ORTHOGONAL ADAPTED BASIS OF $T^*(Osc^k M)$

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Dedicated to Prof. M.C. Chaki and Prof. R. Miron

Abstract

Lately a big attention has been payed on the higher order geometry. Some relevant papers are mentioned in the references. R. Miron and Gh. Atanasiu in [16], [17] studied the geometry of $Osc^k M$. R. Miron in [19] gave the comprehensive theory of higher order geometry and its application. Here the transformation group is slightly different from that used in [19] and it will change the geometry. The adapted basis will have different form. Such an adapted basis is constructed that $T^*_{V_0}, T^*_{V_1}, \ldots, T^*_{V_k}$ are mutually orthogonal subspaces of $T^*(Osc^k M)$ with respect to the given metric G.

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1. Adapted basis in $T(Osc^k M)$ and $T^*(Osc^k M)$

Here $Osc^k M$ will be defined as a C^∞ manifold so that the transformations of form (1.1) are allowed. It is formed as a tangent space of higher order of the base manifold M.

Let $E = Osc^k M$ be a $(k+1)n$ dimensional C^∞ manifold. In a local chart (U, φ) a point $u \in E$ has the coordinates:

$$(x^a, y^{1a}, y^{2a}, \ldots, y^{ka}) = (y^{0a}, y^{1a}, y^{2a}, \ldots, y^{ka}) = (y^{\alpha a}),$$

where $x^a = y^{0a}$ and

$$a, b, c, d, e, \ldots = 1, 2, \ldots, n, \quad \alpha, \beta, \gamma, \delta, \kappa, \ldots = 0, 1, 2, \ldots, k.$$

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The following abbreviations will be used:

$$\partial_{\alpha a} = \frac{\partial}{\partial y^{\alpha a}}, \quad \alpha = 1, 2, \dots, k, \quad \partial_a = \partial_{0a} = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}}.$$

If in an other chart (U', φ') the point $u \in E$ has the coordinates $(x^{a'}, y^{1a'}, y^{2a'}, \dots, y^{ka'})$, then in $U \cap U'$ the allowable coordinate transformations are given by:

$$\begin{aligned}
(1) \qquad & x^{a'} = x^{a'}(x^1, x^2, \dots, x^n), \\
& y^{1a'} = (\partial_a x^{a'}) y^{1a} = (\partial_{0a} y^{0a'}) y^{1a}, \\
& y^{2a'} = (\partial_{0a} y^{1a'}) y^{1a} + (\partial_{1a} y^{1a'}) y^{2a}, \\
& y^{3a'} = (\partial_{0a} y^{2a'}) y^{1a} + (\partial_{1a} y^{2a'}) y^{2a} + (\partial_{2a} y^{2a'}) y^{3a}, \\
& \vdots \\
& y^{ka'} = (\partial_{0a} y^{(k-1)a}) y^{1a} + (\partial_{1a} y^{(k-1)a}) y^{2a} + \dots + (\partial_{(k-1)a} y^{(k-1)a}) y^{ka}.
\end{aligned}$$

Theorem 1.1. *The transformations of type (1.1) form a group.*

A nice example of a space E can be obtained if the points $(x^a) \in M$, $\dim M = n$ are considered as the points of the curve $x^a = x^a(t)$, $t \in I$ and $y^{\alpha a}$, $\alpha = 1, 2, \dots, k$ are determined by:

$$(2) \qquad y^{\alpha a} = d_t^\alpha x^a, \quad d_t^\alpha = \frac{d^\alpha}{dt^\alpha}, \; d_t = \frac{d}{dt}.$$

If in $U \cap U'$ the equation $x^{a'} = x^{a'}(x^1(t), x^2(t), \dots, x^n(t))$ is valid, then it is easy to see that

$$(3) \qquad y^{1a'} = d_t^1 x^{a'}, \quad y^{2a'} = d_t^2 x^{a'}, \dots, y^{ka'} = d_t^k x^{a'}$$

satisfy (1.1). In [19] $y^{\alpha a} = \frac{1}{\alpha!} d_t^\alpha x^a$ and it results that the structure group is different from (1.1). As from (1.2) and (1.3) it follows:

$$(4) \quad y^{1a'} = y^{1a'}(x, y^{1a}), \quad y^{2a'} = y^{2a'}(x, y^{1a}, y^{2a}), \dots, y^{ka'} = y^{ka'}(x, y^{1a}, \dots y^{ka})$$

and from the above equation we get (1.1).

