laplace operator Δ on the functions on M has the form:

$$Sp(M,g) = \{0 < \lambda_1 = \lambda_1 = \cdots < \lambda_2 = \lambda_2 = \cdots < \lambda_n = \lambda_n = \cdots < \infty\}, \tag{34}$$

which means that $Sp(M,g)$ is discrete and each eigenvalue has a finite multiplicity.

It has been proved the following theorem [1]:

Theorem 1 *Let M be a compact manifold of dimension $n \geq 3$. There is always a Riemannian metric g such that the spectrum (34) has a lower bound any positive number.*

This theorem states that for a given positive number $\varepsilon > 0$, we can find a Riemannian metric g on M with the property:

$$\lambda_2 > \lambda_1 > \varepsilon > 0. \tag{35}$$

From (28), by means of (30) and (32), we obtain:

$$\|\delta\beta\|^2 = \|df\|^2 =< f, \Delta f >=< \sum \mu_i f_i, \sum \lambda_i \mu_i f_i = \sum \lambda_i (\mu_i f_i)^2 \tag{36}$$

which by virtue of (35) implies:

$$\|\delta\beta\|^2 = \|df\|^2 \geq \sum \varepsilon(\mu_i f_i)^2 = \varepsilon|f|^2. \tag{37}$$

The relation (37) by virtue of (27) yields:

$$\|\delta\beta\|^2 = \|df\|^2 \geq= \varepsilon|f|^2 = \varepsilon|\beta|^2. \tag{38}$$

If $\alpha \in H^2(M, I\!R)$, then the formula (5) takes the form:

$$\frac{1}{2}\Delta|\alpha|^2 = -|\nabla\alpha|^2 + \frac{1}{2}Q_2(\alpha). \tag{39}$$

It can be easily proved that we have:

$$\Delta(|\alpha|^{2m}) = m|\alpha|^{2m-2}\nabla(|\alpha|^2) - (m-1)|\alpha|^{2m-4}(d|\alpha|^2)^2, \tag{40}$$

which by integration implies:

$$\int_M |\alpha|^{2m-2}|\nabla\alpha|^2 \geq 0. \tag{41}$$

The relation (39), by integration and take under the consideration (41), yields:

$$\int_M |\alpha|^{2m-2}|\nabla\alpha|^2\omega \leq \frac{1}{2} \leq \frac{1}{2}(\alpha)\omega. \tag{42}$$

The inequality (42), by means of (12) and (25), becomes:

$$\frac{m^{m-3}}{2(m-1)}\,\|\delta\beta\|^2 \leq \frac{1}{2}\int [|\nabla\alpha|^{2m-2}[-4(m-1)k|\alpha|^2 + \frac{1}{3}(1-k)]\omega, \tag{43}$$

which by means of (38) takes the form:

$$\int_M [[\frac{1}{2}|\alpha|^{2m-2}[4(m-1)k|\alpha|^2 - \frac{1}{3}(1-k)] + \frac{m^{m-3}}{2(m-1)}\varepsilon|\beta|^2]\omega \leq 0. \tag{44}$$

The relation (44), after some calculations, takes the form:

$$\int_M [6(m-1)^2k|\alpha|^{2m} - \gamma(1-m)(1-k)|\alpha|^{2m-2} + 3m^{m-3}\delta_1|\beta|^2]\omega \leq 0. \tag{45}$$

We construct the following function:

$$f = 6(m-1)^2k|\alpha|^{2m} - (m-1)(1-k)\gamma|\alpha|^{2m-2} + 3m^{m-3}\delta_1|\beta|^2, \tag{46}$$

whose restriction at the point P gives:

$$f_P = 6(m-1)^2k|\alpha|_P^{2m} - (m-1)(1-k)\gamma_P|\alpha|_P^{2m-2} + 3m^{m-3}\delta_1|\beta|_P^2. \tag{47}$$

From (47), by virtue of the inequalities

$$\gamma_P \leq \frac{m-1}{2}|\alpha|_P^2, \ |\beta|_P^2 \leq \frac{1}{m^m}|\alpha|_P^{2m}, \tag{48}$$

we conclude that

$$f_P \geq |\beta|_P^2 \left[6(m-1)^2 k - \frac{(m-1)^2}{2}(1-k) + \delta_1 \frac{3}{m^3}\right]. \tag{49}$$

If

$$k > \frac{m^3(m-1)^2 - 2\delta_1}{11(m-1)^2 m^3}, \tag{50}$$

then $f_P \geq 0$, which contradicts the inequality (44) and therefore $f_P = 0$, if

$$\alpha_{12} = \alpha_{34} = ... = \alpha_{2m-12m} = 0 \tag{51}$$

and hence

$$\alpha = 0. \tag{52}$$

From the above we obtain the theorem:

Theorem 2 *Let M be a compact orientable positively k-pinched manifold of dimension $n = 2m$. If $k > \{m^3(m-1)^2 - 2\delta_1\}/\{11(m-1)^2 m^3\}$, where δ_1 is a lower bound of the first eigenvalue of the Laplace operator, then there exists no element different from zero of the cohomology group $H^2(M, \mathbb{R})$ such that its exterior m-power belongs to a zero class.*

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