GENERALIZATION OF THE GRASSMANN MANIFOLDS

Kostadin Trenčevski

Abstract

In this paper are introduced two kinds of compact analytical (real and complex) manifolds, denoted by $G_{(p_1,p_2,...,p_r;m)}(F)$ and $G^*_{(p_1,p_2,...,p_r;m)}(F)$, where F is the the field of real or complex numbers, $p_1,...,p_r \geq 1$ and $p_1+...+p_r \leq$m. In special case when $p_1 = ... = p_r = 1$, we obtain the ordinary Grassmann manifold $G_{(r,m)}(F)$.

Mathematics Subject Classification: 53C25
Key words: Grassmann manifold, generalized Grassmann manifold.

1 Introduction

There are two generalizations of the Grassmann manifolds in the known literature [1]: 1^o [3,4,5,10,11,12,14,15] as set of completely geodesic submanifolds of the Riemannian manifolds or pseudo-Riemannian manifolds, 2^o [2,6,7,8,9] as Grassmann bundles. In this paper we also give a generalization of the Grassmann manifolds.

First, we give an alternative approach for the Grassmann manifolds. We denote by F the field of real or complex numbers.

The Grassmann manifold $G_{(n,m)}(F)$ is defined as the set of n-dimensional subspaces of F^m $(m > n)$, with a certain topology. Indeed, the set M of all $n\times m$ matrices of rank n is principal bundle with the structure group $GL(n, F)$ and the base manifold is the Grassmann manifold $G_{(n,m)}(F)$. The principal bundle is non-trivial.

Using the elementary transformations on rows, each matrix $A \in M(n, m)$ of rank n can be transformed into a *canonical form* such that:

i) $0 \leq t_1 < t_2 < \cdots < t_n < m$,

where t_i denotes the number of the first zero coordinates of the i-th vector,

ii) its row vectors are orthogonal,

Editor Gr.Tsagas *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1998,* 124-132

iii) each row vector has the norm 1 and the first non-zero coordinate is a positive real number.

Note that the manifold of $n\times m$ matrices with rank n is equivalent to $GL(n,F)\times P$, where P is the set of all matrices in canonical form. Indeed, each $n\times m$ matrix of rank n can be written as $A\cdot X$, for $A\in GL(n,F)$ and $X\in$P. Moreover, if $A\cdot X=B\cdot Y$, for $A,B\in GL(n,F)$ and $X,Y\in P$, then $(B^{-1}\cdot A)\cdot X=Y$ and hence $B^{-1}\cdot A$ is upper-triangular matrix since X and Y satisfy the condition i). But $B^{-1}\cdot A$ is a diagonal matrix, because X and Y satisfy i) and ii) and finally $B^{-1}\cdot A=I$, because X and Y satisfy i), ii) and iii). Thus $A=B$ and $X=Y$.

According to the inner product of n-form [13], it is convenient to introduce the following metric of the Grassmann manifold $G_{(n,m)}(F)$. Let Σ_1 and Σ_2 are two subspaces of $G_{(n,m)}(F)$, and assume that Σ_1 is generated by the orthonormal vectors $\mathbf{a}_1,\cdots,\mathbf{a}_n$ and Σ_2 is generated by the orthonormal vectors $\mathbf{b}_1,\cdots,\mathbf{b}_n$. Then we have:

(1.1) $$d(\Sigma_1,\Sigma_2)=\inf_{\lambda\in F,|\lambda|=1}\|(\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_n)-\lambda(\mathbf{b}_1\wedge\cdots\wedge\mathbf{b}_n)\|=$$

$$=\inf_{\lambda\in F,|\lambda|=1}\sqrt{2-\Delta\lambda-\bar{\Delta}\bar{\lambda}},$$

where

(1.2). $$\Delta=\begin{vmatrix}\mathbf{a}_1\cdot\bar{\mathbf{b}}_1 & \mathbf{a}_1\cdot\bar{\mathbf{b}}_2 & \cdots & \mathbf{a}_1\cdot\bar{\mathbf{b}}_n\\ \mathbf{a}_2\cdot\bar{\mathbf{b}}_1 & \mathbf{a}_2\cdot\bar{\mathbf{b}}_2 & \cdots & \mathbf{a}_2\cdot\bar{\mathbf{b}}_n\\ \cdots & \cdots & \cdots & \cdots\\ \mathbf{a}_n\cdot\bar{\mathbf{b}}_1 & \mathbf{a}_n\cdot\bar{\mathbf{b}}_2 & \cdots & \mathbf{a}_n\cdot\bar{\mathbf{b}}_n\end{vmatrix},$$

So, a metric is defined in this way on the Grassmann manifold $G_{(n,m)}(F)$. According to the corresponding Cauchy-Schwartz inequality, it holds that $|\Delta|\le 1$ and moreover $|\Delta|=1$ if and only if the subspaces Σ_1 and Σ_2 coincide.

Note that this distance does not depend on the choice of the orthonormal systems and in this case the orthonormal systems can be chosen to be the row vectors from the canonical forms. Hence one obtain a metric d over the set P of matrices in canonical forms such that the topology space (P,d) coincides with the topology of the Grassmann manifold $G_{(n,m)}(F)$. This metric does not depend on the choice of the orthonormal coordinate system in F^m.

2 Generalized Grassmann manifold $G_{(p_1,\dots,p_r;m)}(F)$

Let $p_1,p_2,\cdots,p_r$ be positive integers such that $p_1+p_2+\cdots+p_r=n\le$m. We consider a set of linearly independent vectors

$$\mathbf{a}_{11},\mathbf{a}_{12},\cdots,\mathbf{a}_{1p_1},\mathbf{a}_{21},\mathbf{a}_{22},\cdots,\mathbf{a}_{2p_2},\cdots,\mathbf{a}_{r1},\mathbf{a}_{r2},\cdots,\mathbf{a}_{rp_r},$$

of F^m and we denote the matrix with these n row-vectors by A. If

i) $t_1<t_2<\cdots<t_r$, where $t_i=\min\{t(\mathbf{a}_{i1}),t(\mathbf{a}_{i2}),\cdots,t(\mathbf{a}_{ip_i})\}$ and $t(\mathbf{a})$ denotes the number of the first zero coordinates of $\mathbf{a}$,

ii) each two different vectors of these n vectors are orthogonal,

iii) each row vector has the norm 1 and the first non-zero coordinate is a positive real number,

then the considered matrix A is in *canonical form.* The set of all matrices in canonical form can be endowed with a topology as follows.

Suppose that the matrices A and B are in canonical forms. For each sequence $\alpha = j_1, \cdots, j_r$, $1 \leq j_i \leq p_i$, we have the subspaces Σ_α and Π_α generated by the vectors $\mathbf{a}_{1j_1}, \mathbf{a}_{2j_2}, \cdots, \mathbf{a}_{rj_r}$ and $\mathbf{b}_{1j_1}, \mathbf{b}_{2j_2}, \cdots, \mathbf{b}_{rj_r}$ respectively. Notice that there are exactly $N = p_1!p_2! \cdots p_r!$ such sequences α. Now we define a function $d(A, B)$ as sum of all N distances $d(\Sigma_\alpha, \Pi_\alpha)$ in the Grassmann manifold (Section 1). It is easy to verify that d is a metric in the set of all matrices in canonical form and hence it induces a topology. Indeed, if $d(A, B) = 0$, then for $\alpha = s_1, s_2, \cdots, s_r$, where s_u is chosen such that $t(\mathbf{a}_{us_u}) = t_u$, it follows that $d(\Sigma_\alpha, \Pi_\alpha) = 0$ and hence $\mathbf{a}_{is_i} = \mathbf{b}_{is_i}$, because the matrices with row vectors $\mathbf{a}_{1s_1}, \cdots, \mathbf{a}_{rs_r}$ and $\mathbf{b}_{1s_1}, \cdots, \mathbf{b}_{rs_r}$ are in canonical forms from Section 1. Now, in order to prove that $\mathbf{a}_{ij} = \mathbf{b}_{ij}$, it is sufficient to consider the sequence $\beta = s_1, \cdots, s_{i-1}, j, s_{i+1}, \cdots, s_r$ and to use that $\Sigma_\beta = \Pi_\beta$. The other axioms are obvious.