Let us introduce the notations:

$$(5) \qquad {}^{(0)}A_a^{a'} = \partial_a x^{a'}, \quad {}^{(\alpha)}A_a^{a'} = d_t^{\alpha\,(0)} A_a^{a'} = \frac{d^{\alpha\,(0)} A_a^{a'}}{dt^\alpha}, \quad \alpha = 1, 2, \dots, k.$$

The natural basis $\bar{B}^*$ of $T^*(E)$ is

$$\bar{B}^* = \{dy^{0a}, dy^{1a}, \dots, dy^{ka}\}.$$

The elements of $\bar{B}^*$ are not transformed as tensors ([19], [9]).

The adapted basis B^* of $T^*(E)$ is given by

$$(6) \qquad B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \dots, \delta y^{ka}\},$$

where:

$$
\begin{aligned}
\delta y^{0a} &= dx^a = dy^{0a},\\
\delta y^{1a} &= dy^{1a} + M^{1a}_{0b}dy^{0b},\\
\delta y^{2a} &= dy^{2a} + M^{2a}_{1b}dy^{1b} + M^{2a}_{0b}dy^{0b},\\
&\vdots\\
\delta y^{ka} &= dy^{ka} + M^{ka}_{(k-1)b}dy^{(k-1)b} + M^{ka}_{(k-2)b}dy^{(k-2)b} + \cdots + M^{ka}_{0b}dy^{0b}.
\end{aligned} \tag{7}
$$

Theorem 1.2. *The necessary and sufficient conditions that* $\delta y^{\alpha a}$ *are transformed as d-tensor field, i.e.*

$$\delta y^{\alpha a'} = \frac{\partial x^{a'}}{\partial x^a}\delta y^{\alpha a}, \quad \alpha = 0, 1, \dots, k$$

are given by the following equations:

$$
\begin{aligned}
M^{(\alpha+\beta)a}_{\alpha b}(\partial_a x^{b'}) &= M^{(\alpha+\beta)b'}_{\alpha c'}\partial_{\alpha b}y^{\alpha c'} + M^{(\alpha+\beta)b'}_{(\alpha+1)c'}\partial_{\alpha b}y^{(\alpha+1)c'} + \cdots\\
& \quad M^{(\alpha+\beta)b'}_{(\alpha+\beta-1)c'}\partial_{\alpha b}y^{(\alpha+\beta-1)c'} + \partial_{\alpha b}y^{(\alpha+\beta)c'},\\
1 &\le \beta,\ \alpha+\beta \le k.
\end{aligned} \tag{8}
$$

The proof is given in [9].

From (1.8), after some calculation, we get:

$$
\begin{aligned}
M^{(\alpha+\beta)a}_{\alpha b}\ {}^{(0)}A^{b'}_a &= \binom{\alpha}{\alpha}M^{(\alpha+\beta)b'}_{\alpha c'}\ {}^{(0)}A^{c'}_b + \binom{\alpha+1}{\alpha}M^{(\alpha+\beta)b'}_{(\alpha+1)c'}\ {}^{(1)}A^{c'}_b + \cdots +\\
& \binom{\alpha+\beta-1}{\alpha}M^{(\alpha+\beta)b'}_{(\alpha+\beta-1)c'}\ {}^{(\beta-1)}A^{c'}_b + \binom{\alpha+\beta}{\alpha}\ {}^{(\beta)}A^{b'}_b.
\end{aligned}
$$

From (1.5) and (1.8) it follows:

$$M^{(\alpha+\beta)a}_{\alpha b} = M^{(\alpha+\beta)a}_{\alpha b}(x, y^1, \dots, y^\beta). \tag{9}$$

This equation is important when the integrability conditions are examined.

The adapted basis B^* defined by (1.6) and (1.7) is different from that introduced in [16], [17], [19]. The advantage of the present basis B^* is that the functions $M^{\beta b}_{\alpha a}$ can be determined in such a way that the elements of B^* are mutually orthogonal vectors with respect to the given nondegenerated positive definite symmetric metric tensor.

The natural basis $\bar{B}$ of $T(E)$ is $\bar{B} = \{\partial_{0a}, \partial_{1a}, \dots, \partial_{ka}\}$..The transformation law of its elements are given in [19].

