GENERALIZED ALGEBROIDS AND STRONGLY HOMOTOPY LIE ALGEBRAS

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Abstract

Generalized algebroids put together Lie algebroids, prealgebroids and Courant algebroids and were defined earlier by one of the authors. The aim of the paper is to show that they are strongly homotopy Lie algebras. In the case of Courant algebroids, the sh Lie structure defined in the paper is quite different from the construction of D. Roytenberg and A. Weinstein.

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1 Introduction

The *Lie algebroids* are a natural generalization of the tangent bundles of manifold as well as of the Lie algebras. Used earlier by many authors, the study of Lie algebroids were considerably improved by J. Pradines in [10], which observed that they are infinitesimal versions of Lie groupoids in a functorial manner. In the same paper there are considered an algebraic version of Lie algebroids, called Lie pseudoalgebra, and infinitesimal modules, which are algebraic versions of *prealgebroids* (as called in [8]), which fulfill the same conditions as Lie algebroids, but not a null Jacobiator. Other algebroids are the *Lie bialgebroids* defined in [7] by K.C.H.Mackenzie and P.Xu, which considered vector bundles θ with Lie algebroid structures on θ and θ^*, with a natural compatibility condition. The *Courant algebroids* are defined in [5] by Z.J.Liu, A.Weinstein and P.Xu, which proved that Lie bialgebroids are Courant algebroids. In the same paper, some general, but concrete new problems are proposed, some referring to the questions: what has a Courant algebroid in common with a Lie algebroid and find mathematical structures that Courant algebroids fit in. Trying to give some possible answers to this questions, the *generalized algebroids* are defined in [9]. They are generalizations of all algebroids mentioned before. In the same paper,

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a generalized Lie groupoid is also considered, having as an infinitesimal version a generalized algebroid.

Recently, in [11] there were proved that Courant algebroids are sh Lie algebras. We prove in the paper that generalized algebroids with antisymmetric brackets are also sh Lie algebras in at least two ways. Using some ideas from [1] we construct effectively sh Lie algebras structures for a large class of algebraic structures, called *general pseudo-Lie algebras*. In the case of Courant algebroids, our constructions is quite different as in [11].

2 Generalized algebroids

In this section we give the definition and some related constructions concerning generalized algebroids. More details can be found in [9]. Consider a vector bundle $\theta = (E, p, M)$. An *anchor* is a vector bundle map $a : \theta \to \tau M = (TM, \pi, M)$ and a *bracket* is an $I\!R$-bilinear map $[\cdot,\cdot] : \Gamma(\theta) \times \Gamma(\theta) \to \Gamma(\theta)$. We say that $\mathcal{M}$ is *closed* if $[X, Y] \in \mathcal{M}$ whenever X or $Y \in \mathcal{M}$. We call the *derived module* of $\mathcal{M} \subset \Gamma(\theta)$ as being the $\mathcal{F}(M)$-module $\mathcal{D}er(\mathcal{M}) \subset \Gamma(\theta)$ which is the intersection of all closed submodules of $\Gamma(\theta)$ which contain $\mathcal{M}$.

Definition 1 *Let θ be a vector bundle, $a : \theta \to \tau M$ be an anchor, $[\cdot,\cdot]$ be a bracket on θ and $\mathcal{S} \subset \Gamma(\theta)$ be an $\mathcal{F}(M)$-submodule such that $a(X) = 0$, $(\forall)X \in \mathcal{S}$.*

*We say that $(\theta, a, [\cdot,\cdot])$ is an $\mathcal{S}$-*algebroid *(or a* generalized algebroid *if no confusion arise) if the following properties are satisfied:*

(GA1) $\mathcal{J}(X, Y, Z) \in \mathcal{S}$, $(\forall)X, Y, Z \in \Gamma(\theta)$;

(GA2) $[X, f \cdot Y] - f \cdot [X, Y] - a(X)(f)Y$, $[f \cdot X, Y] - f \cdot [X, Y] + a(X)(f)Y \in \mathcal{S}$, $(\forall)X, Y \in \Gamma(\theta)$, $f \in \mathcal{F}(M)$;