In this section we prove that these spaces are analytical manifolds and we will denote them by $G_{(p_1,p_2,\ldots,p_r;m)}(F)$.

Theorem 2.1. *The set of all matrices in canonical form is an analytical (real or complex) compact manifold.*

Proof. The proof is by induction of n.

If $n = 1$ the corresponding space is the compact analytical manifold FP^{m-1}.

Suppose that the theorem holds for n or less than n row vectors. Let X be any $(n+1) \times m$ matrix of rank $n+1$ in canonical form defined in Section 1 and A is non-singular quadratic matrix of order $n+1$. Then $A \cdot X$ is in canonical form if and only if the matrix A is in canonical form. To see this, it is sufficient to denote the row vectors of X as basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$.

Consider a neighborhood of $A \cdot X$, where $A \in G_{(p_1,\ldots,p_r;n+1)}(F)$ and $X \in G_{(n+1,m)}(F)$, since $G_{(n+1,m)}(F)$ considers as a set of matrices in canonical form (Section 1) endowed with a metric d which is invariant under the $m \times m$ transformations of right over X. Without loss of generality we can assume that $t_1 = 0, t_2 = 1, t_3 = 2, \cdots, t_{n+1} = n$ for the matrix X in canonical form. In this case the topology in a small neighborhood of $X \in G_{(n+1,m)}(F)$ coincides with the usual topology of $(n+1) \times m$ matrices with row vectors in FP^{m-1}.

Thus for each $X \in G_{(n+1,m)}(F)$ there exists sufficiently small neighborhood U such that

$$G_{(p_1,\ldots,p_r;n+1)}(F) \times U \to G_{(p_1,\ldots,p_r;m)}(F)$$

is an analytical embedding. The manifold $G_{(n+1,m)}(F)$ is a compact analytical one and it can be covered by $\binom{m}{n+1}$ such open neighborhoods U. So, without loss of generality, we assume that $m = n+1$, in order to prove that the space of $(n+1) \times m$ canonical matrices is a compact analytical manifold. Indeed, $G_{(p_1,\ldots,p_r;m)}(F)$ is equivalent, but not homeomorphic with $G_{(p_1,p_2,\ldots,p_r;n+1)}(F) \times G_{(n+1,m)}(F)$.

We consider an $(n+1)\times(n+1)$ matrix P in canonical form and denote by $\bar{P}$ and $\tilde{P}$ the submatrices of the first p_1 row vectors and the last $n+1-p_1$ row vectors respectively. Since P is non-singular and since P is in canonical form, at least one of the first coordinates of the vectors in $\bar{P}$ is non-zero. For example suppose that the first coordinate of $\mathbf{a}_{11}$ is non-zero. Having in mind the metric of P and the first zero coordinates of the vectors $\mathbf{a}_{ij}$ for $i>1$, considering the sequence of type $\alpha=1,j_2,\cdots,j_r$ we conclude that if the matrices P and P' are close according to that metric, then $\mathbf{a}_{11}$ is close to $\mathbf{a}'_{11}$ with respect to the ordinary metric in the projective space and $\tilde{P}$ is close to $\tilde{P}'$ with respect to the hereditary topology. Further, if we consider a sequence of type $\alpha=j,j_2,\cdots,j_r$ for $j>1$, we conclude that $\mathbf{a}_{1j}$ is close to $\mathbf{a}'_{1j}$ for any $2\le j\le p_1$ with respect to the ordinary topology in the projective spaces. Moreover there is no any constraint for the first zero positions of these vectors. The converse conclusions also hold. Thus we obtain that *the matrices P and P' in canonical form are closed, if and only if $\tilde{P}$ is closed to $\tilde{P}'$ with respect to the hereditary topology and $\bar{P}$ is closed to $\bar{P}'$ with respect to the ordinary topology of matrices with projective row vectors.* Note also that $\bar{P}$ with the topology of the matrices of projective row vectors is just the special case of the topology of $G_{(p_1;n+1)}(F)$, i.e. for $r=1$, and the topology of $\tilde{P}$ with respect to the hereditary topology of P is the special case $G_{(p_2,\cdots,p_r;n)}$.

