PERIODIC SOLUTIONS IN FIELD THEORY AND POINCARÉ INVARIANCE (GROUP ANALYSIS)

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Abstract

Poincare-invariant systems with strong coupling are considered. Quantizations are made in the presence of two-periodic classical field. Dne develops a scheme of perturbation theory using Bogoliubov group variables and taking into account the consevations lows.

Doubly periodic solutions for the Lagrange–Euler equation of the $(1 + 1)$-dimensional scalar φ^4 theory are studied. Provided that nonlinear term is small, the Poincare asymptotic method is used in order to find asymptotic solutions in the standing wave form. Using the Jacobi elliptic function *cn* as a zero approximation, it is proved that one can solve the problem of the main resonance which appear in the case of a zero mass and one can construct a uniform expansion.

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1.1 Introduction

We consider self-acting scalar field in the (1+1)–dimensioned space-time with the following Lagrangian:

$$L(x) = \frac{1}{2} g^{\alpha\beta}\Phi_\alpha\Phi_\beta - g^2 V\Big(\frac{1}{g}\Phi\Big), \qquad (I.1.1)$$

$$x^a = (x^0, x^1) = (t, x), \qquad \Phi_\alpha \equiv \frac{\partial\Phi}{\partial x^\alpha}, \qquad g^{\alpha\beta} = diag(1, -1).$$

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The Lagrangian is invariant with respect to the Poincaré group of transformations. Dimensionless parameter g is assumed to be large. In this case the main effect of interaction is the generation of a classical field.

Let's define *Bogoliubov transformation* as

$$f(x) = gv(x^{'}) + u(x^{'}), \tag{I.1.2}$$

where $v(x^{'})$ is some fixed function, and variables $x^{'}$ are connected with x by the inverted Poincaré transformation $x^{'\alpha} = A^{\alpha}_{\beta}(\phi)(x^{\beta} - \tau^{\beta})$. Here $A^0_0 = A^1_1 = ch\phi$, $A^1_0 = A^0_1 = -sh\phi$, $\tau^a = (\tau^{\alpha}, \phi)$, $a = 0, 1, 2$, $\alpha = 0, 1$.The number of independent variables became at 3 variables more in the right part of equation cause τ^{α}, which are considered to be independent. Let's restrict the choice of the functions $u(x^{'})$ to equalize the number of variables. We will use the following procedure: let's choose some space-like curve C, in which three function $N^a(x^{'})$ and normal derivatives $N^a_n(x^{'})$ and $u_n(x^{'})$ are given, and we demand to be fulfilled of the following conditions:

$$\omega(N^a, u) = \int_C d\lambda \Big(N^a_n(x^{'})u(x^{'}) - N^a(x^{'})u_n(x^{'}) \Big) = 0. \tag{I.1.3}$$

We can obtain equations, which define group variables as functional of $f(x)$ and $f_n(x)$ on the C in the differential form:

$$\frac{\delta \tau^a}{\delta f(x)} = -\frac{1}{g} Q^a_b \tilde{N}^b_n(x^{'}), \qquad \frac{\delta \tau^a}{\delta f_n(x)} = \frac{1}{g} Q^a_b \tilde{N}^b(x^{'}),$$

where Q^a_b are the solution of the equation:

$$Q^a_b = \delta^a_b - \frac{1}{g} R^a_c Q^c_b.$$

Here $\tilde{N}^a$ is a linear combination of N^a, such that the equations $\omega(\tilde{N}^a, M_b) = 0$ are true; and R^a_c is a c=number, calculated with a help of $v(x^{'})$ and $u(x^{'})$.

The operators $\hat{q}(x)$ and $\hat{p}(x)$:

$$\hat{q}(x) = \frac{1}{\sqrt{2}} \Big(f_n(x) + i\frac{\delta}{\delta f(x)} \Big), \qquad \hat{p}(x) = \frac{1}{\sqrt{2}} \Big(f(x) - i\frac{\delta}{\delta f_n(x)} \Big),$$

are defined in the space L of functionals F, where the scalar product is defined as:

$$< F_1 | F_2 >= \int Df Df_n F_{1n}[f, f_n] F_2[f, f_n].$$

The operators $\hat{p}$ and $\hat{q}$ are self-conjugated. They satisfy the formal commutation relation:

$$[\hat{q}(x), \hat{p}(x^{'})] = i\delta(x - x^{'}).$$

So we can treat $\hat{q}(x)$ and $\hat{p}(x)$ as operators of coordinate and momentum and we can develop the secondary quantization scheme. But straightforward use of this procedure leads us to the doubling of numbers of possible field states. We use the following scheme: *we use Bogoliubov transformation (1.2) and, in spite of appearance of exceed states, we will develop scheme of perturbation theory. Then reduction of states number will be made*, so it will depend on dynamic system equations.