(GA3) $[a(X), a(Y)] = a([X, Y])$, $(\forall)X, Y \in \Gamma(\theta)$;

(GA4) $[X, Z] \in \mathcal{S}$, *whenever X or Z are in $\mathcal{S}$.*

where $\mathcal{J}(X, Y, Z) \overset{not}{=} [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$, $(\forall)X, Y, Z \in \Gamma(\theta)$ *is the Jacobi map of the bracket.*

Notice that the bracket need not to be skew-symmetric. It is easy to see that the conditions *(GA2)* and *(GA4)* together the fact that $\mathcal{S}$ is an $\mathcal{F}(M)$-module implies that $\mathcal{S}$ is closed. For an $\mathcal{S}$-algebroid, denote as

$$\mathcal{M}_{\mathcal{J}} = \{\mathcal{J}(X, Y, Z) : X, Y, Z \in \Gamma(\theta)\}. \tag{1}$$

and

$$\mathcal{M}_{\mathcal{L}} = \{[X, fY] - f[X, Y] - a(X)(f)Y, [f \cdot X, Y] - f \cdot [X, Y] + a(X)(f)Y : (\forall)X, Y \in \Gamma(\theta),\ f \in \mathcal{F}(M)\} \tag{2}$$

It is easy to see that $\mathcal{D}er(\mathcal{M}_{\mathcal{J}})$, $\mathcal{D}er(\mathcal{M}_{\mathcal{L}}) \subset \mathcal{D}er(\mathcal{M}_{\mathcal{J}} \cup \mathcal{M}_{\mathcal{L}}) \subset \mathcal{S}$. If $\mathcal{S} = \{0\}$ we obtain the classical definition of a Lie algebroid.

We consider now the prealgebroid defined by J. Pradines in [10]:

Definition 2 *Let θ be a vector bundle, $a : \theta \to \tau M$ be an anchor and $[\cdot,\cdot]$ be bracket on θ. We say that the triple $(\theta, a, [\cdot,\cdot])$ is a* generalized prealgebroid *provided that:*

(PA0) The bracket is skew-symmetric.

(PA1) $[X, f \cdot Y] = f \cdot [X, Y] + a(X)(f) \cdot Y$, $(\forall) X, Y \in \Gamma(\theta)$, $f \in \mathcal{F}(M)$;

(PA2) $[a(X), a(Y)] = a([X, Y])$, $(\forall) X, Y \in \Gamma(\theta)$.

We say that $(\theta, a, [\cdot,\cdot])$ is a right (left) generalized prealgebroid *if the condition* (PA0) *is removed and the condition* (PA1) *is replaced by:*

PA1L) $[X, f \cdot Y] = f \cdot [X, Y] + a(X)(f) \cdot Y$, $(\forall) X, Y \in \Gamma(\theta)$, $f \in \mathcal{F}(M)$,

respectively by:

PA1R) $[f \cdot X, Y] = f \cdot [X, Y] - a(Y)(f) \cdot X$, $(\forall) X, Y \in \Gamma(\theta)$, $f \in \mathcal{F}(M)$.

A simultaneous left and right generalized algebroid is just a generalized prealgebroid. Notice that for a prealgebroid or a generalized prealgebroid it follows that $\mathcal{M}_{\mathcal{L}} = \{0\}$. We consider now the Courant algebroid, defined by Liu-Weinstein-Xu in [5]:

Definition 3 *A* Courant algebroid *is a vector bundle θ with an anchor a, a skew-symmetric and IR-linear bracket $[\cdot,\cdot]$ and a non-degenerate symmetric and bilinear form $(\cdot,\cdot)$ on θ, such that the following properties are satisfied:*

(CA1) $\mathcal{J}(X, Y, Z) = \mathcal{D}T(X, Y, Z)$, $(\forall) X, Y \in \Gamma(\theta)$;

(CA2) $[a(X), a(Y)] = a([X, Y])$, $(\forall) X, Y \in \Gamma(\theta)$;