If $r=1$, in Example 3 it is proved that $G_{(n;m)}(F)$ is a compact analytical manifold. If $r>1$, then $n+1>p_1>0$ and thus, by an inductive assumption, each of the topological spaces of matrices $\bar{P}$ and $\tilde{P}$ is compact analytical manifold. The required manifold of matrices P consists of pairs $(\bar{P},\tilde{P})$ under the condition of orthogonality in F^{n+1} of their row vectors and thus it is also compact analytical manifold. □

Notice that $\dim G_{(p_1,\ldots,p_r;m)}(F)=nm-\binom{n+1}{2}-(p_2+2p_3+\cdots+(r-1)p_r)$.

Example 1. In the special case $p_1=p_2=\cdots=p_r=1$ we obtain the Grassmann manifold $G_{(r,m)}(F)$.

Example 2. Let $F=R$, $r=2, p_1=2, p_2=1$ and $m=3$. Then the manifold of canonical vectors consists of the following cells:

$$C_1=\begin{bmatrix} x & * & * \\ y & * & * \\ 0 & * & * \end{bmatrix},\quad C_2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix},\quad C_3=\begin{bmatrix} 0 & * & * \\ 1 & 0 & 0 \\ 0 & * & * \end{bmatrix},$$

where $x,y>0$. The cell C_1 is homeomorphic to $R\times S^1$ because for fixed ratio $\lambda=x/y\in R^+$ it is homeomorphic to S^1. The cells C_2 and C_3 are homeomorphic to S^1. Thus the Euler characteristic of the manifold is $\chi=0$. The manifold can be described as follows: each point consists of two orthogonal lines p and q through the coordinate origin in R^3 (the third line which is orthogonal to p and q is uniquely determined) such that q lies in the yz-plane. It is homeomorphic to the Klein's bottle.

Example 3. This example completes the proof of the Theorem 2.1. Let $r=1$ and $p_1=n$. Then the space of canonical forms is the manifold of $n\times m$ matrices of rank n, consisting of n orthogonal vectors from the projective space FP^{m-1}. This manifold is base manifold for the principal bundle of matrices consisting of orthonormal vectors

($F = R$ or $F = C$) with the structure group:

$$\left\{ \begin{bmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u_n \end{bmatrix} ; |u_1| = \cdots = |u_n| = 1 \right\}.$$

Now, let us parameterize the manifold $G_{(n;m)}(F)$. For an arbitrary sequence $i_1, i_2, \cdots, i_n$ of different numbers in the set $\{1, \cdots, m\}$ we consider the $n \times n$ submatrix $A_{i_1 i_2 \ldots i_n}$ of A consisting of columns nominated by $i_1, i_2, \cdots, i_n$. For each such sequence we choose arbitrary $n(m-n)$ elements of the matrix A which do not belong to $A_{i_1 i_2 \ldots i_n}$, we choose units on the main diagonal of $A_{i_1 i_2 \ldots i_n}$ and also we choose $\binom{n}{2}$ elements under the main diagonal such that the elements over the main diagonal can be chosen uniquely. It can be made such that the row vectors of the obtained matrix are orthogonal. Indeed, it means that the $n \times n$ submatrix $A_{i_1 i_2 \ldots i_n}$ satisfies:

a) the elements on the main diagonal are non-zeros,

b) all k submatrices ($1 \le k \le n$) obtained by deleting the first $n-k$ rows and $n-k$ columns are non-zero.