Integrals of motion generated by Lagrangian (1.1) symmetry group are energy-momentum vector and Lorentz transformation generator:

$$P_0 = \beta \int_C d\lambda \Big(\frac{1}{2}u^2(x^{'}) - \frac{1}{2}u_\lambda^2(x^{'}) + V\Big) - \alpha \int_C \big(u_\lambda(x^{'})u_n(x^{'})\big),$$

$$P_1 = -\alpha \int_C d\lambda \Big(\frac{1}{2}u^2(x^{'}) - \frac{1}{2}u_\lambda^2(x^{'}) + V\Big) + \beta \int_C \big(u_\lambda(x^{'})u_n(x^{'})\big),$$

$$M = \int_C \lambda d\lambda \Big(\frac{1}{2}u_n^2(x^{'}) - \frac{1}{2}u_\lambda^2(x^{'}) - V\Big).$$

Here α and β are defining $t^{'}$ and $x^{'}$ parameters on the C:

$$t^{'} = \alpha\lambda, \qquad x^{'} = \beta\lambda.$$

Let us denote:

$$H = \int_C \Big(\frac{1}{2}u^2(x^{'}) - \frac{1}{2}u_\lambda^2(x^{'}) + V\Big), \qquad P = \int_C \big(u_\lambda(x^{'})u_n(x^{'})\big).$$

Then integrals of motion can be represented as series with respect to inverted powers of coupling constant:

$$O = g^2 O_{-2} + g O_{-1} + O_0 + \frac{1}{g} O_1 + \dots\,.$$

Now we can quantize and substitute $u(x^{'})$, $u_\lambda(x^{'})$, $u_n(x^{'})$ as follows:

$$u(x^{'}) \longrightarrow \hat{q}(x), \qquad u_\lambda(x^{'}) \longrightarrow \hat{q}_\lambda(x), \qquad u_n(x^{'}) \longrightarrow \hat{p}(x).$$

In the series (4.1) operators O_{-2} are C-numbers and operators O_{-1} are linear with respect to $u(x^{'})$, $u_n(x^{'})$, $\frac{\partial}{\partial u(x^{'})}$, $\frac{\partial}{\partial u_n(x^{'})}$. There are not normalizable eigenvectors of these operators, so it is required to set them to zero for perturbation theory construction. Let's explore, if it is possible. Supposing that some boundary conditions are accomplished, then the following equation is obtained:

$$F_{nn}(x^{'}) - F_{\lambda\lambda}(x^{'}) + V^{'}(F) = 0,$$

thus the operators O_{-1} are equal to zero.

Here $F(x^{'})$ is connected linearly with a classical component $v(x^{'})$.

Hereinafter we assume $F(x)$ to be solution of the wave equation $F_{tt} - F_{xx} + V^{'}(F) = 0$, and $F(x^{'})$ and $F_n(x^{'})$ on C are the solution of the Cauchy problem on C.

The number of independent variables has been doubled, due to considering $f(x)$ and $f_n(x)$ as independent. Cause of additional condition (1.3), which connect $u(x^{'})$ and $u_n(x^{'})$, the number of independent variables (minus group variables) become equal $(2*\infty-3)$. To reduce them to $(\infty-3)$, one needs also three conditions. They are as follows:

$$\omega(\tilde{N}^a, w) = 0, \qquad \omega(M_a, w) = 0.$$

Here:

$$u(x^{'}) = w(x^{'}) + \tilde{N}^a(x^{'})r_a, \qquad u_n(x^{'}) = w_n(x^{'}) + \tilde{N}_n^a(x^{'})r_a.$$

Necessary reduction of the states number can be made in the following way: let's suppose that the field condition is defined by functionals of $w(x^{'})$ and $w_n(x^{'})$, in which $\dfrac{\delta}{\delta w(x')}$ and $\dfrac{\delta}{\delta w_n(x')}$ become as follows:

$$\frac{\delta}{\delta w(x')} \longrightarrow \frac{\delta}{\delta w(x')} - iw_n(x^{'}), \qquad \frac{\delta}{\delta w_n(x')} \longrightarrow -iw(x^{'}). \tag{$I.1.4$}$$

After the reduction of independent variables, it becomes:

(3 group parameters)+(3 variables r_a)+ ($\infty-3$–dimensioned function w space)

The variables r_a have not physical sense. They have appeared as a rest of the state space reduction in the terms of Bogoliubov group variables. Below we will show that separation of these variables is connected with integrals of motion structure in the zero-point order, so it is dynamic by nature.

1.2 An example of field variables reduction

Without loss of generality we can consider the case when $F(x^{'})$ satisfy to the boundary condition $F_n(x^{'}) = 0$ on C.

