NON-LINEAR d-CONNECTIONS

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Abstract

The aim of this paper is to define the non-linear d-connections and to study some Lagrangeans associated with a non-linear connection. In the case of a Finsler space a non-linear d-connection is obtained. When the metric is a Riemannian one, the case of the metric linear d-connection considered by R. Miron is recovered.

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1 The relative fundamental forms of two supplementary vector subbundles

Let $\xi' = (E', \pi_1, M)$ and $\xi'' = (E'', \pi_2, M)$ be two supplementary vector subbundles of the vector bundle $\xi = (E, \pi, M)$, $P' : \xi \to \xi'$ and $P'' : \xi \to \xi''$ be the canonical projections and $I' : \xi' \to \xi$ and $I'' : \xi'' \to \xi$ be the canonical inclusions. Let as also denote the kernel vector bundles by $\mathcal{V}'\xi = \ker P''_*$ and $\mathcal{V}''\xi = \ker P'_*$. There is a reduction as Whitney sum $V\xi = \mathcal{V}'\xi \oplus \mathcal{V}''\xi$ and denote by $Q' : V\xi \to \mathcal{V}'\xi$ and $Q'' : V\xi \to \mathcal{V}''\xi$ the canonical projection.

Definition 1.1 We say that $I'^*\mathcal{V}''\xi$ is the *normal ξ''-bundle* of the vector bundle ξ', and the morphism $\tau E' \overset{T'}{\to} I'^*\mathcal{V}''\xi$ obtained as the composition of the following sequence of vector bundle morphisms

$$\tau E' \xrightarrow{I'_*} I'^*\tau E \xrightarrow{I'^*C} I'^*V\xi \xrightarrow{I'^*Q''} I'^*\mathcal{V}''\xi, \tag{1}$$

i.e. $T' = I'^*Q'' \circ I'^*C \circ I'_*$, is the *fundamental ξ''-form* of ξ'.

In an analogous way one define $I''^*\mathcal{V}'\xi$ as the *normal ξ'-bundle* of ξ'' and $T'' = I''^*Q' \circ I''^*C \circ I''_*$ as the *fundamental ξ'-form* of ξ''.

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Consider now vectorial coordinates on E which are adapted (i.e., local coordinates on E adapted to the vector bundle structure) which induce also on E' and E'' adapted vectorial coordinates. More precisely, around every $y \in E$, $\pi(y) = x \in M$, $P'(y) = y'$, $P''(y) = y''$ we have as adapted coordinates: $x : (x^i)$ on M, $y' : (x^i, y^\alpha)$ on E', $y'' : (x^i, y^u)$ on E'', $y : (x^i, y^\alpha, y^u)$ on E, where $i = \overline{1,m}$, $\alpha = \overline{1,k_1}$, $u = \overline{1,k_2}$ and $k = k_1 + k_2$. The change rules are: $x^{i'} = x^{i'}(x^i)$, $y^{\alpha'} = h^{\alpha'}_\alpha(x^i)y^\alpha$, $y^{u'} = h^{u'}_u(x^i)y^u$.

Using local coordinates as above, the local form of T', the fundamental ξ''-form of ξ', is obtained performing the following compositions:

$$((x^i, y^\alpha), (X^j, Y^\beta)) \stackrel{I'_*}{\longrightarrow} ((x^i, y^\alpha, 0), (X^j, Y^\beta, 0)) \stackrel{I'^*C}{\rightarrow}$$

$$((x^i, y^\alpha), (Y^\beta + X^j N^\beta_j(x^i, y^\alpha, 0), X^j N^{\bar\beta}_j(x^i, y^\alpha, 0))) \stackrel{I'^*Q''}{\longrightarrow}$$

$$((x^i, y^\alpha), X^j N^{\bar\beta}_j(x^i, y^\alpha, 0)),$$

thus T' has the local form:

$$((x^i, y^\alpha), (X^j, Y^\beta)) \stackrel{T'}{\longrightarrow} ((x^i, y^\alpha), (N^{\bar\alpha}_j(x^i, y^\alpha, 0)X^j)). \tag{2}$$

In the sequel we use the following result proved in [3]:

Proposition 1.1 *Let C' and C'' be two non-linear connections on the supplementary vector subbundles ξ' and ξ'' of the vector bundle ξ. If P' and P'' are considered as above, then there is a unique non-linear connection C on the vector bundle ξ such that $P'_*(H\xi) = H\xi'$ and $P''_*(H\xi) = H\xi''$, where $H\xi$, $H\xi'$ and $H\xi''$, the horizontal bundles of the non-linear connections C, C' and C'' respectively.*

If the connections C' and C'' are linear connections, then the connection C is linear, too.