Note that we have an open neighborhood as subset of Euclidean space for each sequence $i_1, i_2, \cdots, i_n$. Now the following question appears: *If the columns of A nominated by $i_1, i_2, \cdots, i_n$ are linearly independent, does there exist a permutation τ of $i_1, i_2, \cdots, i_n$, such that the considered $n \times m$ matrix A belongs to the neighborhood induced by the sequence $\tau(i_1), \tau(i_2), \cdots, \tau(i_n)$?* The answer is affirmative and it is sufficient to prove that for any non-singular $n \times n$ matrix can be chosen a permutation τ of its columns such that the obtained matrix satisfy the conditions a) and b). It is easy to prove this statement by induction on n, by decomposing the corresponding determinant with respect to the first row. This parameterization of $G_{(n;m)}(F)$ makes it an analytical manifold. This manifold is also a flag manifold.

3 Generalized Grassmann manifold $G^*_{(p_1,\ldots,p_r;m)}(\mathbf{F})$.

Let $p_1, p_2, \cdots, p_r$ be r positive integers and let $p_1 + p_2 + \cdots + p_r = n \le m$. We consider a set of linearly independent vectors

$$\mathbf{a}_{11}, \mathbf{a}_{12}, \cdots, \mathbf{a}_{1p_1}, \mathbf{a}_{21}, \mathbf{a}_{22}, \cdots, \mathbf{a}_{2p_2}, \cdots, \mathbf{a}_{r1}, \mathbf{a}_{r2}, \cdots, \mathbf{a}_{rp_r},$$

of F^m, and we denote the matrix with these n row-vectors by A.

If the following conditions hold:

i) there exists a block decomposition of A of submatrices A_{ij}, ($1 \le i, j \le r$), such that $A_{ij} = 0$ for $i > j$ and $rank(A_{ii}) = p_i$ ($1 \le i \le r$),

ii) each two different row vectors of A are orthogonal,

iii) each row vector has the norm 1 and the first non-zero coordinate is a positive real number,

then we say that the considered n vectors are in a *canonical form.*

We are going to define a topology over the set of all n vectors in canonical form. Without loss of generality we can suppose that these n vectors are taken to be row vectors of an $n \times m$ matrix and then we say that the corresponding matrix is in canonical form. If A and B are two matrices in canonical form, with row vectors $\mathbf{a}_{ij}$ and $\mathbf{b}_{ij}$, for $1 \le i \le r$ and $1 \le j \le p_i$, then we define:

$$d(A,B) = \inf_{\lambda \in F, |\lambda|=1} \sqrt{2 - \Delta\lambda - \bar{\Delta}\bar{\lambda}} \tag{2.1),}$$

where

$$\Delta = det \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1r} \\ P_{21} & P_{22} & \cdots & P_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ P_{r1} & P_{r2} & \cdots & P_{rr} \end{bmatrix},$$