Let us denote the adapted basis of $T(E)$ by B, where:

$$B = \{\delta_{0a}, \delta_{1a}, \delta_{2a}, \dots, \delta_{ka}\} = \{\delta_{\alpha a}\} \tag{10}$$

and

$$
(11)\qquad
\begin{array}{rcl}
\delta_{0a} & = & \partial_{0a} - N_{0a}^{1b}\partial_{1b} - N_{0a}^{2b}\partial_{2b} - \cdots - N_{0a}^{kb}\partial_{kb},\\
\delta_{1a} & = & \partial_{1a} - N_{1a}^{2b}\partial_{2b} - \cdots - N_{1a}^{kb}\partial_{kb},\\
\vdots & & \\
\delta_{ka} & = & \partial_{ka}.
\end{array}
$$

Theorem 1.3. *([9]) The necessary and sufficient conditions that B be dual to B^* ((1.6) and (1.10)) when $\bar{B}$ is dual to $\bar{B}^*$ i.e.*

$$
< \delta_{\alpha a}\delta^{\beta b} >= \delta_{\alpha}^{\beta}\delta_{a}^{b}
$$

are the following relations:

$$
(12)\qquad
\begin{array}{rcl}
N_{\alpha a}^{(\alpha+\beta)b} & = & M_{\alpha a}^{(\alpha+\beta)b} - M_{(\alpha+1)c}^{(\alpha+\beta)b}N_{\alpha a}^{(\alpha+1)c} - \\
& & M_{(\alpha+2)c}^{(\alpha+\beta)b}N_{\alpha a}^{(\alpha+2)c} - \cdots - M_{(\alpha+\beta-1)c}^{(\alpha+\beta)b}N_{\alpha a}^{(\alpha+\beta-1)c}.
\end{array}
$$

Theorem 1.4. *([9]) The necessary and sufficient conditions that $\delta_{\alpha a}$ with respect to (1.1) are transformed as d-tensors are the following formulae:*

$$
(13)\qquad
\begin{array}{rcl}
N_{\alpha a'}^{(\alpha+\beta)b'}(\partial_a x^{a'}) & = & N_{\alpha a}^{(\alpha+\beta)c}\partial_{(\alpha+\beta)c}y^{(\alpha+\beta)b'} + \\
& & N_{\alpha a}^{(\alpha+\beta-1)c}\partial_{(\alpha+\beta-1)c}y^{(\alpha+\beta)b'} + \cdots + \\
& & N_{\alpha a}^{(\alpha+1)c}\partial_{(\alpha+1)c}y^{(\alpha+\beta)b'} - \partial_{\alpha a}y^{(\alpha+\beta)b'}.
\end{array}
$$

The other form of (1.13) is:

$$
\begin{array}{rcl}
N_{\alpha a'}^{(\alpha+\beta)b'} \; {}^{(0)}A_a^{a'} & = & \binom{\alpha+\beta}{\alpha+\beta} N_{\alpha a}^{(\alpha+\beta)c} \; {}^{(0)}A_c^{b'} + \binom{\alpha+\beta}{\alpha+\beta-1} N_{\alpha a}^{(\alpha+\beta-1)c} \; {}^{(1)}A_c^{b'} \\
& & + \cdots + \binom{\alpha+\beta}{\alpha+1} N_{\alpha a}^{(\alpha+1)c} \; {}^{(\beta-1)}A_c^{b'} - \binom{\alpha+\beta}{\alpha} \; {}^{(\beta)}A_a^{b'}.
\end{array}
$$

From (1.12) and (1.9) we get:

$$
(14)\qquad N_{\alpha a}^{(\alpha+\beta)b} = N_{\alpha a}^{(\alpha+\beta)b}(x, y^1, y^2, \ldots, y^{\beta}).
$$

Theorem 1.5. *The basis vectors of $\bar{B}$ are connected with the basis vectors of B by:*

$$
(15)\qquad
\begin{array}{rcl}
\partial_{0a} & = & \delta_{0a} + M_{0a}^{1b}\delta_{1b} + M_{0a}^{2b}\delta_{2b} + \cdots + M_{0a}^{kb}\delta_{kb},\\
\partial_{1a} & = & \delta_{1a} + M_{1a}^{2b}\delta_{2b} + \cdots + M_{1a}^{kb}\delta_{kb},\\
\vdots & & \\
\partial_{ka} & = & \delta_{ka}.
\end{array}
$$

Proof. From (1.11) and (1.12) it follows (1.15).