In that case:

$$M_\alpha(x^{'}) = -\frac{1}{\sqrt{2}}e_\alpha F_\lambda(x^{'}), \qquad M_{n\alpha}(x^{'}) = \frac{1}{\sqrt{2}}n_\alpha F_{nn}(x^{'}),$$

$$M_2(x^{'}) = 0, \qquad M_{2n} = \frac{1}{\sqrt{2}}\big(\lambda F_{nn}(x^{'}) + F_\lambda(x^{'})\big).$$

Suppose that we chose N^a as follows:

$$\tilde{N}_\alpha(x^{'}) = An_\alpha F_{nn}(x^{'}), \qquad \tilde{N}_n^\alpha(x^{'}) = Be_\alpha F_\lambda(x^{'}),$$

$$\tilde{N}^2(x^{'}) = b_1\lambda F_{nn}(x^{'}) + b_2F_\lambda(x^{'}), \qquad \tilde{N}_n^2(x^{'}) = 0,$$

where $\vec{e}$ is a vector along line C, $\vec{n}$ is a normal vector and b_1, b_2 are appropriate numbers (see [2]). We can obtain the following relationships between F and its normal derivative $F(x^{'}) = F_{nn}(x^{'})\dfrac{\int FF_{nn}}{\int F_{nn}^2}$,

thus the second normal derivative of classical component is proportional to the function itself:

$$F_{nn}(x^{'}) = \omega^2 F(x^{'}). \tag{$I.2.1$}$$

1.3 Zero-point order with respect to g.

The integrals of motion at the zero-point order with respect to g are:

$$H_0 = in^{\alpha}\frac{\partial}{\partial\tau^{\alpha}} + H_{01} + H_{02} + H_{03},$$

$$P_0 = ie^{\alpha}\frac{\partial}{\partial\tau^{\alpha}} + P_{01} + P_{02} + P_{03},$$

$$M_0 = i\frac{\partial}{\partial\tau^2} + i\Big(\tau^0\frac{\partial}{\partial\tau^1} + \tau^1\frac{\partial}{\partial\tau^0}\Big) + M_{01} + M_{02} + M_{03},$$

where

$$\begin{aligned} H_{01} &= \frac{1}{2}\int\big(\hat{P}^2(x^{'}) + \hat{Q}^2_{\lambda}(x^{'}) + V^{''}(F)\hat{Q}^2(x^{'})\big),\\ H_{02} &= i\int\Big(w_n(x^{'})\frac{\delta}{\delta w(x^{'})} + w_{nn}(x^{'})\frac{\delta}{\delta w_n(x^{'})}\Big), \end{aligned}$$

$$\begin{aligned} H_{03} &= \frac{1}{2}\int(p^2(x^{'}) + q^2_{\lambda}(x^{'}) + V^{''}(F)q^2(x^{'})) + ir_c\frac{\partial}{\partial r_b}\int +\\ &\Big(\tilde{N}^c_n(x^{'})\frac{\delta r_b}{\delta u(x^{'})} - \tilde{N}^c_{nn}(x^{'})\frac{\delta r_b}{\delta u_n(x^{'})}\Big) +\\ &r_c r_a\int\big(\tilde{N}^c_n(x^{'})\tilde{N}^a_n(x^{'}) - \tilde{N}^c_{nn}(x^{'})\tilde{N}^a(x^{'})\big), \end{aligned}$$

$$\begin{aligned} P_{01} &= \int\hat{Q}_{\lambda}(x^{'})\hat{P}(x^{'}),\\ P_{02} &= i\int\Big(w_n(x^{'})\frac{\delta}{\delta w(x^{'})} + w_{\lambda n}\frac{\delta}{\delta w_n(x^{'})}\Big), \end{aligned}$$

$$\begin{aligned} P_{03} &= \int q_{\lambda}(x^{'})p(x^{'}) + ir_c\frac{\partial}{\partial r_b}\int\Big(\tilde{N}^c_{\lambda}(x^{'})\frac{\delta r_b}{\delta u(x^{'})} - \tilde{N}^c_{\lambda n}(x^{'})\frac{\delta r_b}{\delta u_n(x^{'})}\Big) +\\ &r_c r_a\int\big(\tilde{N}^c_{\lambda}(x^{'})\tilde{N}^a_n(x^{'}) - \tilde{N}^c_{\lambda n}(x^{'})\tilde{N}^a(x^{'})\big), \end{aligned}$$

$$\begin{aligned} M_{01} &= \frac{1}{2}\int\lambda\big(\hat{P}^2(x^{'}) + \hat{Q}^2_{\lambda}(x^{'}) + V^{''}(F)\hat{Q}^2(x^{'})\big),\\ M_{02} &= i\int\Big(\lambda w_n(x^{'})\frac{\delta}{\delta w(x^{'})} + (\lambda w_{nn}(x^{'}) + w_{\lambda}(x^{'}))\frac{\delta}{\delta w_n(x^{'})}\Big), \end{aligned}$$