(CA3) $[X, fY] = f\,[X, Y] + a(X)(f)Y - (X, Y)\,\mathcal{D}f$, $(\forall) X, Y \in \Gamma(\theta)$, $f \in \mathcal{F}(M)$;

(CA4) $(\mathcal{D}f, \mathcal{D}g) = 0$, $(\forall) f, g \in \mathcal{F}(M)$;

(CA5) $a(X)\,((Y, Z)) = ([X, Y] + \mathcal{D}(X, Y), Z) + (X, [X, Z] + \mathcal{D}(X, Z))$, $(\forall) X, Y, Z \in \Gamma(\theta)$;

where $\mathcal{J}$ is the Jacobi map,

$$T(X, Y, Z) = \frac{1}{3}\left(([X, Y], Z) + ([Y, Z], X) + ([Z, X], Y)\right)$$

and $\mathcal{D} : \mathcal{F}(M) \to \Gamma(\theta)$ is defined by $(\mathcal{D}f, X) = \frac{1}{2} a(X)f$ for every $(\forall) X, Y, Z \in \Gamma(\theta)$ and $f \in \mathcal{F}(M)$.

Taking the new brackets:

$$[X,Y]_1 = [X,Y] + \mathcal{D}(X,Y),\ [X,Y]_2 = [X,Y] - \mathcal{D}(X,Y), \tag{3}$$

then $(\theta, a, [\cdot,\cdot]_1)$ is a generalized right prealgebroid and respectively $(\theta, a, [\cdot,\cdot]_2)$ is a generalized left prealgebroid (see [5] for comments concerning the first bracket).

The following results from [9] give some characterizations for generalized algebroids, easy to handle.

Proposition 1 *Let θ be a vector bundle, $a : \theta \to \tau M$ be an anchor and $[\cdot,\cdot]$ be a bracket on θ which has the property $a([X,Y]) = [a(X).a(Y)]$, $(\forall)X, Y \in \Gamma(\theta)$.*

Then there is an $\mathcal{S}$-algebroid structure $(\theta, a, [\cdot,\cdot])$, with $\mathcal{S} = \mathcal{D}er(\mathcal{M}_{\mathcal{J}} \cup \mathcal{M}_{\mathcal{L}})$, where $\mathcal{M}_{\mathcal{J}}$ and $\mathcal{M}_{\mathcal{L}}$ are given by the formulas (1) and (2) respectively.

Corollary 1 *If $(\theta, a, [\cdot,\cdot])$ is an $\mathcal{S}$-algebroid then $(\theta, a, [\cdot,\cdot])$ is a $\mathcal{D}er(\mathcal{M}_{\mathcal{J}} \cup \mathcal{M}_{\mathcal{L}})$-algebroid and $\mathcal{D}er(\mathcal{M}_{\mathcal{J}} \cup \mathcal{M}_{\mathcal{L}})$ is the minimal $\mathcal{S}$.*

Corollary 2 *A (generalized) prealgebroid $(\theta, a, [\cdot,\cdot])$ is an $\mathcal{S}$ -algebroid, with the minimal $\mathcal{S} = \mathcal{D}er(\mathcal{M}_{\mathcal{J}})$.*

Corollary 3 *A Courant algebroid is an $\mathcal{S}$ -algebroid in three ways: 1) $(\theta, a, [\cdot,\cdot])$ is an $\mathcal{S}$-algebroid with an antisymmetric bracket, 2) $(\theta, a, [\cdot,\cdot]_1)$ is a generalized right prealgebroid, and 3) $(\theta, a, [\cdot,\cdot]_2)$ is a generalized left prealgebroid. All of them have the minimal $\mathcal{S} = \mathcal{D}er(\mathcal{M}_{\mathcal{C}})$, where $\mathcal{M}_{\mathcal{C}} = \{\mathcal{D}(f) \mid f \in \mathcal{F}(M)\}$.*

Notice that the generalized algebroid is a progress for 'Open Problem 3' from [5]: What is the geometric meaning of such asymmetric bracket, satisfying most of the axioms of a Lie algebroid ?'