Theorem 1.6. *The basis vectors of $\bar{B}^*$ are connected with the basis covectors of B^* by :*

(16)
$$\begin{aligned}
dy^{0a} &= && \delta y^{0a}\\
dy^{1a} &= && \delta y^{1a} - N^{1a}_{0e}\delta y^{0e},\\
dy^{2a} &= && \delta y^{2a} - N^{2a}_{1e}\delta y^{1e} - N^{2a}_{0e}\delta y^{0e},\\
&\vdots\\
dy^{ka} &= \delta y^{ka} - N^{ka}_{(k-1)e}\delta y^{(k-1)e} - && \cdots - N^{ka}_{1e}\delta y^{1e} - N^{ka}_{0e}\delta y^{0e}.
\end{aligned}$$

Proof. From (1.7) and (1.12) it follows (1.16).

2. Orthogonal adapted basis in $T^*(Osc^kM)$

Let us denote by $T^*_H = T^*_{V_0}, T^*_{V_1}, T^*_{V_2}, \ldots, T^*_{V_k}$ the subspaces of $T^*(E) = T^*(Osc^kM)$ spanned by $\{\delta y^{0a}\}, \{\delta y^{1a}\}, \{\delta y^{2a}\}, \ldots, \{\delta y^{ka}\}$ respectively. Then

$$T^*(E) = T^*_{V_0} \oplus T^*_{V_1} \oplus T^*_{V_2} \oplus \cdots \oplus T^*_{V_k},$$
$$dim T^*(E) = n(k+1), \quad dim T^*_{V_\alpha} = n, \quad \alpha = 0, 1, 2, \ldots, k.$$

Let us suppose that G is a symmetric nondegenerated positive definite metric tensor and in the basis $\bar{B}^*$ is given by:

(17)
$$G = \bar{g}_{\alpha a \beta b} dy^{\alpha a} \otimes dy^{\beta b},$$

where the summation is going over all indices. In the matrix form G can be written in the following way:

(18)
$$G = \begin{bmatrix} dy^{0a} \\ dy^{1a} \\ \vdots \\ dy^{ka} \end{bmatrix}^T \begin{bmatrix} \bar{g}_{0a0b} & \bar{g}_{0a1b} & \cdots & \bar{g}_{0akb} \\ \bar{g}_{1a0b} & \bar{g}_{1a1b} & \cdots & \bar{g}_{1akb} \\ \vdots & & & \\ \bar{g}_{ka0b} & \bar{g}_{ka1b} & \cdots & \bar{g}_{kakb} \end{bmatrix} \otimes \begin{bmatrix} dy^{0b} \\ dy^{1b} \\ \vdots \\ dy^{kb} \end{bmatrix}.$$

In the basis B^* we have:

(19)
$$G = g_{\alpha b\ \beta b} \delta y^{\alpha a} \otimes \delta y^{\beta b}.$$

If in (2.2) d is substituted by δ, $\bar{g}$ by g we obtain the matrix representation of G in the basis B^*.

From (1.7) it is clear, that there are so many adapted basis B^* of $T^*(E)$ as many functions $M^{(\alpha+\beta)a}_{\alpha b}$ can be found so that the coordinate transformations of type (1.1) satisfy (1.8). In this section we shall determine such an adapted basis B^* of $T^*(E)$ in which $T^*_{V_0}, T^*_{V_1}, T^*_{V_2}, \ldots, T^*_{V_k}$ are mutually orthogonal subspaces of $T^*(E)$ with respect to the given metric G ((2.1)). This condition will be satisfied if $g_{\alpha a\ \beta b} = 0$, $\forall \alpha \neq \beta$.