P_{ij} for $i \ne j$ is $p_i \times p_j$ matrix such that $(P_{ij})_{uv} = \mathbf{a}_{iu} \cdot \bar{\mathbf{b}}_{jv}$ and $(P_{ii})_{uv} = \delta_{uv}\mathbf{a}_{iu} \cdot \bar{\mathbf{b}}_{iv}$. In order to prove that d is a metric, first we note that it is easy to see that $|\Delta| \le 1$ and $|\Delta| = 1$ if and only if the row vectors of A generate the same vector space as the row vectors of B and $\mathbf{a}_{iu} \cdot \bar{\mathbf{b}}_{iv} = 0$ for $u \ne v$, i.e. the $\mathbf{a}_{iu}$ and $\mathbf{b}_{iv}$ are orthogonal for $u \ne v$. In order to see that this it is sufficient we assume, without loss of generality , that $\mathbf{b}_{ij}$ are the orthonormal vectors $\mathbf{e}_1, \cdots, \mathbf{e}_n$. Then the vectors which are row vectors of Δ have the modules ≤ 1 and if $|\Delta| = 1$, it follows that these vectors have unit module and they are orthogonal. Hence it follows that the row vectors of A generate the same vector space as the row vectors of B and $\mathbf{a}_{iu}$ and $\mathbf{b}_{iv}$ are orthogonal for $u \ne v$.

Now we prove that d is a metric.

i)It is obviously that $d(A,B) \ge 0$. If $A = B$, then $\Delta = 1$ and hence $d(A,B) = 0$. Conversely, if $d(A,B) = 0$, then $|\Delta| = 1$ and according to the previous discussion it follows that A generates the same vector space as the row vectors of B and $\mathbf{a}_{iu}$ and $\mathbf{b}_{iv}$ are orthogonal for $u \ne v$. Now, using the canonical form, we will prove that $A = B$. Indeed, let us decompose the matrices A and B as $M \cdot X$ and $N \cdot Y$, where M and N are n-quadratic matrices and X and Y are $n \times m$ matrices in canonical form from Section 1. Since the row vectors of A and B generate the same vector subspace, it follows that $X = Y$ and moreover the matrices M and N are in canonical forms (see Proposition 3.2). Thus, it is sufficient to prove that $M = N$, i.e. to prove the statement for $m = n$. Now, according to the proof of the Proposition 3.1 it follows that M and N are block-diagonal matrices. Thus for each i, $1 \le i \le r$, both sets of vectors $\mathbf{a}_{i1}, \cdots, \mathbf{a}_{ip_i}$ and $\mathbf{b}_{i1}, \cdots, \mathbf{b}_{ip_i}$ generate the same vector subspace. Using the orthogonality conditions, it follows that $\mathbf{a}_{ij}$ is collinear with $\mathbf{b}_{ij}$ for $1 \le i \le r$ and $1 \le j \le p_i$. Using the condition iii) of canonical form, it follows that $M = N$.

ii) From the definition of d it follows that $d(A,B) = d(B,A)$.

iii) The inequality $d(A,B) + d(B,C) \ge d(A,C)$ can be proved as follows. The matrices A, B and C can be replaced by A', B' and C' of dimension $n \times m'$ of sufficiently large number m', such that for each two matrices, for example A' and B', the scalar product of $\mathbf{a}'$ and $\mathbf{b}'$ is close to the scalar product of the corresponding vectors $\mathbf{a}$ and $\mathbf{b}$ of A and B if these two vectors belong to other sets of partition of $\{1, \cdots, n\}$ into r subsets, or they correspond to same indices. Otherwise, the scalar product of $\mathbf{a}'$ and $\mathbf{b}'$ is zero. Now, according to the Section 1, $d(A',B') + d(B',C') \ge d(A',C')$ and the metric d coincides in this case to the mapping d considered in this section. Using that $d(A',B')$ is close to $d(A,B)$, $d(B',C')$ is close to $d(B,C)$ and

$d(A', C')$ is close to $d(A', C')$, it follows that $d(A,B) + d(B,C) \geq d(A,C)$. Thus d is a metric.

In order words, the metric d in set of canonical forms can be described such that two matrices A and B are close if the subspaces generated by the rows of the matrices A and the rows of B are close (in the sense of the metric of the Grassmann manifold) and $\mathbf{a}_{iu}$ and $\mathbf{b}_{iv}$ are approximately orthogonal for $1 \leq i \leq r$ and $u \neq v$.