For a vector bundle ξ denote as $\Gamma(\xi)$ its $\mathcal{F}(M)$-module of sections. The following proposition gives an interpretation for vanishing fundamental ξ'-form of ξ''.

Proposition 1.2 *Let C be a non-linear connection on the vector bundle ξ and ξ', ξ'' be two supplementary vector subbundles of ξ.*

Then T', the fundamental ξ''-form of ξ', vanishes iff the covariant derivation associated with the connection C sends the $\Gamma(\xi')$ in $\Gamma(\xi')$.

Proof. The formula (2) shows that T', the fundamental ξ''-form of ξ' vanishes if and only if the local functions $(x^i, y^\alpha) \to N^{\bar\alpha}_j(x^i, y^\alpha, 0)$ vanish.

Consider $X = X^i \dfrac{\partial}{\partial x^i} \in \mathcal{X}(U)$ and $A = A^\alpha s_\alpha + A^{\bar\alpha} s_{\bar\alpha} \in \Gamma(\xi_{|U})$ on an open domain $U \subset M$ of some local adapted coordinates . The local form of the covariant derivative of the connection C, is:

$$\nabla_X A = X^i \left(\frac{\partial A^\alpha}{\partial x^i} + N^\alpha_i(x^i, A^\alpha, A^{\bar\alpha}) s_\alpha + N^{\bar\alpha}_i(x^i, A^\alpha, A^{\bar\alpha}) s_{\bar\alpha} \right).$$

When $A' = A^\alpha s_\alpha \in \Gamma(\xi')$ then

$$\nabla_X A' = X^i \left(\frac{\partial A^\alpha}{\partial x^i} + N_i^\alpha(x^i, A^\alpha, 0) s_\alpha + N_i^{\bar{\alpha}}(x^i, A^\alpha, 0) s_{\bar{\alpha}} \right),$$

thus $\nabla_X A' \in \Gamma(\xi')$ iff $N_i^{\bar{\alpha}}(x^i, A^\alpha, 0) = 0$. It holds for every section $A' \in \Gamma(\xi')$ iff the local functions $(x^i, y^\alpha) \to N_j^{\bar{\alpha}}(x^i, y^\alpha, 0)$ vanish, thus the conclusion of the Proposition follows.□

Corollary 1.1 *Let C be a non-linear connection on the vector bundle ξ and ξ' and ξ'' be two supplementary vector subbundles of ξ.*

Then T', the fundamental ξ''-form of ξ', vanishes iff the following relation is fulfilled:

$$P''(\nabla_X(P'A)) = 0, (\forall) X \in \mathcal{X}(M), A \in \Gamma(\xi).$$

Corollary 1.2 *Let ξ be a vector bundle, C be a non-linear connection on ξ, ∇ be the covariant derivative of C, ξ' and ξ'' be two supplementary vector subbundles of ξ, $P' : \xi \to \xi'$ and $P'' : \xi \to \xi''$ be the canonical projections and T', T'' be the fundamental forms.*

Then the following statements are equivalent:

1. *The fundamental forms T' and T'' vanish;*
2. $P'(\nabla_X(P''A)) = 0$, $P''(\nabla_X(P'A)) = 0$, $(\forall) X \in \mathcal{X}(M)$, $A \in \Gamma(\xi)$.

2 Non-linear d-connections on fibred manifolds

The above construction is applied in this section in order to define a non-linear d-connection on a fibred manifold.

Definition 2.1 Let $\xi = (E, \pi, M)$ be a *fibred manifold*, i.e. $\pi : E \to M$ is a surjective submersion.

The *vertical bundle* of ξ is the kernel bundle $V\xi = \ker \pi_*$, thus $V\xi$ is a vector subbundle of the tangent bundle τE of E.