3 Strongly homotopy Lie algebra

Strongly homotopy Lie algebras have been introduced by J. Stasheff in [12]. They were intensively studied in the recent period of time, being recognized in many and various mathematical objects. We use [1, 3] as basic references.

Let V be a graded vector space. Let $T(V)$ denote the tensor algebra of V in the category of graded vector spaces, and let $\Lambda(V)$ denote its exterior algebra in the same category.

Definition 4 *A* strongly homotopy Lie algebra *(shLA, or L_∞-algebra) is a graded vector space V together with a collection of linear maps $l_k : \Lambda^k V \to V$ of degree $k-2$, $k \geq 1$, satisfying the following relation for each $n \geq 1$ and for all homogeneous $x_1, \ldots, x_n \in V$:*

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} (-1)^{\sigma} \epsilon(\sigma) l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0, \tag{4}$$

where $\epsilon(\sigma)$ *is the* Koszul sign *(arising from the fundamental convention of supermathematics that a minus sign is introduced whenever two consecutive odd elements are permuted), and* σ *runs over all* $(i, n-i)$*-unshuffles (i.e. permutations satisfying* $\sigma(1) < \ldots < \sigma(i)$ *and* $\sigma(i+1) < \ldots < \sigma(n)$*) with* $i \geq 1$.

For $n = 1$ this means that l_1 is a differential on V. Let us denote l_2 as a bracket $[\cdot, \cdot]$. For $n = 2$, we have: $l_1([x,y]) = [l_1(x), y] + (-1)^{|x|}[x, l_1(y)]$ (equivalently, $l_2 : \Lambda^2(V) \to V$ is a chain map of complexes); $n = 3$ gives the super-Jacobi identity for l_2:

$$\begin{aligned}(-1)^{|x||z|}\left[[x,y],z\right] + (-1)^{|y||z|}\left[[z,x],y\right] + (-1)^{|x||y|}\left[[y,z],x\right] = \\ (-1)^{|x||z|+1} \cdot \Big\{ l_1 l_3(x,y,z) + l_3(l_1(x),y,z) + (-1)^{|x|} l_3(x, l_1(y), z) \\ + (-1)^{|x|+|y|} l_3(x,y,l_1(z)) \Big\}.\end{aligned} \tag{5}$$

(see [3, Example 2.2]), and higher l_k's can be interpreted as higher homotopies, in analogous ways. The equation (4) is usually written in the more succinct equivalent form:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} l_j l_i = 0. \tag{6}$$

More details concerning shLA's can be found in [1, 3, 11].

Consider now two vector spaces $\mathcal{M}_1$ and $\mathcal{M}_2$. We can construct the following resolution of $\mathcal{M}_2$:

$$\cdots \to 0 \to \mathcal{X}_2 \stackrel{d_2}{\to} \mathcal{X}_1 \stackrel{d_1}{\to} \mathcal{X}_0 \stackrel{p}{\to} \mathcal{M}_2 \to 0 \tag{7}$$

where $\mathcal{X}_2 = \mathcal{M}_2$, $\mathcal{X}_1 = \mathcal{X}_\prime = \mathcal{M}_1 \oplus \mathcal{M}_2$, $d_2(m_2) = 0 \oplus m_2$, $d_1(m_1 \oplus m_2) = m_1$ and $p(m_1 \oplus m_2) = m_2$.

Proposition 2 *For two vector spaces* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *consider the resolution (7) of* $\mathcal{M}_2$ *and suppose that there is a bilinear and antisymmetric bracket* $[\cdot,\cdot]_0$ *on* $\mathcal{X}_\prime = \mathcal{M}_1 \oplus \mathcal{M}_2$ *such that* $\mathcal{M}_2$ *is closed (i.e.* $[X_0, M_2]_0 \subset M_2$*) and* $\mathcal{J}(x_0, y_0, z_0) \in M_2$, $(\forall) x_0, y_0, z_0 \in X_0$.

