EINSTEIN EQUATIONS FOR A GENERALIZED LAGRANGE SPACE OF ORDER 2 IN INVARIANT FRAMES

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Abstract

The study of higher order Lagrange spaces founded on the notion of bundle of velocities of order k has been given by Radu Miron and Gheorghe Atanasiu in [2].

The bundle of accelerations correspond in this study to k=2.

The notion of invariant geometry of order 2 was introduced by the author in [4].

In this paper we shall give the Maxwell equations of a generalized Lagrange space of order 2 in invariant frames.

AMS Subject Classification: 53C05.
Key words: 2-osculator bundle, invariant frames, Maxwell equations

1. General Invariant Frames

Let us consider the bundle $E = Osc^2M$, a nonlinear connection N with the coefficients $\left(\underset{(1)}{N^i{}_j} \,,\, \underset{(2)}{N^i{}_j} \right)$ and the duals $\left(\underset{(1)}{M^i{}_j} \,,\, \underset{(2)}{M^i{}_j} \right)$.

The invariant frames adapted to the direct decomposition

$$T_u(\, Osc^2M \,) \;=\; N_0(u) \oplus N_1(u) \oplus V_2(u) \quad \forall u \in E \tag{1}$$

will be $\Re \;=\; (\, e^{(0)i}{}_{\alpha} \,,\, e^{(1)i}{}_{\alpha} \,,\, e^{(2)i}{}_{\alpha} \,)$ and the dual $\Re^* \;=\; (\, f^{(0)\alpha}{}_{i} \,,\, f^{(1)\alpha}{}_{i} \,,\, f^{(2)\alpha}{}_{i} \,)$.

The duality conditions are:

$$<\; e^{(A)i}{}_{\alpha} \,,\, f^{(B)\alpha}{}_{j} \;> \;=\; \delta^i_j \, \delta^A_B \,, \quad (\, A, B = 0, 1, 2\,).$$

Editor Gr.Tsagas *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1998,* 76-82

In this frame the adapted basis has the representation:

$$\frac{\delta}{\delta x^i} = f^{(0)\alpha}_{\ \ i} \frac{\delta}{\delta s^{(0)\alpha}} \quad \frac{\delta}{\delta y^{(1)i}} = f^{(1)\alpha}_{\ \ i} \frac{\delta}{\delta s^{(1)\alpha}} \quad \frac{\delta}{\delta y^{(2)i}} = f^{(2)\alpha}_{\ \ i} \frac{\delta}{\delta s^{(2)\alpha}}$$

and the cobasis

$$\delta x^i = e^{(0)i}_{\ \ \alpha} \delta s^{(0)\alpha} \quad ; \delta y^{(1)i} = e^{(1)i}_{\ \ \alpha} \delta s^{(1)\alpha} \quad ; \delta y^{(2)i} = e^{(2)i}_{\ \ \alpha} \delta s^{(2)\alpha} \tag{2}$$

and we have the relations:

$$\left\langle \frac{\delta}{\delta s^{(A)\alpha}} , \delta s^{(B)\beta} \right\rangle = \delta^\beta_\alpha \, \delta^B_A \, , \qquad (A, B = 0, 1, 2).$$

This representation lead us to an invariant frames transformation group with the analitycal expressions

$$\overline{e}^{(A)i}_{\ \ \alpha} = \overset{A}{C}{}^\beta_\alpha \, (x, y^{(1)}, y^{(2)}), e^{(A)i}_{\ \ \beta} \quad ; \quad f^{(B)\alpha}_{\ \ j} = \overline{\overset{B}{C}}{}^\alpha_\beta \, \overline{f}^{(B)\beta}_{\ \ j} \, ,$$

isomorphic with the multiplicative nonsingular matrix group

$$\begin{pmatrix} \overset{0}{C}{}^\alpha_\beta & 0 & 0 \\ 0 & \overset{1}{C}{}^\alpha_\beta & 0 \\ 0 & 0 & \overset{2}{C}{}^\alpha_\beta \end{pmatrix} .$$