From (2.1), (2.3) and (1.16) we get:

(20)
$$\begin{aligned}
& g_{\alpha a\ \beta b} \delta y^{\alpha a} \otimes \delta y^{\beta b} = \bar{g}_{\gamma c\ \delta d} (\delta y^{\gamma c} - N^{\gamma c}_{(\gamma-1)c} \delta^{(\gamma-1)e}_y - \cdots - N^{\gamma c}_{0e} \delta y^{0e}) \otimes \\
& (\delta y^{\delta d} - N^{\delta d}_{(\delta-1)e} \delta^{(\delta-1)e}_y - \cdots - N^{\delta d}_{0e} \delta y^{0e}),\\
& \alpha, \beta, \gamma, \delta = 0, 1, 2, \ldots, k.
\end{aligned}$$

For the beginning we shall take $k = 3$. After longer calculation, using the symmetry of the metric tensor $G(g_{\alpha a\ \beta b} = g_{\beta b\ \alpha a}, \bar{g}_{\alpha a\ \beta b} = \bar{g}_{\beta b\ \alpha a})$, we obtain the coefficients of $\delta y^{\alpha a} \otimes \delta y^{\beta b}$ in the following way:

$$\begin{aligned} \delta y^{0a} \otimes \delta y^{0b} : g_{0a\ 0b} &= \bar{g}_{0a\ 0b} - \bar{g}_{1f\ 0b} N^{1f}_{0a} - \bar{g}_{2f\ 0b} N^{2f}_{0a} - \bar{g}_{3f\ 0b} N^{3f}_{0a} \\ &\quad - N^{1d}_{0b}(\bar{g}_{0a\ 1d} - \bar{g}_{1f\ 1d} N^{1f}_{0a} - \bar{g}_{2f\ 1d} N^{2f}_{0a} - \bar{g}_{3f\ 1d} N^{3f}_{0a}) \\ \text{(a)} \qquad &\quad - N^{2d}_{0b}(\bar{g}_{0a\ 2d} - \bar{g}_{1f\ 2d} N^{1f}_{0a} - \bar{g}_{2f\ 2d} N^{2f}_{0a} - \bar{g}_{3f\ 2d} N^{3f}_{0a}) \\ &\quad - N^{3b}_{0b}(\bar{g}_{0a\ 3d} - \bar{g}_{1f\ 3d} N^{1f}_{0a} - \bar{g}_{2f\ 3d} N^{2f}_{0a} - \bar{g}_{3f\ 3d} N^{3f}_{0a}), \end{aligned}$$

$$\begin{aligned} \delta y^{0a} \otimes \delta y^{1b} : g_{0a\ 1b} &= \bar{g}_{0a\ 1b} - \bar{g}_{1f\ 1b} N^{1f}_{0a} - \bar{g}_{2f\ 1b} N^{2f}_{0a} - \bar{g}_{3f\ 1b} N^{3f}_{0a} \\ \text{(b)} \qquad &\quad - N^{2d}_{1b}(\bar{g}_{0a\ 2d} - \bar{g}_{1f\ 2d} N^{1f}_{0a} - \bar{g}_{2f\ 2d} N^{2f}_{0a} - \bar{g}_{3f\ 2d} N^{3f}_{0a}) \\ &\quad - N^{3d}_{1b}(\bar{g}_{0a\ 3d} - \bar{g}_{1f\ 3d} N^{1f}_{0a} - \bar{g}_{2f\ 3d} N^{2f}_{0a} - \bar{g}_{3f\ 3d} N^{3f}_{0a}), \end{aligned}$$

$$\begin{aligned} \delta y^{0a} \otimes \delta y^{2b} : g_{0a\ 2b} &= \bar{g}_{0a\ 2b} - \bar{g}_{1f\ 2b} N^{1f}_{0a} - \bar{g}_{2f\ 2b} N^{2f}_{0a} - \bar{g}_{3f\ 2b} N^{3f}_{0a} \\ \text{(c)} \qquad &\quad - N^{3d}_{2b}(\bar{g}_{0a\ 3d} - \bar{g}_{1f\ 3d} N^{1f}_{0a} - \bar{g}_{2f\ 3d} N^{2f}_{0a} - \bar{g}_{3f\ 3d} N^{3f}_{0a}), \end{aligned}$$

$$\text{(d) } \delta y^{0a} \otimes \delta y^{3b} : g_{0a\ 3b} = \bar{g}_{0a\ 3b} - \bar{g}_{1f\ 3b} N^{1f}_{0a} - \bar{g}_{2f\ 3b} N^{2f}_{0a} - \bar{g}_{3f\ 3b} N^{3f}_{0a},$$