Then a SHLA structure of (7) can be considered on $\mathcal{X}_*$ *with* $l_1 = d$ *and the higher structure maps given by:*

$l_2\left((a_0 \oplus b_0) \wedge (c_0 \oplus d_0)\right) = [a_0 \oplus b_0, c_0 \oplus d_0]_0$ *in degree* 0;
$l_2\left((a_1 \oplus b_1) \wedge (c_0 \oplus d_0)\right) = 0 \oplus \left[(0 \oplus b_1), (c_0 \oplus d_0)\right]_0$ *in degree* 1;
$l_2 = 0$ *in degree* > 1;
$l_3\left((a_0 \oplus b_0) \wedge (c_0 \oplus d_0) \wedge (e_0 \oplus f_0)\right) =$
$0 \oplus \left(-\mathcal{J}_0\left((a_0 \oplus b_0), (c_0 \oplus d_0), (e_0 \oplus f_0)\right)\right)$ *in degree* 0;
$l_3 = 0$ *in degree* > 0;
$l_n = 0$ *for* $n > 3$.

Proof. We follow [1] (see also [11] for a concrete computation; we use the exterior notation as in this paper instead of the tensor product used in [1]).

We define inductively the bracket, using the law indices in order to denote the degree 0 or 1. Consider $x_0 = a_0 \oplus b_0$ and $y_0 = c_0 \oplus d_0 \in X_0$ and take

$l_2\left((a_0 \oplus b_0) \wedge (c_0 \oplus d_0)\right) = [a_0 \oplus b_0, c_0 \oplus d_0]_0$ in degree 0. It is easy to see that the conditions in the hypothesis $[X_0, M_2]_0 \subset M_2$ and $\mathcal{J}(x_0, y_0, z_0) \in M_2$, $(\forall) x_0, y_0, z_0 \in X_0$ are just the conditions *(i)* respectively *(ii)* in [1]. Using [1, Theorem 7], it follows that these conditions extend to an sh Lie structure on the resolution space X_*. It remains to prove that $(l_n)_{n\geq 0}$ of this space is just that considered in the conclusion of the proposition.

First let $n = 2$. Using successively $l_2 l_1 = l_1 l_2$, the derivation law and the previous definitions, we obtain: In degree 1: $l_1 l_2((a_1 \oplus b_1) \wedge (c_0 \oplus d_0)) = l_2 l_1((a_1 \oplus b_1) \wedge (c_0 \oplus d_0)) = l_2((0 \oplus b_1) \wedge (c_0 \oplus d_0)) = 0 \oplus [(0 \oplus b_1), (c_0 \oplus d_0)]_0 = l_1(0 \oplus [(0 \oplus b_1), (c_0 \oplus d_0)]_0)$, so we can define $l_2\left((a_1 \oplus b_1) \wedge (c_0 \oplus d_0)\right) = -l_2\left((c_0 \oplus d_0) \wedge (a_1 \oplus b_1)\right) = 0 \oplus [(0 \oplus b_1), (c_0 \oplus d_0)]_0$ in degree 1. In degree 2 we have first $l_1 l_2(a_2 \wedge (c_0 \oplus d_0)) = l_2 l_1(a_2 \wedge (c_0 \oplus d_0)) = l_2((a_2 \oplus 0) \wedge (c_0 \oplus d_0)) = 0 \oplus [0, (c_0 \oplus d_0)]_0 = 0$, so we can define $l_2(a_2 \wedge (c_0 \oplus d_0)) = 0$. We have also in degree 1: $l_1 l_2((a_1 \oplus b_1) \wedge (c_1 \oplus d_1)) = l_2 l_1((a_1 \oplus b_1) \wedge (c_1 \oplus d_1)) = l_2((0 \oplus b_1) \wedge (c_1 \oplus d_1)) + l_2((a_1 \oplus b_1) \wedge (0 \oplus d_1)) = -0 \oplus [d_1, (0 + b_1)]_0 + 0 \oplus [d_1, (0 + b_1)]_0 = 0$, so we can define $l_2((a_1 \oplus b_1) \wedge (c_1 \oplus d_1)) = 0$. In degree $k \geq 3$, we have $X_k = \{0\}$ and since l_2 has degree 0, it follows that $l_2 = 0$ in this case.