A N-linear connection D has in the frame $\Re$ the coefficients:

$$\overset{0A}{L}{}^\gamma_{\beta\alpha} = f^{(A)\gamma}_{\ \ m} \left(\frac{\delta e^{(A)m}_{\ \ \beta}}{\delta s^{(0)\alpha}} + e^{(0)i}_{\ \ \alpha} e^{(A)j}_{\ \ \beta} L^m_{ij} \right), \quad (A = 0, 1, 2),$$

$$\overset{BA}{C}{}^\gamma_{\beta\alpha} = f^{(A)\gamma}_{\ \ m} \left(\frac{\delta e^{(A)m}_{\ \ \beta}}{\delta s^{(B)\alpha}} + e^{(B)i}_{\ \ \alpha} e^{(A)j}_{\ \ \beta} \underset{(B)}{C^m_{ij}} \right), \quad (A = 0, 1, 2 \, ; \, B = 1, 2).$$

Definition 1. *If the vector field $X \in \mathcal{X}(E)$ has the invariant components $X^{(A)\alpha}$ $(A = 0, 1, 2)$ and we denote by $'\rangle'$ and $'\overset{B}{)}'$ the $h-$ and the v_B, $B = 1, 2$, covariant invariant derivative operators then*

$$\begin{aligned} X^{(A)\alpha}{}_{)\beta} &= \frac{\delta X^{(A)\alpha}}{\delta s^{(0)\beta}} + \overset{0A}{L}{}^\alpha_{\varphi\beta} \, X^{(A)\varphi}, \\ X^{(A)\alpha} \overset{(B)}{)}_\beta &= \frac{\delta X^{(A)\alpha}}{\delta s^{(B)\beta}} + \overset{BA}{C}{}^\alpha_{\varphi\beta} \, X^{(A)\varphi}. \end{aligned} \tag{3}$$

The definition of the Lie bracket conduces us to the introduction of the non-holonomy coefficients of Vranceanu:

$$\left[\frac{\delta}{\delta s^{(A)\alpha}}\, ,\, \frac{\delta}{\delta s^{((B)\beta}}\right] = \underset{(AB)}{\overset{0}{W}}{}^{\gamma}_{\alpha\beta}\, \frac{\delta}{\delta s^{(0)\gamma}} + \underset{(AB)}{\overset{1}{W}}{}^{\gamma}_{\alpha\beta}\, \frac{\delta}{\delta s^{(1)\gamma}} + \underset{(AB)}{\overset{2}{W}}{}^{\gamma}_{\alpha\beta}\, \frac{\delta}{\delta s^{(2)\gamma}},$$

$(A, B = 0, 1, 2, \quad A \leq B)$.

2. Einstein Equations

Let us consider a metric tensor G on Osc^2M, the invariant frames $\overline{\Re}$ and $\overline{\Re^*}$ so that the quadratic form associated to the metric has the canonical representation

$$\omega^{(A)} = (\omega^{(A)1})^2 + .. + (\omega^{(A)i_A})^2 - (\omega^{(A)i_{A+1}})^2 + - .. - (\omega^{(A)n})^2,$$

where:

$$\omega^{(A)} = g^{(A)}_{\ ij}\delta y^{(A)i}\delta y^{(A)j}.$$

We introduce the Vranceanu's symbols:

$$\varepsilon^{(A)}_{\ \alpha\beta} = \begin{cases} \delta_{\alpha\beta} & \alpha \leq i_A\ ,\ \beta \leq i_A, \\ -\delta_{\alpha\beta} & \alpha > i_A\ ,\ \beta > i_A, \\ 0 & \text{in rest.} \end{cases}$$

Theorem 2.1. *The Vranceanu's symbols* $\varepsilon^{(A)}_{\ \alpha\beta}$ *represents the invariant componenets of the tensors* $g^{(A)}_{\ ij}$ *in the frame* $\overline{\Re}$.