A *connection* C on ξ is a left splitting of the canonical inclusion $V\xi \xrightarrow{i} \tau E$, i.e. it is a vector bundle morphism $C : \tau E \to V\xi$ so that $C \circ i = 1_{V\xi}$.

A *non-linear F-connection* on ξ is a pair (C, N), where C is a connection on ξ and N is a non-linear connection on the manifold E.

A *non-linear d-connection* on ξ is an F-connection (C, N), where C is a connection on ξ and N is a non-linear connection on E, such that the vertical distribution $V\xi$ of ξ and the horizontal distribution $H\xi$ of the connection C are parallel with respect to the covariant derivative of the non-linear connection N.

This definition extends the notion of linear d-connection (as in [1, 2]). We list below some properties of linear d-connections which are still valid in the case of non-linear d-connections.

Proposition 2.1 *A non-linear F-connection (C, N) on ξ is a non-linear d-connection iff the property*

$$vD_X hY = 0, hD_X vY = 0, (\forall)X, Y \in \mathcal{X}(E)$$

holds, where D is the covariant derivative associated with the non-linear connection N.

Proof. It follows from Corollary 1.2.□

Proposition 2.2 *Let ξ be a fibred manifold.*

Every non-linear d-connection (C, N) on ξ defines in a canonical way a non-linear connections on the supplementary vector bundles $V\xi$ and $H\xi$.

A connection C on ξ and two non-linear connections on the supplementary vector subbundles $V\xi$ and $H\xi$, which correspond to the connection C, define in a canonical way a non-linear F-connection on ξ.

Proof. The first statement follows using Proposition 1.2 . The second statement follows using Proposition 1.2 and Proposition 1.1.□

The horizontal bundle $H\xi$ of a connection on the fibred manifold $\xi = (E, \pi, M)$ is isomorphic in a canonical way with the induced vector bundle $\pi^*\tau M$. It follows that the result stated in the above Proposition 2.2 can be also stated as follows:

Proposition 2.3 *Let ξ be a fibred manifold.*

Every non-linear d-connection (C, N) on ξ induces non-linear connections on the vector bundles $V\xi$ and $\pi^*\tau M$.

Conversely, a connection C on ξ and two non-linear connections on the vector bundles $V\xi$ and $\pi^*\tau M$ define in a canonical way a non-linear F-connection on ξ.

This property of non-linear d-connections is used in [3] in order to induce non-linear d-connections on submanifolds.

We consider now adapted coordinates on M, E and TE:

$(x^i), i = \overline{1, m}$ on M;

(x^i, y^α) on E, $(x^i, y^\alpha, X^j, Y^\beta)$ on TE, $\alpha, \beta = \overline{1, n}$, $i, j = \overline{1, m}$.

Using such adapted coordinates, a connection C on ξ defines in a canonical way local bases of the fields on E, which we call *C-adapted local bases.* The local form of the C-adapted local bases is:

$$\left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^\alpha(x^j, y^\beta)\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\alpha} \right\}_{i=\overline{1,m},\ a=\overline{1,n}} .$$

It is easy to see that $\left\{ \frac{\delta}{\delta x^i} \right\}_{i=\overline{1,m}}$ is a local bases for the horizontal sections and $\left\{ \frac{\partial}{\partial y^\alpha} \right\}_{a=\overline{1,n}}$ is a local bases for the vertical sections. The change formulas of the local fields $\dfrac{\delta}{\delta x^i}$ and $\dfrac{\partial}{\partial y^\alpha}$ are:

$$\frac{\delta}{\delta x^i} = \frac{\partial x^{i'}}{\partial x^i}(x^j)\frac{\delta}{\delta x^{i'}}, \quad \frac{\partial}{\partial y^\alpha} = \frac{\partial y^{\alpha'}}{\partial y^\alpha}(x^j, y^\beta)\frac{\partial}{\partial y^{\alpha'}} .$$

We define now the *special components* of a non-linear d-connection (C, N).

