SPECIAL CHARACTERISTICALLY NILPOTENT LIE ALGEBRA OF DIMENSION EIGHT

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Abstract

Let g_8 be a characteristically Nilpotent Lie algebra of dimension eight. The aim of the present paper is to study the group of Lie automorphisms of g_8.

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1. Introduction

Let $\mathbf{g}_8$ be a characteristically Nilpotent Lie algebra of dimension eight over a field F of characteristic zero. This comes from the classification of Nilpotent Lie algebras of dimension eight, whose maximal abelian ideal is 6. The problem of the determination of the group $Aut(g)$ of Lie automorphisms of a characteristically Nilpotent Lie algebra g is an open one. It is known that no characteristically Nilpotent Lie algebra of dimension ≤ 6 exists. G. Favre in [10] has studied the Lie automorphisms of a characteristically Nilpotent Lie algebra of dimension 7. It has already been proved in [11] by professor Tsagas and the author that the group of Lie automorphisms of the characteristically Nilpotent Lie algebras of dimension 7 contains unipotent Lie automorphisms.

In dimension 8 we try to find a subclass of Nilpotent Lie algebras, called *characteristically Nilpotent Lie algebras*, whose group of Lie automorphisms consists of unipotent Lie automorphisms.

In this paper we present some conclusions concerning the Lie algebra $\mathbf{g}_8$. These conclusions are part of the paper [12]. The purpose of this paper is to determine the group of Lie automorphisms of $\mathbf{g}_8$.

The whole paper contains three sections. Each of them is analyzed as follows. The first section is the introduction. Basic elements of Lie algebras, special Lie algebras and the group of Lie automorphisms are given in the second section. The third section includes the study of the group of Lie automorphisms of the characteristically Nilpotent Lie algebra $\mathbf{g}_8$ of dimension eight.

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2. Basic elements

Let g be a Lie algebra over a field F of characteristic zero and of dimension n.

If there exists an integer $q \geq 2$ such that $C^q g = \{0\}$, then the Lie algebra g is called *Nilpotent* of nilpotency q.

A linear mapping f on g is called *derivation*, if it satisfies the relation:

$$f[x,y] = [fx,y] + [x,fy],\ (\forall)x,y \in g.$$

The set of all derivations f on g is denoted by $D(g)$, that is:

$$D(g) = \{f\ /\ f : g \to g,\ \text{linear and } f[x,y] = [fx,y] + [x,fy].$$

The set of derivations $D(g)$ on g is another Lie algebra.

The Lie algebra g is called *characteristically Nilpotent* if the Lie algebra $D(g)$ is Nilpotent.

Let us consider a Lie algebra g over a field F of characteristic zero. The *Lie automorphism* f of g is defined as follows:

$$f : g \to g,\ f \text{ linear mapping}$$

and

$$f : [x,y] \to f([x,y]) = [f(x), f(y)],\ (\forall)x,y \in g.$$

The set of all Lie automorphisms of a given Lie algebra g is denoted by $Aut(g)$ and is defined by:

$$Aut(g) = \{f\ /\ f : g \to g,\ f \text{ is linear, } f([x,y]) = [f(x), f(y)].$$

This is a group with the usual composition of automorphisms as its inner law.

A Lie automorphism is called *unipotent* if there is a base such that its representation with respect to the base $\{x_1, ..., x_n\}$ has the form:

$$\begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ b_{21} & 1 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ b_{n-1\,1} & b_{n-1\,2} & b_{n-1\,3} & ... & 1 & 0 \\ b_{n1} & b_{n2} & b_{n3} & ... & b_{n\,n-1} & 1 \end{bmatrix}.$$

3. The study of the group of Lie authomorphisms of the characteristically nilpotent Lie algebra $\mathbf{g}_8$

Let $\mathbf{g}_8$ be the following characteristically Nilpotent Lie algebra of dimension eight over a field F of characteristic zero:

$$\begin{array}{llll} [x_1,x_2] = x_3 + x_6 & [x_1,x_3] = x_4 & [x_1,x_4] = x_5 & [x_1,x_6] = x_8 \\ [x_2,x_3] = x_5 & [x_2,x_6] = x_7 & [x_2,x_7] = \gamma x_8, & \gamma \neq 0. \end{array}$$

Let Θ be a Lie automorphism of the characteristically Nilpotent Lie algebra $\mathbf{g}_8$ and $\{x_1, ..., x_8\}$ be a base of $\mathbf{g}_8$.