Consider now $n = 3$. According to [1, Proposition 4], in this case we ask that l_3 satisfies the condition $l_1 l_3 + l_2 l_2 + l_3 l_1 = 0$, which must be read as in (5). In degree 0 we have $l_1 l_3(x_0 \wedge y_0 \wedge z_0) = -J(x_0, y_0, z_0) = l_1(-J(x_0, y_0, z_0))$, so we can define $l_3(x_0 \wedge y_0 \wedge z_0) = 0 \oplus (-J(x_0, y_0, z_0))$. In degree 1 we have $0 = l_1 l_3((a_1 \oplus b_1) \wedge y_0 \wedge z_0) + l_3((0 \oplus b_1) \wedge y_0 \wedge z_0) + \mathcal{J}((0 \oplus b_1) \wedge y_0 \wedge z_0) = l_1 l_3((a_1 \oplus b_1) \wedge y_0 \wedge z_0)$, so we can define $l_3((a_1 \oplus b_1) \wedge y_0 \wedge z_0) = 0$. In degree $k \geq 2$, we have $X_{k+1} = \{0\}$ and since l_3 has degree 1, it follows that $l_3 = 0$ in this case.

For $n = 4$, we prove first $l_2 l_3 = l_3 l_2$, hence $l_4 = 0$. We have to compute only in degree 0.

$$l_2 l_3(x_0 \wedge y_0 \wedge z_0 \wedge t_0) = l_2(l_3(x_0 \wedge y_0 \wedge z_0) \wedge t_0) \pm (3,1) - unshuffles =$$

$$-\sum \left(\pm \left[\mathcal{J}\left(x_0, y_0, z_0\right), t_0\right]_0\right),$$

where the sign is determined by the unshuffles. But it is 0, since it belongs to $M_1 \cap M_2 = \{0\}$. On the other hand,

$$l_3 l_2(x_0 \wedge y_0 \wedge z_0 \wedge t_0) = l_3(l_2(x_0 \wedge y_0) \wedge z_0 \wedge t_0) \pm (2,2) - unshuffles =$$

$$-\sum \left(\pm \mathcal{J}\left([x_0, y_0], z_0, t_0\right)\right),$$

which vanish for the same reason as above.

Counting the degrees for $n > 4$ we have $l_n = 0$.□

Definition 5 We call the sh Lie algebra structure given in Proposition 2 as a *split sh Lie algebra* structure associated with $X_0 = M_1 \oplus M_2$.

4 Generalized pseudoalgebras

Let **k** be a field, $\mathcal{A}$ a commutative **k**-algebra, $\mathcal{D}er\,\mathcal{A}$ be the Lie **k**-algebra of derivations on $\mathcal{A}$ which has the bracket $[\cdot,\cdot]_{\mathcal{D}er\,\mathcal{A}}$ and $\mathcal{V}$ be an $\mathcal{A}$-module that is obvious an **k**-vector space. A $I\!R$-linear map $[\cdot,\cdot]_{\mathcal{V}} : \mathcal{V}\times\mathcal{V}\to\mathcal{V}$ is called a *bracket*, and an $\mathcal{A}$-module map $\rho:\mathcal{V}\to\mathcal{L}$ is called an *anchor* on $\mathcal{V}$. Notice that the bracket is not necessarily antisymmetric.

Definition 6 Let $[\cdot,\cdot]_{\mathcal{D}er\,\mathcal{A}}$ be a bracket on the $\mathcal{A}$-module $\mathcal{V}$, and $\mathcal{M}\subset\mathcal{V}$ be a subset.

1. The subset $\mathcal{M}$ of $\mathcal{V}$ is *closed* if $[X,Y]_{\mathcal{V}}\in\mathcal{M}$, whenever X or $Y\in\mathcal{M}$.
2. The *derived subspace* $\mathcal{D}er_{\mathcal{V}}\mathcal{M}$, is the intersection of all closed $\mathcal{A}$-submodules of $\mathcal{V}$ which contain $\mathcal{M}$.
3. The *Jacobiator* is $\mathcal{J}(X,Y,Z)\overset{not}{=}[[X,Y]_{\mathcal{V}},Z]_{\mathcal{V}}+[[Y,Z]_{\mathcal{V}},X]_{\mathcal{V}}+[[Z,X]_{\mathcal{V}},Y]_{\mathcal{V}}$, $(\forall)X,Y,Z\in\mathcal{V}$.