Using Proposition 2.2 the non-linear connections N^v and N^h are induced on the vector bundles $V\xi$ and $H\xi$ respectively. We denote as $\{L^i_k, C^i_\gamma\}_{\substack{i,k=\overline{1,m}\\ \gamma=\overline{1,n}}}$ and $\{L^\alpha_i, C^\alpha_\gamma\}_{\substack{i=\overline{1,m}\\ \alpha,\gamma=\overline{1,n}}}$ the components of the non-linear connections N^h and N^v respectively, considered using C-adapted bases. We say that the local functions

$$\left\{L^i_k, L^\alpha_j, C^i_\gamma, C^\alpha_\gamma\right\}_{\substack{i,j=\overline{1,m}\\ \alpha,\gamma=\overline{1,n}}}$$

are the *special components* of the non-linear d-connection (C, N). Notice that they are not local functions on TE; $\{L^i_k = L^i_k(x^i, y^\alpha, X^j), C^i_\gamma = C^i_\gamma(x^i, y^\alpha, X^j)\}$ are local functions on HE, and $\{L^\alpha_i = L^\alpha_i(x^i, y^\alpha, Y^\beta), C^\alpha_\gamma = C^\alpha_\gamma(x^i, y^\alpha, Y^\beta)\}$are local functions on VE.

The change rules of the special components, using C-adapted bases, are

$$L^{i'}_{k'}\frac{\partial x^{k'}}{\partial x^k} + C^{i'}_{\gamma'}\frac{\partial y^{\gamma'}}{\partial x^k} = L^i_k\frac{\partial x^{i'}}{\partial x^i} - \frac{\partial^2 x^{i'}}{\partial x^j\partial x^k}X^j, \tag{3}$$

$$L^{\alpha'}_{i'}\frac{\partial x^{i'}}{\partial x^k} + C^{\alpha'}_{\gamma'}\frac{\partial y^{\gamma'}}{\partial x^k} = L^\alpha_k\frac{\partial y^{\alpha'}}{\partial y^\alpha} - \frac{\partial^2 y^{\alpha'}}{\partial x^j\partial y^\gamma}Y^\gamma, \tag{4}$$

$$C^{i'}_{\gamma'}\frac{\partial y^{\gamma'}}{\partial y^\gamma} = C^i_\gamma\frac{\partial x^{i'}}{\partial x^i}, \tag{5}$$

$$C^{\alpha'}_{\gamma'}\frac{\partial y^{\gamma'}}{\partial y^\gamma} = C^\alpha_\gamma\frac{\partial y^{\alpha'}}{\partial y^\alpha} - \frac{\partial^2 y^{\alpha'}}{\partial y^\beta\partial y^\gamma}Y^\beta, \tag{6}$$

where the local functions $L^{i'}_{k'}$ and $C^{i'}_{k'}$ have as variables $(x^{i'}, y^{a'}, X^{j'})$, the local functions $L^{\alpha'}_{\gamma'}$ and $C^{\alpha'}_{\gamma'}$ have as variables $(x^{i'}, y^{a'}, Y^{\beta'})$, the local functions L^i_k and C^i_α have as variables (x^i, y^a, X^j), and the local functions L^α_i and C^α_γ have as variables (x^i, y^a, Y^β).

Notice that for an arbitrary fibred manifold the local functions $\dfrac{\partial y^{\alpha'}}{\partial y^\alpha}$ have as variables (x^j, y^β). But in the particular case of a vector bundle the local functions $y^{\alpha'}$ are 1-homogenous in y^α, thus the local functions $\dfrac{\partial y^{\alpha'}}{\partial y^\alpha}$ have as variables only (x^i).

2.1 The fibred manifolds with a vertical induced bundle

Definition 2.2 [5]A *fibred manifold with a vertical induced bundle* is a fibred manifold $\xi = (E, \pi, M)$ which enjoys the property that there is a vector bundle $\eta = (F, q, M)$ and an isomorphism $V\xi \cong \pi^*\eta$.