Proposition 2.1. *The frame* $\overline{\Re}$ *defined above is pseudoorthogonal.*

Proposition 2.2. *The invariant components* $\varepsilon^{(A)}_{\ \alpha\beta}$ *of the metric tensor* g *of the total space* E *satisfy the relations:*

$$\varepsilon^{(A)}_{\ \ \alpha\beta)\gamma} = 0 \quad , \quad \varepsilon^{(A)}_{\alpha\beta}\overset{(B)}{)}_{\gamma} = 0, \qquad \textit{(A=0,1,2; B=1,2).}$$

Proposition 2.3. *The invariant components of the canonical metrical N-linear connection* $C\Gamma(N)$ *satisfy the relations:*

$$\overset{0A}{L}{}^{\varphi}_{\alpha\gamma}\,\varepsilon^{(A)}_{\varphi\beta} + \overset{0A}{L}{}^{\varphi}_{\beta\gamma}\,\varepsilon^{(A)}_{\alpha\varphi} = 0,$$

$$\overset{BA}{C}{}^{\varphi}_{\alpha\gamma}\,\varepsilon^{(A)}_{\varphi\beta} + \overset{BA}{C}{}^{\varphi}_{\beta\gamma}\,\varepsilon^{(A)}_{\alpha\varphi} = 0.$$

The calculus of the Ricci's tensor and the scalar curvature permit us to formulate the following result:

Theorem 2.2. *The Einstein equations have the following invariant expressions in the frame* $\overline{\Re}$ *:*

$$R_{\alpha\beta} - \frac{1}{2}\varepsilon^{(0)}{}_{\alpha\beta}R = \kappa \overset{(0)}{T}_{\alpha\beta} ,$$

$$\underset{(1)}{S_{\alpha\beta}} - \frac{1}{2}\varepsilon^{(1)}{}_{\alpha\beta}R = \kappa \overset{(1)}{T}_{\alpha\beta} , \quad \underset{(2)}{S_{\alpha\beta}} - \frac{1}{2}\varepsilon^{(2)}{}_{\alpha\beta}R = \kappa \overset{(2)}{T}_{\alpha\beta} ,$$

$$\overset{(B)}{\underset{(A)}{P_{\alpha\beta}}} = (-1)^{B+1}\kappa \overset{(0B)}{\underset{(A)}{T_{\alpha\beta}}} , \quad \overset{(B)}{\underset{(12)}{P_{\alpha\beta}}} = (-1)^{B+1}\kappa \overset{(12)}{\underset{(B)}{T_{\alpha\beta}}} , \ (A,B=1,2),$$

where $T_{\alpha\beta}$ *and* $\underset{(A)}{T_{\alpha\beta}}$ *are d-tensor fields and represent the invariant components of the energy-impulse tensor.*

Theorem 2.3. *Some of the equations which give the conservation law with respect to the canonical metrical N-linear connection* $C\Gamma(N)$ *are given by:*

$$\left(R^{\alpha}_{\beta} - \frac{1}{2}R\delta^{\alpha}_{\beta}\right)_{)\alpha} + \underset{(1)}{P^{\alpha}_{\beta}}\overset{(1)}{{}_{)\alpha}} + \underset{(21)}{P^{\alpha}_{\beta}}\overset{(2)}{{}_{)\alpha}} = 0,$$

$$\left(\underset{(1)}{S^{\alpha}_{\beta}} - \frac{1}{2}R\delta^{\alpha}_{\beta}\right)\overset{(1)}{{}_{)\alpha}} - \overset{(2)}{\underset{(1)}{P^{\alpha}_{\beta)\alpha}}} + \overset{(1)}{\underset{(12)}{P^{\alpha}_{\beta}}}\overset{(2)}{{}_{)\alpha}} = 0,$$

$$\left(\underset{(2)}{S^{\alpha}_{\beta}} - \frac{1}{2}R\delta^{\alpha}_{\beta}\right)\overset{(2)}{{}_{)\alpha}} - \overset{(1)}{\underset{(2)}{P^{\alpha}_{\beta)\alpha}}} + \overset{(2)}{\underset{(12)}{P^{\alpha}_{\beta}}}\overset{(1)}{{}_{)\alpha}} = 0,$$

where $R^{\alpha}{}_{\beta} = \varepsilon^{(0)\alpha\gamma}R_{\gamma\beta}$.