$$\begin{aligned} \delta y^{1a} \otimes \delta y^{1b} : g_{1a\ 1b} &= \bar{g}_{1a\ 1b} - \bar{g}_{2f\ 1b} N^{2f}_{1a} - \bar{g}_{3f\ 1b} N^{3f}_{1a} \\ \text{(e)} \qquad &\quad - N^{2d}_{1b}(\bar{g}_{1a\ 2d} - \bar{g}_{2f\ 2d} N^{2f}_{1a} - \bar{g}_{3f\ 2d} N^{3f}_{1a}) \\ &\quad - N^{3d}_{1b}(\bar{g}_{1a\ 3d} - \bar{g}_{2f\ 3d} N^{2f}_{1a} - \bar{g}_{3f\ 3d} N^{3f}_{1a}), \end{aligned}$$

$$\begin{aligned} \delta y^{1a} \otimes \delta y^{2b} : g_{1a\ 2b} &= \bar{g}_{1a\ 2b} - \bar{g}_{2f\ 2b} N^{2f}_{1a} - \bar{g}_{3f\ 2b} N^{3f}_{1a} \\ \text{(f)} \qquad &\quad - N^{3d}_{2b}(\bar{g}_{1a\ 3d} - \bar{g}_{2f\ 3d} N^{2f}_{1a} - \bar{g}_{3f\ 3d} N^{3f}_{1a}), \end{aligned}$$

$$\text{(g) } \delta y^{1a} \otimes \delta y^{3b} : g_{1a\ 3b} = \bar{g}_{1a\ 3b} - \bar{g}_{2f\ 3b} N^{2f}_{1a} - \bar{g}_{3f\ 3b} N^{3f}_{1a},$$

$$\begin{aligned} \delta y^{2a} \otimes \delta y^{2b} : g_{2a\ 2b} &= \bar{g}_{2a\ 2b} - \bar{g}_{3f\ 2b} N^{3f}_{2a} \\ \text{(h)} \qquad &\quad - N^{3d}_{2b}(\bar{g}_{2a\ 3d} - \bar{g}_{3f\ 3d} N^{3f}_{2a}), \end{aligned}$$

$$\text{(i) } \delta y^{2a} \otimes \delta y^{3b} : g_{2a\ 3b} = \bar{g}_{2a\ 3b} - \bar{g}_{3f\ 3b} N^{3f}_{2a},$$

$$\text{(j) } \delta y^{3a} \otimes \delta y^{3b} : g_{3a\ 3b} = \bar{g}_{3a\ 3b}.$$

From the above equation we have:

Theorem 2.1. *$T^*_{V_3}$ is orthogonal to $T^*_{V_2}(g_{2a\ 3b} = 0)$ iff*

$$\bar{g}_{3f\ 3b} N^{3f}_{2a} = \bar{g}_{2a\ 3b}. \qquad (21)$$

Proposition 2.1. *$T^*_{V_3}$ is orthogonal to $T^*_{V_1}(g_{1a\ 3b}=0)$ iff*

$$\bar{g}_{1a\ 3b} - \bar{g}_{2f\ 3b}N^{2f}_{1a} - \bar{g}_{3f\ 3b}N^{3f}_{1a} = 0. \tag{22}$$

Proposition 2.2. *If $T^*_{V_3}$ is orthogonal to $T^*_{V_1}$, then $T^*_{V_1}$ is orthogonal to $T^*_{V_2}(g_{1a\ 2b} = 0)$ iff*

$$\bar{g}_{1a\ 2b} - g_{2f\ 2b}N^{2f}_{1a} - \bar{g}_{3f\ 2b}N^{3f}_{1a} = 0.$$

Theorem 2.2. *$T^*_{V_3}$ is orthogonal to $T^*_{V_1}$ and $T^*_{V_1}$ is orthogonal to T^*_{V2} iff N^{2f}_{1a} and N^{3f}_{1a} are the solutions of the matrix equation*

$$\begin{bmatrix} \bar{g}_{2b\ 2f} & \bar{g}_{2b\ 3f} \\ \bar{g}_{3b\ 2f} & \bar{g}_{3b\ 3f} \end{bmatrix} \begin{bmatrix} N^{2f}_{1a} \\ N^{3f}_{1a} \end{bmatrix} = \begin{bmatrix} \bar{g}_{1a\ 2b} \\ \bar{g}_{1a\ 3b} \end{bmatrix}. \tag{23}$$