In the case of a fibred manifold with a vertical induced bundle we consider the vertical sections $\{s_\alpha\}$ induced by the isomorphism $V\xi \cong \pi^*\eta$, instead of C-adapted local bases of vertical sections. The formulas which give the change rule of these local adapted bases are:

$$\frac{\delta}{\delta x^i} = \frac{\partial x^{i'}}{\partial x^i}(x^j)\frac{\delta}{\delta x^{i'}}, \quad s_\alpha = g_\alpha^{\alpha'}(x^i)s_{\alpha'} \, .$$

It follows that the formulas which give the change rule of the special components of a non-linear d-connection on a fibred manifold with a vertical induced bundle, using local adapted bases, are:

$$L_{k'}^{i'}\frac{\partial x^{k'}}{\partial x^k} + C_{\gamma'}^{i'}\frac{\partial y^{\gamma'}}{\partial x^k} = L_k^i\frac{\partial x^{i'}}{\partial x^i} - \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k}X^j,$$

$$L_{i'}^{\alpha'}\frac{\partial x^{i'}}{\partial x^k} + C_{\gamma'}^{\alpha'}\frac{\partial y^{\gamma'}}{\partial x^k} = L_k^\alpha g_\alpha^{\alpha'} - \frac{\partial g_\gamma^{\alpha'}}{\partial x^j}Y^\gamma,$$

$$C_{\gamma'\gamma}^{i'g\gamma'} = C_\gamma^i\frac{\partial x^{i'}}{\partial x^i},$$

$$C_{\gamma'}^{\alpha'}g_\gamma^{\gamma'} = C_\gamma^\alpha g_\alpha^{\alpha'}.$$

They are simply obtained by replacing $\dfrac{\partial y^{\alpha'}}{\partial y^\alpha}(x^i, y^\alpha) = g_\alpha^{\alpha'}(x^i)$.

Remarks

1) In this case the components C_γ^i and C_γ^α have a tensor change rule.

2) If one gives two non-linear connections on the vertical and horizontal bundle respectively, the F-connection given by the second part of Proposition 2.2 has as local components exactly the special components. This F-connection is a non-linear d-connection iff the fundamental forms vanish i.e. the following relations hold true in this case:

$$L_k^i(x^i, y^a, 0) = 0,\ C_\alpha^i(x^i, y^a, 0) = 0,\ L_i^\alpha(x^i, y^a, 0) = 0,\ C_\gamma^\alpha(x^i, y^a, 0) = 0. \qquad (7)$$

3 Non-linear d-connections of Finsler type

If $\xi = (E, \pi, M)$ is a fibred manifold and C is a connection on ξ, then there is a natural decomposition $\tau E = V\xi \oplus H\xi$. As in the first section, this decomposition induces a decomposition $V\tau E = \mathcal{V}\xi \oplus \mathcal{H}\xi$, where $\mathcal{V}\xi = \mathcal{V}'\tau E$ and $\mathcal{H}\xi = \mathcal{V}''\xi$

Definition 3.1 If $\xi = (E, \pi, M)$ is a fibred manifold, C is a connection on ξ and $\mathcal{L}$ is a Lagrangian on the manifold E, then we say that $\mathcal{L}$ is *adapted* to the connection C if the vector subbundles $\mathcal{V}\xi$ and $\mathcal{H}\xi$ are orthogonal with respect to the metric defined by $\mathcal{L}$.

In the sequel we suppose that $\xi = (E, \pi, M)$ is a vector bundle.

Consider now local coordinates on M, E and TE, adapted to the vector bundles structures on E and TE and some new local coordinates on TE, which are adapted to the decomposition $\tau E = H\xi \oplus V\xi$. We denote this new set of coordinates as $(\bar{x}^i, \bar{y}^\alpha, \bar{X}^j, \bar{Y}^\beta) = (x^i, y^\alpha, X^j, \bar{Y}^\beta)$, where $\bar{x}^i = x^i$, $\bar{y}^\alpha = y^a$, $\bar{X}^j = X^j$, $\bar{Y}^\beta = Y^\beta - N_i^\beta(\bar{x}^j, \bar{y}^\gamma)$. They have the change rule:

$$x^{i'} = x^{i'}(x^i), y^{a'} = g_a^{a'}(x^i)y^a, X^{i'} = \frac{\partial x^{i'}}{\partial x^i}X^i, \bar{Y}^{b'} = g_b^{b'}(x^i)\bar{Y}^b.$$