$$
\begin{aligned}
M_{03} \quad = \quad & \frac{1}{2}\int \lambda\big(p^2(x^{'}) + q_\lambda^2(x^{'}) + V^{''}(F)q^2(x^{'})\big) + \\
& ir_c\frac{\partial}{\partial r_b}\int \Big(\lambda\tilde{N}_n^c(x^{'})\frac{\delta r_b}{\delta u(x^{'})} - (\lambda\tilde{N}_{nn}^c(x^{'}) + \tilde{N}_\lambda^c(x^{'}))\frac{\delta r_b}{\delta u_n(x^{'})}\Big) + \\
& r_c r_a\int \big(\lambda\tilde{N}_n^c(x^{'})\tilde{N}_n^a(x^{'}) - (\lambda\tilde{N}_{nn}^c(x^{'}) + \tilde{N}_\lambda^c(x^{'})\tilde{N}^a(x^{'}))\big).
\end{aligned}
$$

Operators $\hat{Q}$ and $\hat{P}$ have sense for coordinates and momentum for the systems described by variables w, w_n. q and p are c-numbers calculated using the same variables. Explicit expressions for those values can be found in [1].

The operators O_0 act at the space like $F[w, w_n]F[r]$, so O_{01} and O_{02} act at the space $F[w, w_n]$, but operators O_{03} act at the space $F[r]$. Those spaces are orthogonal.

H_{02} is the displacement operator along the normal to C. Being Hamiltonian of the system described by the operators $\hat{Q}(x^{'})$ and $\hat{P}(x^{'})$, the operator H_{01} has the same sense. It is possible to show that H_{02} differs from H_{01} only by the sign, so they annihilate each to other:

$$H_0 = in^\alpha\frac{\partial}{\partial\tau^\alpha} + H_{03}.$$

Analogously one can show that the operator P_{02} is the displacement operator along C and it is annihilated together with operator P_{01}, which is the system momentum, and operator P_0 is:

$$P_0 = ie^\alpha\frac{\partial}{\partial\tau^\alpha} + P_{03}.$$

The operator M_{02} describes the rotation of angle ϕ and it is compensated by the Lorentz rotation operator M_{01}, so the operator M_0 looks like:

$$M_0 = i\frac{\partial}{\partial\tau^2} + i\Im^\alpha\frac{\partial}{\partial\tau^\alpha} + M_{03}.$$

The operator O_{03} contains only exceed variables r_a, $\frac{\partial}{\partial r_a}$ which can be removed with the help of an appropriate choice of state vector:

$$f = \exp(\alpha x_0^2 + \beta x_2^2 + \gamma x_2^2 + \mu x_0 x_2),$$

addends that depend on r_a commute, average of P_{03} and M_{03} are equal zero, average of H_{03} can be sat to zero by appropriate renormalization. So those addends can be removed.

(Here x_a are linear combinations of r_a, $\alpha, \beta, \gamma, \mu$ are well chosen numbers, see [1].)

After removing of exceed variables integrals of motion in the zero-point order, one has:

$$P_\alpha = i\frac{\partial}{\partial\tau^\alpha}, \qquad M = i\frac{\partial}{\partial\tau^2} + i\Im^\alpha\frac{\partial}{\partial\tau^\alpha},$$

which satisfy the Poincaré group permutation relations:

$$[P_0, P_1] = 0, \qquad [M, P_\alpha] = -g_{0\alpha}P_1 + g_{1\alpha}P_0.$$

Heisenberg equations

$$\frac{\partial Z}{\partial x^\alpha} = i[P_\alpha, Z], \qquad \Big(x^0\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^0}\Big)Z = i[M, Z]$$

for the field $\psi(x)$ are as follows:

$$\frac{\partial Z}{\partial x^\alpha} = -\frac{\partial Z}{\partial \tau^\alpha}, \tag{I.3.2a}$$

$$\Big(x^0\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^0}\Big)Z - \Big(\frac{\partial}{\partial\phi} + \tau^0\frac{\partial}{\partial\tau^1} + \tau^1\frac{\partial}{\partial\tau^0}\Big)Z \tag{I.3.2b}$$

with a boundary conditions:

$$\psi|_c = q, \qquad \psi_n|_c = p.$$

The solutions of the equations (3.2a) are the functions:

$$Z = Z(x^\alpha - \tau^\alpha),$$

and solutions of the equations (3.2b) are the functions:

$$Z = Z\Big(A^\alpha_\beta(\phi)\big(x^\alpha - \tau^\alpha\big)\Big).$$

The field operator $\psi(x)$ looks like:

$$\psi(x) = gF(x^{'}) + \hat\Phi(x^{'}) + \hat\phi_a\frac{\partial}{\partial r_a} + \frac{1}{g}A(x^{'}, \tau),$$

here $\hat\Phi(x^{'})$ is the solution of the wave equation:

$$\hat\Phi_{tt} - \hat\Phi_{xx} + Y(x)\hat\Phi = 0$$

with a boundary condition on the C:

$$\hat\Phi = \hat Q(x^{'}), \qquad \hat\Phi_t = \hat P(x^{'}).$$

The permutation function looks like:

$$D(x^{'}, x^{''}) = \Big[\psi(x^{'}), \psi(x^{''})\Big] = i\delta\left(x^{'} - x^{''}\right).$$

2.