Notice that the derived space of a subset is a closed $\mathcal{A}$-submodule of $\mathcal{V}$. It consists of the elements in $\mathcal{V}$ which are $\mathcal{A}$-combinations of elements which are obtained from elements from $\mathcal{M}$ by multiplying with elements from $\mathcal{A}$ and applying successively the bracket with elements from $\mathcal{V}$.

Definition 7 Let $\mathcal{L}$ be a Lie **k**-algebra, $\mathcal{V}$ be a **k**-vector space, $[\cdot,\cdot]_{\mathcal{V}}$ be a bracket and ρ be an anchor on $\mathcal{V}$. If the relation

$$[\rho(a),\rho(b)]_{\mathcal{L}}=\rho([a,b]_{\mathcal{V}}),\ (\forall)a,b\in\mathcal{V}$$

holds, then we say that the $\mathcal{A}$-module $\mathcal{V}$ is a *generalized pseudoalgebra* (*g.p.a.*).

Theorem 1 *Let $\mathcal{V}$ be a generalized pseudoalgebra with a linear bracket, let us denote as $\mathcal{M}_{\mathcal{J}}=\{\mathcal{J}(a,b,c):a,b,c\in\mathcal{V}\}$ and suppose that there is a submodule $\mathcal{V}_1\subset\mathcal{V}$ such that $\mathcal{D}er_{\mathcal{V}}\mathcal{M}_{\mathcal{J}}\subset\mathcal{V}_1$ and $\mathcal{V}_1$ has a split $\mathcal{V}_2$ in $\mathcal{V}$: $\mathcal{V}=\mathcal{V}_1\oplus\mathcal{V}_2$. Take $M_1=\mathcal{V}_2$ and $M_2=\mathcal{V}_1$ in (7).*

Then a split sh Lie algebra structure can be defined on $\mathcal{V}$.

If $\xi=(E,\pi,M)$ is a vector bundle, then $\Gamma(\xi)$ and $\mathcal{X}(M)$ are modules over the real algebra $\mathcal{F}(M)$ of real smooth functions and $\mathcal{D}er\,\mathcal{F}(M)=\mathcal{X}(M)$ is a real Lie algebra. A bracket $[\cdot,\cdot]_{\Gamma(\xi)}$ on $\Gamma(\xi)$ and an anchor ρ on $\Gamma(\xi)$ which fulfills the condition

$$[\rho(s),\rho(t)]=\rho\left([s,t]_{\Gamma(\xi)}\right),\ (\forall)s,t\in\Gamma(\xi),$$

define a generalized pseudoalgebra structure on $\Gamma(\xi)$. According to [9], a generalized algebroid structure can be defined on ξ.

Corollary 4 *If $(\xi,\rho,[\cdot,\cdot])$ is a generalized algebroid structure then a split sh Lie algebra structure associated with $\Gamma(\xi)$ can be defined.*

In the case of Courant algebroids the elements in $\mathcal{S}$ have the form $\sum_i f_i \mathcal{D} g_i$ (finite sum with $f_i, g_i \in \mathcal{F}(M)$). Beside the SHLA structure constructed in [11] for a Courant algebroid, we can construct a new one, using a little different resolution. Consider the natural map $1 \otimes \mathcal{D} : \mathcal{F}(M) \otimes_{\mathbb{R}} \mathcal{F}(M) \to \Gamma(\xi)$ given by $(1 \otimes \mathcal{D})(\sum_i f_i \otimes g_i) \overset{def}{=} \sum_i f_i \mathcal{D}(g_i)$ and the following resolution of $\mathcal{H}_1 = coker\ (1 \otimes \mathcal{D})$:

$$\cdots \to 0 \to Y_2 \overset{d_2}{\to} Y_1 \overset{d_1}{\to} Y_0 \to H_1 \to 0 \tag{8}$$

where $Y_0 = \Gamma(E)$, $Y_1 = \mathcal{F}(M)$, $X_2 = \ker \mathcal{D}$, $d_1 = (1 \otimes \mathcal{D})$ and d_2 is the inclusion $\iota : \ker \mathcal{D} \to \mathcal{F}(M)$.