Proposition 2.3. *$T^*_{V_3}$ is orthogonal to $T^*_{V_0}$ iff*

$$\bar{g}_{0a\ 3b} - \bar{g}_{1f\ 3b}N^{1f}_{0a} - g_{2f\ 3b}N^{2f}_{0a} - \bar{g}_{3f\ 3b}N^{3f}_{0a} = 0. \tag{24}$$

Proposition 2.4. *If $T^*_{V_0}$ is orthogonal to $T^*_{V_3}$, then $T^*_{V_0}$ is orthogonal to $T^*_{V_2}$ iff*

$$\bar{g}_{0a\ 2b} - g_{1f\ 2b}N^{1f}_{0a} - \bar{g}_{2f\ 2b}N^{2f}_{0a} - \bar{g}_{3f\ 2b}N^{3f}_{0a} = 0. \tag{25}$$

Proposition 2.5. *If $T^*_{V_0}$ is orthogonal to $T^*_{V_3}$ and $T^*_{V_2}$, then $T^*_{V_0}$ is orthogonal to $T^*_{V_1}$ iff*

$$\bar{g}_{0a\ 1b} - \bar{g}_{1f\ 1b}N^{1f}_{0a} - \bar{g}_{2f\ 1b}N^{2f}_{0a} - \bar{g}_{3f\ 1b}N^{3f}_{0a} = 0. \tag{26}$$

Theorem 2.3. *$T^*_{V_0}$ is orthogonal to $T^*_{V_1}$, $T^*_{V_2}$ and $T^*_{V_3}$ iff N^{1f}_{0a}, N^{2f}_{0a} and N^{3f}_{0a} are the solutions of the following equation:*

$$\begin{bmatrix} \bar{g}_{1b\ 1f} & \bar{g}_{1b\ 2f} & \bar{g}_{1b\ 3f} \\ \bar{g}_{2b\ 1f} & \bar{g}_{2b\ 2f} & \bar{g}_{2b\ 3f} \\ \bar{g}_{3b\ 1f} & \bar{g}_{3b\ 2f} & \bar{g}_{3b\ 3f} \end{bmatrix} \begin{bmatrix} N^{1f}_{0a} \\ N^{2f}_{0a} \\ N^{3f}_{0a} \end{bmatrix} = \begin{bmatrix} \bar{g}_{0a\ 1b} \\ \bar{g}_{0a\ 2b} \\ \bar{g}_{0a\ 3b} \end{bmatrix}. \tag{27}$$

Theorem 2.4. *The necessary and sufficient conditions that the subspaces $T^*_{V_0}$, $T^*_{V_1}$, $T^*_{V_2}$ and $T^*_{V_3}$ of $T^*(Osc^3M)$ formed by $\{\delta y^{0a}\}$, $\{\delta y^{1a}\}$, $\{\delta y^{2a}\}$ and $\{\delta y^{3a}\}$ respectively are mutually orthogonal with respect to the given metric G (given by (2.2)) are the equations (2.5), (2.7) and (2.11).*

Theorem 2.5. *When $T^*_{V_0}$, $T^*_{V_1}$, $T^*_{V_2}$ and $T^*_{V_3}$ are mutually orthogonal subspaces of $T^*(Osc^3M)$, with respect to the metric G, then:*

$$\begin{aligned} g_{0a\ 0b} &= \bar{g}_{0a\ 0b} - \bar{g}_{1f\ 0b}N^{1f}_{0a}\bar{g}_{2f\ 0b}N^{2f}_{0a} - \bar{g}_{3f\ 0b}N^{3f}_{0a}, \\ g_{1a\ 1b} &= \bar{g}_{1a\ 1b} - \bar{g}_{2f\ 1b}N^{2f}_{1a} - \bar{g}_{3f\ 1b}N^{3f}_{1a}, \\ g_{2a\ 2b} &= \bar{g}_{2a\ 2b} - \bar{g}_{3f\ 2b}N^{3f}_{2a}, \\ g_{3a\ 3b} &= \bar{g}_{3a\ 3b}. \end{aligned} \tag{28}$$