3. An example of computation for Einstein equations

In this section we compute the Einstein equations in the particular case of the generalized Lagrange space $GL^{(2)n}$ ($M,\ g_{ij}(x, y^{(1)}, y^{(2)}) = e^{2\sigma(x,y^{(1)},y^{(2)})}\gamma_{ij}(x)$

In some special cases it is not necessary to choose the frames where the quadratic forms ω have canonical shape but in this cases we must consider the restrictions $e^{(0)\imath}{}_{\alpha} = e^{(1)\imath}{}_{\alpha} = e^{(2)\imath}{}_{\alpha} = e^{i}_{\alpha}$. For the considerated generalized Lagrange space let us take the canonical nonlinear connection N with the coefficients:

$$\underset{(1)}{N^{i}{}_{j}} = \gamma^{\imath}_{kj}y^{(1)k},$$

$$\underset{(2)}{N^i_{\ j}} = \frac{1}{2}\left(\frac{\partial\gamma^i_{kj}}{\partial x^r} - \gamma^i_{rh}\gamma^h_{kj}\right) y^{(1)k}y^{(1)r} + \gamma^i_{kj}y^{(2)k}$$

and the Berwald connection $B\Gamma(N) = \left(\gamma^i_{jk},\ 0,\ 0\right)$.

Theorem 3.1 *The invariant components of the Berwald connection are obtained from $B\Gamma(N)$ by a deviation induced by the invariant frames.*

Direct calculus lead us to the following expressions of the invariant components of the Berwald connection:

$$\overline{B\Gamma(N)} = \left(\gamma^\eta_{\beta\alpha} + \frac{1}{2}\underset{(00)}{\overset{0}{W^\eta_{\beta\alpha}}}\ ,\ \frac{1}{2}\underset{(11)}{\overset{1}{W^\eta_{\beta\alpha}}}\ ,\ \frac{1}{2}\underset{(22)}{\overset{2}{W^\eta_{\beta\alpha}}}\right).$$

Theorem 3.2 *In invariant frames the canonical metrical N-linear connection $\overline{C\Gamma(N)}$ (with respect to the metrical tensor proposed) have the coefficients*

$$\overline{C\Gamma(N)} = \left(L^\gamma_{\beta\alpha},\ \underset{(1)}{C^\gamma_{\beta\alpha}}\ \ \underset{(2)}{C^\gamma_{\beta\alpha}}\right),$$

with

$$L^\gamma_{\beta\alpha} = \overset{*}{L}{}^\gamma_{\beta\alpha} + \underset{0}{\Lambda^\gamma_{\beta\alpha}}\ ,$$

$$\underset{(A)}{\overset{*}{C}{}^\gamma_{\beta\alpha}} + \underset{A}{\Lambda^\gamma_{\beta\alpha}} \qquad \textit{(A=1,2)},$$

where $\overset{*}{L}{}^\gamma_{\beta\alpha}$*,and* $\underset{(A)}{\overset{*}{C}{}^\gamma_{\beta\alpha}}$ *are the coefficients of Berwald connection in invariant frames and* $\underset{A}{\Lambda^\gamma_{\beta\alpha}}$ *are the invariant components of the deviation tensor* $\underset{A}{\Lambda^i_{jk}}$ *induced by the metric, which have the expressions:*

$$\underset{(B)}{\Lambda^i_{jk}} = \delta^i_k \overset{(B)}{\sigma_j} + \delta^i_j \overset{(B)}{\sigma_k} - \gamma_{jk}\overset{(B)}{\sigma^i},$$

where

$$\overset{(B)}{\sigma_j} = \frac{\delta\sigma}{\delta y^{(B)j}}\ ,\quad \overset{(B)}{\sigma^i} = \gamma^{is}\overset{(B)}{\sigma_s}\ ,\quad \textit{(B=1,2)}$$