Proposition 3 *Consider a Courant algebroid structure on a vector bundle ξ. Then a SHLA structure can be defined on the total vector space $\mathcal{Y}_*$ of (8) with $l_1 = d$ and the higher structure maps given by the following formulas:*

$l_2 (e_1 \wedge e_2) = [e_1, e_2]$ *in degree* 0;
$l_2 \left(e \wedge \left(\sum_i f_i \otimes g_i \right) \right) = \sum_i f_i \langle e, \mathcal{D} g_i \rangle$ *in degree* 1;
$l_2 = 0$ *in degree* > 1;
$l_3 (e_1 \wedge e_2 \wedge e_3) = -T(e_1, e_2, e_3)$ *in degree* 0;
$l_3 = 0$ *in degree* > 0;
$l_n = 0$ *for* $n > 3$.

The proof is essentially the same as [11, Theorem 4.3]. Notice that in the case of the generalized algebroid structure considered on ξ in Corollary 3, the splitting SHLA given by Corollary 4 is constructed using a different resolution from (8) of $\mathcal{H}_1$.

We give now a different method to construct a SHLA using a bracket. Consider $\mathcal{V} \subset \mathcal{V}'$ a vector subspace and $[\cdot, \cdot] : \mathcal{V}' \times \mathcal{V}' \to \mathcal{V}'$ a bracket on $\mathcal{V}'$ such that the Jacobiator of the bracket take the values in $\mathcal{V}'$ and $\mathcal{V}'$ is closed. Denote as $Z_0 = \mathcal{V}'$, $Z_1 = \mathcal{V}' \otimes \mathcal{V}$, and consider also the map $f : Z_1 \to Z_0$ given by $f(v' \otimes v) = [v', v]$. Denoting as $Z_2 = \ker f$ we can consider the following resolution of *coker* f:

$$\cdots \to 0 \to Z_2 \overset{d_2}{\to} Z_1 \overset{d_1}{\to} Z_0 \to coker\ f \to 0 \tag{9}$$

where $d_1 = f$ and d_2 is the inclusion.

Proposition 4 *Using the above notations, a SHLA structure can be defined on the total vector space $\mathcal{Z}_*$ of (9) with $l_1 = d$ and the higher structure maps given by the following formulas:*

$l_2 (v_1 \wedge v_2) = [v_1, v_2]$ *in degree* 0;
$l_2 ((v \otimes v') \wedge w) = [v, v'] \otimes w$ *in degree* 1;
$l_2 ((v_1 \otimes v_1') \wedge (v_2 \otimes v_2')) = [v_2, v_2'] \otimes [v_1, v_1'] - [v_1, v_1'] \otimes [v_2, v_2']$ *and in rest* 0 *in degree* 2;
$l_2 = 0$ *in degree* > 2;
$l_n = 0$ *for* $n \geq 3$.

The importance of this construction for Courant algebroids comes from the following result:

Proposition 5 *Consider a Courant algebroid structure on* ξ*, take* $\mathcal{V} = \mathcal{S}$ *(the sections which have the form* $\sum_i f_i \mathcal{D} g_i$ *with a finite sum with* $f_i, g_i \in \mathcal{F}(M)$*) and* $\mathcal{V}' = Im\mathcal{D}$*. Then* $f : \mathcal{V} \otimes \mathcal{V}' \to \mathcal{V}'$*,* $f(v \otimes v') = [v, v']$ *is well defined and is locally exact.*

It leads us to the supposition that for a large class of Courant algebroids the vector space *coker* f in resolution (9) is finite dimensional.

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