The local bases of vector fields on TE which corresponds to these coordinates have the form $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial X^j}, \frac{\partial}{\partial \bar{Y}^b}\right\}$. They change according to the formulas:

$$\frac{\partial}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i}(x^j)\frac{\partial}{\partial x^{i'}} + \frac{\partial g_a^{a'}}{\partial x^i}y^a\frac{\partial}{\partial y^{a'}} + \frac{\partial^2 x^{j'}}{\partial x^i \partial x^j}X^j\frac{\partial}{\partial X^{j'}} + \frac{\partial g_b^{b'}}{\partial x^i}\bar{Y}^b\frac{\partial}{\partial \bar{Y}^{b'}},$$

$$\frac{\partial}{\partial y^a} = g_a^{a'}\frac{\partial}{\partial y^{a'}}, \ \frac{\partial}{\partial X^j} = \frac{\partial x^{j'}}{\partial x^j}\frac{\partial}{\partial X^{j'}}, \ \frac{\partial}{\partial \bar{Y}^b} = g_b^{b'}\frac{\partial}{\partial \bar{Y}^{b'}}.$$

Notice that the local vector fields $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^\alpha(x^j, y^\beta)$, considered on TE, change according to the formulas:

$$\frac{\delta}{\delta x^i} = \frac{\partial x^{i'}}{\partial x^i}(x^j)\frac{\delta}{\delta x^{i'}} + \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}X^j\frac{\partial}{\partial x^{i'}} + \frac{\partial g_a^{a'}}{\partial x^i}y^a\frac{\partial}{\partial y^{a'}}.$$

The local coordinates on HE are $\{x^i, y^a, X^j\}$ and the local coordinates on VE are $\{x^i, y^a, Y^b\}$.

A straightforward computation leads to the following result:

Theorem 3.1 *Let N be a non-linear connection on the vector bundle $\xi = (E, \pi, M)$ and $\mathcal{L}$ a Lagrangian on E adapted to the connection N. We denote by $\mathcal{L}^1$ the restriction of $\mathcal{L}$ to $H\xi$ (denoting the local components of the metric tensor $g_{ij} = \frac{\partial^2 \mathcal{L}^1}{\partial X^i \partial X^j}$) and by $\mathcal{L}^2$ the restriction of $\mathcal{L}$ to $V\xi$ (denoting the local components of the metric tensor $h_{ab} = \frac{\partial^2 \mathcal{L}^2}{\partial Y^b \partial Y^b}$). Then:*

1. *The local functions on HE:*

$$L_k^i(x^i, y^a, X^j) = \frac{1}{4}\frac{\partial}{\partial X^k}\left(g^{ij}\left(\frac{\delta\partial\mathcal{L}^1}{\delta x^p \partial X^j}X^p - \frac{\delta\mathcal{L}^1}{\delta x^j}\right)\right), \ L_b^i(x^i, y^a, X^j) = \frac{1}{4}g^{ik}\frac{\partial^2 \mathcal{L}^1}{\partial y^b \partial x^k}$$

are the local components of a non-linear connection Γ^h *on the vector bundle* $H\xi$, *and the local functions on* VE:

$$L_k^a(x^i, y^a, X^j) = \frac{1}{2}h^{ac}\left(\frac{\delta\partial\mathcal{L}^2}{\delta x^i \partial Y^c} - \frac{\partial N_i^b}{\partial y^c}\frac{\delta\mathcal{L}^2}{\partial Y^b}\right) + \frac{1}{2}\frac{\partial\mathcal{L}^2}{\partial Y^b}$$

$$L_b^a(x^i, y^a, X^j) = \frac{1}{4}\frac{\partial}{\partial Y^b}\left(h^{ac}\left(\frac{\partial^2\mathcal{L}^2}{\partial y^d \partial Y^c}Y^d - \frac{\partial\mathcal{L}^2}{\partial y^c}\right)\right)$$

are the local components of a non-linear connection Γ^V *on the vector bundle* $H\xi$.

2. *The non-linear connections* Γ^h *and* Γ^V *give together a non-linear connection* Γ *on the manifold* TE, *thus* (N, Γ) *is an F-connection on* ξ.

3. *If* $\mathcal{L}$ *define a Finsler metric, then* (N, Γ) *is a non-linear d-connection.*

In the case when $\mathcal{L}$ define a Riemannian metric on E, then using the local form of the metric as $g_{ij}(x^i, y^a)dx^i \otimes dx^j + h_{ab}(x^i, y^a)\delta y^a \otimes \delta y^b$, we obtain the metric linear d-connection defined by R.Miron and M.Anastasiei in [1, Corolarul 1.2, pag. 96].

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