Let us introduce the notations:

$$
(29)\qquad
\begin{aligned}
\bar{G}_{\beta k} &= \begin{bmatrix} \bar{g}_{\beta b\ \beta f} & \cdots & \bar{g}_{\beta b\ kf} \\ \vdots & & \\ \bar{g}_{kb\ \beta f} & \cdots & \bar{g}_{kb\ kf} \end{bmatrix},\\
\bar{G}^{\beta k} &= (G_{\beta k})^{-1} = \begin{bmatrix} \bar{g}^{\beta b\ \beta f} & \cdots & \bar{g}^{\beta b\ kf} \\ \bar{g}^{kb\ \beta f} & \cdots & \bar{g}^{kb\ kf} \end{bmatrix},\\
\bar{G}_{\alpha,\beta k} &= \begin{bmatrix} \bar{g}_{\alpha a\ \beta b} \\ \vdots \\ \bar{g}_{\alpha a\ kb} \end{bmatrix}.
\end{aligned}
$$

From (2.13) it is clear that $\bar{G}_{\beta k}$ and $\bar{G}^{\beta k}$ are matrices of type $(k-\beta+1)\times(k-\beta+1)$ and $\bar{G}_{\alpha,\beta k}$ is a matrix of type $(k-\beta+1)\times 1$. The elements of all three matrices are submatrices of type $n\times n$.

As in all propositions and theorems of this section it was supposed that $k=3$, so equations (2.5), (2.7) and (2.11) can be writtten in the form:

$$
(30)\qquad N^{3b}_{2a} = \bar{g}^{3b\ 3e} g_{2a\ 3b} = \bar{G}^{33}\bar{G}_{2,33},
$$

$$
(31)\qquad \begin{bmatrix} N^{2e}_{1a} \\ N^{3e}_{1a} \end{bmatrix} = \begin{bmatrix} \bar{g}^{2e\ 2b} & \bar{g}^{2e\ 3b} \\ \bar{g}^{3e\ 2b} & \bar{g}^{3e\ 3b} \end{bmatrix} \begin{bmatrix} \bar{g}_{1a\ 2b} \\ \bar{g}_{1a\ 3b} \end{bmatrix} = \bar{G}^{23}\bar{G}_{1,23},
$$

$$
(32)\qquad \begin{bmatrix} N^{1e}_{0a} \\ N^{2e}_{0a} \\ N^{3e}_{0a} \end{bmatrix} = \begin{bmatrix} \bar{g}^{1e\ 1b} & \bar{g}^{1e\ 2b} & \bar{g}^{1e\ 3b} \\ \bar{g}^{2e\ 1b} & \bar{g}^{2e\ 2b} & \bar{g}^{2e\ 3b} \\ \bar{g}^{3e\ 1b} & \bar{g}^{3e\ 2b} & \bar{g}^{3e\ 3b} \end{bmatrix} \begin{bmatrix} \bar{g}_{0a\ 1b} \\ \bar{g}_{0a\ 2b} \\ \bar{g}_{0a\ 3b} \end{bmatrix} = \bar{G}^{13}\bar{G}_{0,13}.
$$

The matrices on the right hand side in (2.14), (2.15) and (2.16) are the correspondent inverse matrices which appeare in (2.5), (2.7) and (2.11).

Now we have:

Theorem 2.4'. *The necessary and sufficient conditions that the subspaces $T^*_{V_0}$, $T^*_{V_1}$, $T^*_{V_2}$ and $T^*_{V_3}$ of $T^*(Osc^3M)$ formed by $\{\delta y^{0a}\}$, $\{\delta y^{1a}\}$, $\{\delta y^{2a}\}$ and $\{\delta y^{3a}\}$ respectively be mutually orthogonal with respect to the given metric G are the equations (2.14), (2.15) and (2.16).*

The main result is the following theorem:

Theorem 2.6. *If in $T^*(Osc^kM)$ the metric tensor G is given by (2.2), then there exists one and only one adapted basis $\{\delta y^{0a}, \delta y^{1a}, \dots, \delta y^{ka}\}$ such that the subspaces $T^*_{V_0}, T^*_{V_1}, \dots, T^*_{V_k}$ of $T^*(Osc^kM)$ are mutually orthogonal. The vectors of such unique base are determined by (1.16) and the coefficients N are given by:*

$$
\begin{bmatrix} N^{\alpha e}_{\alpha-1} \\ N^{(\alpha+1)e}_{\alpha-1} \\ N^{ke}_{\alpha-1} \end{bmatrix} = \bar{G}^{\alpha k}\bar{G}_{\alpha-1,\alpha k} \quad \alpha = 1,2,\dots,k.
$$

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