Proposition 3.1. *If $m = n$, then the space $G^*_{(p_1,\dots,p_r;m)}(F)$ is analytical compact manifold homeomorphic to*

$$G_{p_1}(F) \times G_{p_2}(F) \times \cdots \times G_{p_r}(F) = \begin{bmatrix} G_{p_1} & 0 & 0 & \cdots & 0 \\ 0 & G_{p_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & G_{p_r} \end{bmatrix},$$

where $G_{p_i}(F)$ is the manifold consisting of $p_i \times p_i$ matrices with the orthogonal row vectors in FP^{p_i-1}.

Proof. Note that $n = m$ implies that all the submatrices $A_{ii}(1 \leq i \leq r)$ are quadratic $p_i \times p_i$ matrices. Since each row of this matrix has norm 1, $A \in O(n)$ or $A \in U(n)$. Since $A_{ij} = 0$ for $i > j$ it follows that $A_{ij} = 0$ for $i < j$. Thus, we obtain the required form in the special case $m = n$.

It also can be verified in this case that the topology of $G^*_{(p_1,p_2,\dots,p_r;m)}(F)$ is the same as the topology of $G_{p_1}(F) \times \cdots \times G_{p_r}(F)$. Since $G_{p_i}(F)$ is a compact analytical manifold for each $i \in \{1, \cdots, m\}$ (see Example 3 in Section 2), we obtain that $G^*_{(p_1,\dots,p_r;m)}(F)$ is also a compact analytical manifold. □

Proposition 3.2. *There is an equivalence between the sets*

$$G^*_{(p_1,\dots,p_r;m)}(F) \text{ and } \Big[G_{p_1}(F) \times G_{p_2}(F) \times \cdots \times G_{p_r}(F)\Big] \times G_{(n,m)}(F).$$

Proof. Notice that is Section 1 we have proved that each $n \times m$ matrix of rank n can uniquely be written as $A \cdot X$, where $A \in GL(F)$ and $X \in G_{(n,m)}(F)$, i.e. the set of $n \times m$ matrices in canonical form defined in Section 1. One can verify that $A \cdot X$ has a canonical form (defined in this Section) if and only if A has canonical form (defined in this Section). Since A is quadratic matrix in canonical form, it belongs to $G_{p_1}(F) \times G_{p_2}(F) \times \cdots \times G_{p_r}(F)$ according to the Proposition 3.1. Thus, each matrix of $G^*_{(p_1,\dots,p_r;m)}(F)$ can uniquely be written as $A \cdot X$ for $A \in G_{p_1}(F) \times G_{p_2}(F) \times \cdots \times G_{p_r}(F)$ and $X \in G_{(n,m)}(F)$. □

Notice that the equivalence in this proposition does not mean homeomorphic. To see this, it is sufficient to consider the special case $r = 1$.

Having in mind the topology of $G^*_{(p_1,\dots,p_r;m)}(F)$ and the Proposition 3.2, the following theorem holds.

Theorem 3.3. *The space $G^*_{(p_1,\dots,p_r;m)}(F)$ of canonical matrices is an analytical (real or complex) compact manifold.*

Proof. According to the Proposition 3.2, analogously to the proof of the theorem 2.1, for each matrix $X \in G_{(n,m)}(F)$ there exists a neighborhood U of X such that

$$G^*_{(p_1,\dots,p_r;n)}(F) \times U \to G^*_{(p_1,\dots,p_r;m)}(F)$$

is an analytical embedding and moreover $G_{(n,m)}(F)$ can be covered with $\binom{m}{n}$ such neighborhoods U. Since $G_{(n,m)}(F)$ is an analytical compact manifold and $G^*_{(p_1,\ldots,p_r;n)}(F)$ is also an analytical compact manifold according to the Proposition 3.1, it follows that $G^*_{(p_1,\ldots,p_r;m)}(F)$ is an analytical compact manifold, too. □