We have constructed the procedure of quantization close to a non-trivial classical field. This field is a solution of nonlinear differential equation:

$$\frac{\partial^2\varphi(x,t)}{\partial x^2} - \frac{\partial^2\varphi(x,t)}{\partial t^2} - V'(\varphi) = 0,$$

with following boundary conditions:

$$F_t|_{\partial C} = F_x|_{\partial C} = 0.$$

The doubly periodic solutions in the standing wave form satisfy these boundary conditions. In some theories (for example, Sine-Gordon) such solutions are well known. In other theories exact standing wave solutions are not known.

Our investigation is dedicated to the construction of doubly periodic classical fields in the $(1+1)$-dimensional φ^4 theory. Let us consider the Lagrange–Euler equation:

$$\frac{\partial^2\varphi(x,t)}{\partial x^2} - \frac{\partial^2\varphi(x,t)}{\partial t^2} - M^2\varphi(x,t) - \varepsilon\varphi^3(x,t) = 0. \qquad (II.1)$$

We intend to find solutions in the standing wave form:

$$\varphi(x,t) \equiv \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \mathcal{C}_{nj}\sin(n(x-x_0))\sin(j\cdot\omega(t-t_0)), \qquad (II.2)$$

where x_0 and t_0 are constants determined by boundary and initial conditions. The equation (1) is a translation-invariant one, so, without loss of generality, we can restrict our consideration to the case of zero x_0 and t_0. We suppose that the function $\varphi(x,t)$ is 2π-periodic in space and seek its period in time. Exact standing wave solutions of this equation are not known.

The purpose of this report is the construction of standing wave solutions of equation (1) with $M=0$, using asymptotic methods. They can only be applied, provided $\varepsilon \ll 1$.

We use the Poincaré method,: introducing the new time $\tilde{t} \equiv \omega t$ and looking for a double periodic solution of equation (1) $\varphi(x,\tilde{t})$ and the frequency (in time) ω in the form of power series of ε:

$$\varphi(x,\tilde{t},\varepsilon) \equiv \sum_{n=0}^{\infty}\varphi_n(x,\tilde{t})\varepsilon^n,$$
$$\omega(\varepsilon) \equiv 1 + \sum_{n=1}^{\infty}\omega_n\varepsilon^n.$$

After we expand the Lagrange–Euler equation in series of ε powers,we obtain a sequence of equations. Write out two leading equations:

1) To zero order of ε the equation in φ_0:

$$\frac{\partial^2\varphi_0(x,\tilde{t})}{\partial x^2} - \frac{\partial^2\varphi_0(x,\tilde{t})}{\partial\tilde{t}^2} = 0, \qquad (II.3)$$

2) To first order of ε the equation in φ_0, φ_1 and ω_1:

$$\frac{\partial^2\varphi_1(x,\tilde{t})}{\partial x^2} - \frac{\partial^2\varphi_1(x,\tilde{t})}{\partial\tilde{t}^2} = 2\omega_1\frac{\partial^2\varphi_0(x,\tilde{t})}{\partial\tilde{t}^2} + \varphi_0^3(x,\tilde{t}). \qquad (II.4)$$

The equation (3) has many periodic solutions. If we select as solution for this equation the function $\varphi_0(x,\tilde{t}) = \sin(x)\sin(\tilde{t})$, then the second equation hasn't periodic solutions, because the frequency of the external force $\sin(3x)\sin(3\tilde{t})$ is equal to the frequency of its own oscillations and it is impossible to vanish this resonance harmonic, selecting only ω_1. We shall show that right selection of not only the frequency $\omega(\varepsilon)$, but also the function $\varphi_0(x,\tilde{t})$ allows to find a uniform expansion.

The general solution in the standing wave form (2) for equation (3) is the function

$$\varphi_0(x,\tilde{t}) = \sum_{n=1}^{\infty} a_n \sin(nx)\sin(n\tilde{t})$$

with arbitrary a_n. We have to find such coefficients a_n such that the function $\varphi_1(x,\tilde{t})$ is a periodic solution for equation (4). If we select $\varphi_1(x,\tilde{t})$ as a double sum:

$$\varphi_1(x,\tilde{t}) \equiv \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} b_{nj}\sin(nx)\sin(j\tilde{t})$$

with arbitrary b_{nj}, then equation (4) can be presented in the form of Fourier series:

$$\begin{aligned} R(x,\tilde{t}) &\equiv \frac{\partial^2\varphi_1(x,\tilde{t})}{\partial x^2} - \frac{\partial^2\varphi_1(x,\tilde{t})}{\partial\tilde{t}^2} - \\ 2\omega_1\frac{\partial^2\varphi_0(x,\tilde{t})}{\partial\tilde{t}^2} - \varphi_0^3(x,\tilde{t}) &= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} R_{nj}(a,b)\sin(nx)\sin(j\tilde{t}) = 0. \end{aligned}$$

which is equivalent to the following infinite system of the algebraic equations in Fourier coefficients of functions $\varphi_0(x,\tilde{t})$ and $\varphi_1(x,\tilde{t})$:

$$\forall n,j \ : \ R_{nj}(a,b) = 0.$$

This system has a subsystem of the equations in Fourier coefficients of $\varphi_0(x,\tilde{t})$:

$$\begin{aligned} &\forall j \in \mathbb{N} \ : \ R_{jj}(a) \equiv 9a_j^3 + 3a_j^2 a_{3j} + \\ &a_j\left(6\sum_{s\neq j}^{\infty}(2a_s^2 + a_s a_{2j+s}) + 3\sum_{s\neq j}^{2j-1} a_s a_{2j-s} - 32j^2\omega_1\right) + \\ &+3\sum_{s\neq j}^{\infty}\sum_{p\neq j}^{\infty} a_s a_p a_{j+s+p} + \\ &3\sum_{s\neq j}^{\infty}\sum_{\substack{p\neq j\\ p\neq 2j-s}}^{\infty} a_s a_p a_{s+p-j} + \sum_{s=1}^{j-2}\sum_{p=1}^{j-2} a_s a_p a_{j-s-p} = 0. \end{aligned} \qquad (II.5)$$

We have obtained a necessary and sufficient condition for the existence periodic solutions for the equation (4): there exist a periodic function $\varphi_1(x,\tilde{t})$, satisfying equation (4), if and only if the coefficients of $\varphi_0(x,\tilde{t})$ Fourier series satisfy the system (5).

The coefficient a_1 is a parameter, determining the oscillation amplitude. In fact, let $a_j = c_j a_1$ and $\omega_1 = c_\omega a_1^2$; then all polynomials R_{jj} are proportional to a_1^3: $R_{jj}(a) = a_1^3 R_{jj}(c)$ and, therefore, the coefficient a_1 can be selected arbitrarily. Our goal is to find a real solution and we look for $c_j \in \mathbb{R}$.

This system of equations is very difficult to solve. On the one hand all R_{jj} are infinite series and number of equations is infinite too. On the other hand each equation of this system is a nonlinear one. We have restricted ourselves to find a particular solution. To simplify calculations we assume that the function $\varphi_0(x,\tilde{t})$ contains only odd harmonics. Using Jacobi elliptic functions we have found the analytical form of the function, which Fourier series obeys this system.

For arbitrary $q \in (0,1)$ let us define the following sequence:

$$f \stackrel{\text{def}}{=} \{\ \forall n \in \mathbb{N}\ :\ f_{2n-1} = \frac{q^{n-1/2}}{1+q^{2n-1}}, \quad f_{2n} = 0\ \}.$$

The terms of the sequence f are proportional to Fourier coefficients of the Jacobi elliptic function *cn* [1]:

$$\operatorname{cn}(z,k) = \frac{\gamma}{k}\sum_{n=1}^{\infty} f_{2n-1}\cos\left((2n-1)\frac{\gamma z}{4}\right), \quad \text{where} \quad \gamma \equiv \frac{2\pi}{K}\ , \quad z \in \mathbb{R}. \qquad (II.6)$$

Let us clarify introduced designations and point out some properties of elliptic cosine:

1) Basic periods of the doubly periodic function $\operatorname{cn}(z,k)$ are $4K(k)$ and $2K(k) + 2iK'(k)$, where $K(k)$ is a full elliptic integral, $K'(k) \equiv K(k')$ and $k' = \sqrt{1-k^2}$.

2) The parameter q in the Fourier expansion can be expressed in the term of elliptic integrals: $q \equiv e^{-\pi\frac{K'}{K}}$.

3) The Fourier-series expansion of $\operatorname{cn}(z,k)$ doesn't include even harmonics. This expansion is corrected in the following domain of the complex plane: $-K' < \Im m\, z < K'$, in particular, for $z \in \mathbb{R}$.

4) If $z \in \mathbb{R}$ and $k \in (0,1)$, then $\operatorname{cn}(z,k) \in \mathbb{R}$.

5) The function $\operatorname{cn}(z,k)$ is a solution of the following differential equation:

$$\frac{d^2\operatorname{cn}(z,k)}{dz^2} = (2k^2-1)\operatorname{cn}(z,k) - 2k^2\operatorname{cn}^3(z,k). \qquad (II.7)$$

The latest property means that sequence of $\operatorname{cn}(z,k)$ Fourier coefficients is a solution of some infinite system of cubic equations.