It is known that Θ can be represented with respect to the base $\{x_1, ..., x_8\}$ by the matrix:

$$T = \begin{bmatrix} b_{11} & b_{12} & ... & b_{18} \\ b_{21} & b_{22} & ... & b_{28} \\ ... & ... & ... & ... \\ b_{81} & b_{82} & ... & b_{88} \end{bmatrix}. \tag{3.1}$$

If we apply this Lie automorphism Θ on the Lie brackets of the Lie algebra $\mathbf{g}_8$, including those which are zero, then we obtain:

$$\Theta([x_1, x_2]) = \Theta(x_3 + x_6), \quad \Theta([x_1, x_3]) = \Theta(x_4), \quad \Theta([x_1, x_4]) = \Theta(x_5), \tag{3.2}$$

$$\Theta([x_1, x_6]) = \Theta(x_8), \qquad \Theta([x_2, x_3]) = \Theta(x_5), \quad \Theta([x_2, x_6]) = \Theta(x_7), \tag{3.3}$$

$$\Theta([x_2, x_7]) = \Theta(\gamma x_8), \qquad \Theta([x_k, x_\lambda]) = 0, \tag{3.4}$$

where $[x_k, x_\lambda] = 0$ means that all other Lie brackets are zero.

The relations (3.2), (3.3) and (3.4), after the necessary calculations, give:

$$b_{12} = 0,\ b_{13} = b_{23} = 0,\ b_{14} = b_{24} = b_{34} = 0,\ b_{15} = b_{25} = b_{35} = b_{45} = 0,$$
$$b_{16} = b_{26} = b_{36} = b_{46} = b_{56} = 0,\ b_{17} = b_{27} = b_{37} = b_{47} = b_{57} = b_{67} = 0,$$
$$b_{18} = b_{28} = b_{38} = b_{48} = b_{58} = b_{68} = b_{78} = 0,\ b_{21} = 0,$$
$$b_{63} = b_{64} = b_{65} = 0,\ b_{73} = b_{74} = b_{75} = 0,\ b_{84} = b_{85} = 0,$$
$$b_{56} = b_{42} - b_{31} - b_{53},\ b_{43} = b_{54} = b_{32},\ b_{76} = -b_{61},$$
$$b_{86} = b_{62} - b_{71}\gamma - b_{83},\ b_{87} = -b_{61}\gamma,$$
$$b_{11} = b_{22} = b_{33} = b_{44} = b_{55} = b_{66} = b_{77} = b_{88} = 1.$$

Finally the matrix T takes the following form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & 1 & 0 & 0 & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{32} & 1 & 0 & 0 & 0 & 0 \\ b_{51} & b_{52} & b_{53} & b_{32} & 1 & 0 & 0 & 0 \\ b_{61} & b_{62} & 0 & 0 & 0 & 1 & 0 & 0 \\ b_{71} & b_{72} & 0 & 0 & 0 & -b_{61} & 1 & 0 \\ b_{81} & b_{82} & b_{83} & 0 & 0 & b_{62} - b_{71}\gamma - b_{83} & -b_{61}\gamma & 1 \end{bmatrix}$$

From the above we have proved that Θ is unipotent. Now we can state the following theorem:

Theorem. *The characteristically Nilpotent Lie algebra $\mathbf{g_8}$ of dimension eight over a field $\mathbf{F}$ of characteristic zero, has the property that the group $\mathbf{Aut}(\mathbf{g_8})$ consists of unipotent Lie automorphisms.*

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