$(y^{(0)i} = x^i)$

Theorem 3.3 *The deviation tensor* $\underset{(B)}{\Lambda^{\alpha}_{\beta\gamma}}$ *with respect to* $\overline{B\Gamma(N)}$ *is*

$$\underset{(0)}{\Lambda^{\alpha}_{\beta\gamma}} = \frac{1}{2} g^{\alpha\eta} \left(g_{\beta\eta))\gamma} + g_{\gamma\eta))\beta} - g_{\beta\gamma))\eta} \right)$$

$$\underset{(B)}{\Lambda^{\alpha}_{\beta\gamma}} = \frac{1}{2} g^{\alpha\eta} \left(g_{\beta\eta} \overset{(B)}{))}_{\gamma} + g_{\gamma\eta} \overset{(B)}{))}_{\beta} - g_{\beta\gamma} \overset{(B)}{))}_{\eta} \right)$$

A direct calculus of curvature tensors, Ricci's tensors and curvature scalars lead us to the following result:

Theorem 3.4 *Consider the space* $GL^{(2)n}$ *endowed with the metric* $g_{ij} = e^{2\sigma(x,\, y^{(1)},\, y^{(2)})}\gamma_{ij}$ *and the canonical metrical connection* $C\Gamma(N)$. *The Einsein equations in the invariant frames are:*

$$r_{\alpha\beta} + R^{*}_{\alpha\beta} + \tilde{R}_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta}(r + R^{*} + \tilde{R}) = \kappa \overset{0}{T}_{\alpha\beta}\,,$$

$$S^{*}_{\alpha\beta} + \tilde{S}_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta}(r + R^{*} + \tilde{R}) = \kappa \overset{A}{T}_{\alpha\beta}\,,$$

$$\underset{(A)}{\overset{B}{P^{*}_{\alpha\beta}}} + \underset{(A)}{\overset{B}{\tilde{P}^{*}_{\alpha\beta}}} = (-1)^{B} \kappa \underset{(A)}{\overset{0B}{T_{\alpha\beta}}}\,,$$

$$\underset{(12)}{\overset{B}{P^{*}_{\alpha\beta}}} + \underset{(12)}{\overset{B}{\tilde{P}^{*}_{\alpha\beta}}} = (-1)^{B} \kappa \underset{(12)}{\overset{0B}{T_{\alpha\beta}}}\,, \qquad \textit{(A,B=1,2).}$$

Theorem 3.5 *The conservation law with respect to the canonical metrical connection* $C\Gamma(N)$ *in invariant frames are:*

$$\left(R^{\alpha}_{\beta} - \frac{1}{2} R \delta^{\beta}_{\alpha} \right)_{)\beta} + \underset{(1)}{\overset{1}{P^{\beta}_{\alpha}}} \overset{(1)}{)}_{\beta} + \underset{(1)}{\overset{2}{P^{\beta}_{\alpha}}} \overset{(1)}{)}_{\beta} + \underset{(2)}{\overset{1}{P^{\beta}_{\alpha}}} \overset{(2)}{)}_{\beta} + \underset{(2)}{\overset{2}{P^{\beta}_{\alpha}}} \overset{(2)}{)}_{\beta} = 0,$$

$$\left(\underset{(1)}{S^{\beta}_{\alpha}} - \frac{1}{2} R \delta^{\beta}_{\alpha} \right) \overset{(1)}{)}_{\beta} - \underset{(1)}{\overset{2}{P^{\beta}_{\alpha)\beta}}} + \underset{(12)}{\overset{2}{P^{\beta}_{\alpha}}} \overset{(2)}{)}_{\beta} = 0,$$

$$\left(\underset{(2)}{S^{\beta}_{\alpha}} - \frac{1}{2} R \delta^{\beta}_{\alpha} \right) \overset{(2)}{)}_{\beta} - \underset{(2)}{\overset{2}{P^{\beta}_{\alpha)\beta}}} - \underset{(12)}{\overset{2}{P^{\beta}_{\alpha}}} \overset{(2)}{)}_{\beta} = 0,$$

where $'\,_{)}\,'$ *and* $'\,\overset{(B)}{)}\,'$ *(*$B = 1, 2$*) are the* $h-$ *and the* v_B- *covariant invariant derivatives with respect to* $C\Gamma(N)$.

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