On the one hand, it is clear from (6) that the Fourier-series expansion for the function $\operatorname{cn}^3(z,k)$ is:

$$\operatorname{cn}^3(z,k) = \frac{\gamma^3}{4k^3}\sum_{n=1}^{\infty} F_j^{(3)}(f)\cos\left(j\frac{\gamma z}{4}\right), \quad \text{where } j = 1,3,5,\ldots,+\infty\ ;$$

$$F_j^{(3)}(f) \equiv 3f_j^3 + 3f_j^2 f_{3j} + f_j\left(6\sum_{s\neq j}^{\infty}(f_s^2 + f_s f_{2j+s}) + 3\sum_{s\neq j}^{2j-1} f_s f_{2j-s}\right) + $$
$$+ 3\sum_{s\neq j}^{\infty}\sum_{p\neq j}^{\infty} f_s f_p f_{j+s+p} + 3\sum_{s\neq j}^{\infty}\sum_{\substack{p\neq j\\ p\neq 2j-s}}^{\infty} f_s f_p f_{s+p-j} + \sum_{s=1}^{j-2}\sum_{p=1}^{j-2} f_s f_p f_{j-s-p}$$

(in all sums we summarize over only odd numbers).

On the other hand, from differential equation (7) it follows that $F_j^{(3)}(f)$ is proportional to f_j, with coefficients of proportionality depending on j:

$$\forall j \quad : \quad F_j^{(3)}(f) = \left(\frac{2(2k^2-1)}{\gamma^2} + \frac{j^2}{8}\right) f_j. \qquad (II.8)$$

Thus, the sequence f is a nonzero solution of system (8) at all $q \in (0,1)$. The following lemma proves the existence of a preferred value of q.

Lemma. *There exists a value of parameter $q \in (0,1)$ such that the sequence f is a real solution of the system* (5)*; in addition a value of ω_1 also is real.*

Proof. Inserting the sequence f into system (5) : $a_j = f_j$ and using system (8), we obtain:

$$\forall j \quad : \quad R_{jj}(f) = \left\{F_j^{(3)}(f) + f_j\left(6\sum_{n=1}^{\infty} f_n^2 - 32j^2\omega_1\right)\right\} =$$
$$= f_j\left\{6\sum_{n=1}^{\infty} f_n^2 + \frac{2(2k^2-1)}{\gamma^2} + j^2\left(\frac{1}{8} - 32\omega_1\right)\right\} = 0.$$

the system (5) has nonzero solution if and only if

$$\begin{cases} \omega_1 = \dfrac{1}{256}, \\ \displaystyle\sum_{n=1}^{\infty} f_n^2 = \dfrac{(1-2k^2)}{3\gamma^2}. \end{cases}$$

We have obtained the value of ω_1. The second equation of this system is equivalent to the following equation in parameter q:

$$3\sum_{n=1}^{\infty}\left(\frac{q^{n-1/2}}{1+q^{2n-1}}\right)^2 - \left(\frac{1}{4} + \sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\right)^2 + 2\left(\sum_{n=1}^{\infty}\frac{q^{n-1/2}}{1+q^{2n-1}}\right)^2 = 0. \qquad (II.9)$$

This equation has the solution $q = 1.42142623201 \times 10^{-2} \pm 1 \times 10^{-13} \in (0,1)$.

.

□

Now it is easy to construct the required zero approximation of the function $\varphi(x,\tilde{t})$:

$$\varphi_0(x,\tilde{t}) = A\Big\{\mathrm{cn}(\alpha(x-\tilde{t}),k) - \mathrm{cn}(\alpha(x+\tilde{t}),k)\Big\}.$$

For arbitrary $k \in (0,1)$ this function is a real solution of equation (4). If $\alpha = \frac{2K}{\pi}$, then periods $\varphi_0(x,\tilde{t})$ in x and in $\tilde{t}$ are equal to 2π. Using the Fourier-series expansion for the function $\mathrm{cn}(z,k)$ (formula (6)), we obtain the following expansion for the function $\varphi_0(x,\tilde{t})$:

$$\varphi_0(x,\tilde{t}) = 2A\frac{\gamma}{k}\sum_{n=1}^{\infty} f_{2n-1}\sin((2n-1)x)\sin((2n-1)\tilde{t}).$$

If $q = 1.42142623201 \times 10^{-2} \pm 1 \times 10^{-13}$, then q is a solution of equation (9) and the sequence f is a real solution of system (5). The middle value of q corresponds to $k = 0.451075598811$ and $\alpha = 1.0576653982$. All the equations in the system (5) are homogeneous ones, hence, for these values of parameters, the sequence of $\varphi_0(x,\tilde{t})$ Fourier coefficients is also a solution of the system (5), with $\omega_1 = \dfrac{\gamma^2}{64k^2}A^2 = 1.0983600974A^2$.