Notice that

$$\dim G^*_{(p_1,\ldots,p_r;m)}(F) = n(m-n) + \binom{p_1}{2} + \cdots + \binom{p_r}{2},$$

and the manifold is covered with $\binom{m}{n} p_1! \cdot p_2! \cdots p_r!$ coordinate neighborhoods. Hence, if $q_1, \cdots, q_r$ is a permutation of $p_1, \cdots, p_r$, then

$$\dim G^*_{(p_1,\ldots,p_r;m)}(F) = \dim G^*_{(q_1,\ldots,q_r;m)}(F),$$

but we do not know when the manifolds $G^*_{(p_1,\ldots,p_r;m)}(F)$ and $G^*_{(q_1,\ldots,q_r;m)}(F)$ are homeomorphic.

In the special case $p_1 = p_2 = \cdots = p_r = 1$, we obtain the ordinary Grassmann manifold $G_{(r,m)}(F)$.

References

[1] A.A.Borisenko and Yu.A.Nikolaevskii, *Grassmann manifolds and Grassmann image of submanifolds* (in Russian), Uspehi Math. Nauk, 46 (1991), 41-83.

[2] M.J.Ferriera, *A twistorial characterization of conformal branched immersions with parallel mean curvature*, Bull. Soc. Math. Belg.,39 (1987), 47-81.

[3] E.Grecu, *Espaces de Grassman singuli ́eres*, Bull. Math. Soc. Sci. Math. RSR. 25 (1981), 359-365.

[4] T.Ishihara, *On a submanifold of a Riemannian manifold and the Gauss map* , J. Math. Tokushima Univ., 14 (1980), 1-9.

[5] T.Ishihara, *The harmonic Gauss map in a generalized sense*, J. London Math. Soc. 26 (1982), 104-112.

[6] T.Ishihara, *Total curvature of manifolds in self-immersed manifolds*, Manuser Math., 39 (1982), 201-218.

[7] S.E.Kozlov, *On the spherical mappings of surfaces in Riemannian manifolds* (in Russian), Research in surface theory of the Riemannian manifolds, (1984), 85-97.

[8] S.E.Kozlov, *Areas forms of spherical mappings and analogs to the Gauss and Gauss-Bonnet theorems for two dimensional surfaces in four-dimensional Riemannian manifolds* (in Russian), Vestnik L.G.U. Math. Mech. Astr. (1985), 18.

[9] S.E.Kozlov, *Conformal points of the spherical mappings immersed in two-dimensional surfaces* (in Russian), Geometrical questions in theory of functions and sets, Kalinin (1985), 106-119.

[10] I.Maazikas, *On the Riemannian geometry of the Grassmann manifolds of unisotropic subspaces of pseudo-Euclidean spaces* (in Russian), Uch. zap. Tartus. Univ., 342 (1974), 76-82.

[11] I.Maazikas, *Grassmann mapping of congruences of 2-plains in Euclidean spaces* (in Russian), Uch. zap. Tartus. Univ. 355 (1975), 57-75.

[12] I.Maazikas, *Congruences of 2-plains with completely geodesic Grassmann images* (in Russian), Uch. zap. Tartus. Univ., 355 (1975), 76-85.

[13] S.MacLane and G.Birkhoff, *Algebra*, The Macmillan Company, New York.

[14] S.Nishikawa, *The Gauss map of Kaehler immersions*, Tohoku Math. J., 27 (1975), 453-460.

[15] M.Obata, *The Gauss map of immersion of Riemannian manifolds in spaces of constant curvature*, J. Diff. Geom., 2 (1968), 217-223. *The Gauss map of Kaehler immersions*, Tohoku Math. J., 27 (1975), 453-460.

Author's address:

Kostadin Trenčevski
Institute of Mathematics,
St.Cyril and Methodius University,
P.O.Box 162, Skopje, Macedonia
E-mail address: kostatre@iunona.pmf.ukim.edu.mk