Thus we have proved that the function

$$\varphi_0(x,\tilde{t}) = A\Big\{\mathrm{cn}(\alpha(x-\tilde{t}),k) - \mathrm{cn}(\alpha(x+\tilde{t}),k)\Big\},$$

with $k = 0.451075598811$ and $\alpha = 1.0576653982$ is such a standing wave solution of the equation (3) and that equation (4) has a periodic solution.

2.1 The first approximation

Now it is easy to find this periodic solution $\varphi_1(x,\tilde{t})$. Let us designate Fourier coefficients of $\varphi_0^3(x,\tilde{t})$ as D_{nj}: $\varphi_0^3(x,\tilde{t}) \equiv \sum_{n=1}^{\infty} : \sum_{j=1}^{\infty} D_{nj}\sin(nx)\sin(j\tilde{t})$.

The equation (4) gives the following result:

$$\begin{aligned}\varphi_1(x,\tilde{t}) \quad &\equiv \quad \sum_{n=1}^{\infty} : \sum_{j=1}^{\infty} b_{nj}\sin(nx)\sin(j\tilde{t}) = \\ &\sum_{n=1}^{\infty} : \sum_{\substack{j=1\\ j\neq n}}^{\infty} \frac{D_{nj}}{j^2-n^2}\sin(nx)\sin(j\tilde{t}) + \sum_{n=1}^{\infty} b_{nn}\sin(nx)\sin(n\tilde{t}).\end{aligned}$$

It should be noted that the function $\varphi_1(x,\tilde{t})$ with arbitrary diagonal coefficients b_{nn} is a solution of the equation (4) and that all off-diagonal coefficients of $\varphi_1(x,\tilde{t})$ are proportional to A^3.

2.2 The second approximation

Let us consider the equation of second order for ε:

$$\frac{\partial^2\varphi_2(x,\tilde{t})}{\partial x^2} - \frac{\partial^2\varphi_2(x,\tilde{t})}{\partial\tilde{t}^2} = 2\omega_1\frac{\partial^2\varphi_1(x,\tilde{t})}{\partial\tilde{t}^2} + (2\omega_2+\omega_1^2)\frac{\partial^2\varphi_0(x,\tilde{t})}{\partial\tilde{t}^2} + 3\varphi_1(x,\tilde{t})\varphi_0^2(x,\tilde{t}). \tag{II.10}$$

If all diagonal coefficients of $\varphi_1(x,\tilde{t})$ are zeros: $\forall n : b_{nn} = 0$, then $\forall j,n : b_{jn} = -b_{nj}$, and the function $\varphi_1(x,\tilde{t})\varphi_0^2(x,\tilde{t})$ hasn't diagonal harmonics. Hence, selecting $\omega_2 = -\frac{1}{2}\omega_1^2$ we obtain a periodic solution of equation (10):

$$\varphi_2(x,\tilde{t}) \equiv \sum_{n=1}^{\infty}\sum_{\substack{j=1\\ j\neq n}}^{\infty}\frac{H_{nj}}{j^2-n^2}\sin(nx)\sin(j\tilde{t}) + \sum_{n=1}^{\infty} h_{nn}\sin(nx)\sin(n\tilde{t}), \quad \text{where}$$

$$H(x,\tilde{t}) \equiv 2\omega_1\frac{\partial^2\varphi_1(x,\tilde{t})}{\partial\tilde{t}^2} - 3\varphi_1(x,\tilde{t})\varphi_0^2(x,\tilde{t}) \equiv \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} H_{nj}\sin(nx)\sin(j\tilde{t}).$$

It should be noted that all diagonal coefficients h_{nn} are arbitrary numbers.

2.3 Conclusions

Using massless φ^4 theory as an example, we show that a uniform expansion of solutions for quasilinear Klein–Gordon equations can be constructed even in the main resonance case. In order to construct the uniform expansion we have used the Poincaré method and the nontrivial zero approximation: the function $\varphi_0(x,t) = A\Big\{\mathrm{cn}(\alpha(x-\omega t),k) - \mathrm{cn}(\alpha(x+\omega t),k)\Big\}$,with $k = 0.451075598811$ and $\alpha = 1.0576653982$.

Thus, using the Jacobi elliptic function *cn* instead of the trigonometric function cos, we have vanished the main resonance and constructed with accuracy $\mathcal{O}(\varepsilon^3)$ a doubly periodic solution in the standing wave form $\varphi(x,\omega t) = \varphi_0(x,\omega t) + \varepsilon\varphi_1(x,\omega t) + \varepsilon^2\varphi_2(x,\omega t) + \mathcal{O}(\varepsilon^3)$ with the frequency $\omega = 1 + \frac{\gamma^2}{64k^2}A^2\varepsilon - \frac{\gamma^4}{8192k^4}A^4\varepsilon^2 + \mathcal{O}(\varepsilon^3) = 1 + 1.0983600974A^2\varepsilon - 0.6031974518A^4\varepsilon^2 + \mathcal{O}(\varepsilon^3)$.

References

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