SPECIAL CLASSIFICATION OF NILPOTENT LIE ALGEBRAS

P. Barbari and A. Kobotis

Abstract

The aim of the present paper is to consider the current problems of Lie algebras such as their classification, the definition of the structure of the algebra of derivations of Lie algebras and the relation between a Lie algebra g and the Lie algebra of its derivations.

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1. Introduction

Let g be a Lie algebra over a field k of characteristic zero. The purpose of this paper is to consider the current problems of Lie algebras. We refer to the classification of Lie algebras and especially to that of the Nilpotent Lie algebras, we try to define the structure of the Lie algebra of derivations $D(g)$ of g and also to find the relations between g and $D(g)$.

The whole paper contains three sections each of them is analyzed as follows. The first section is the introduction. Basic elements of Lie algebras are given in the second one. In the third section are stated some important theorems of Nilpotent and characteristically Nilpotent Lie algebras as well as, some open problems of the above Lie algebras.

2. Basic elements on Lie algebras

We shall give some basic notions used in this paper.

Let g be a Lie algebra over the field k, of characteristic zero and of dimension n. It is known that from this algebra we can form the following sequences of ideals of g:

$$C^0g = g, \quad C^1g = [g, g], ..., \quad C^qg = [g, C^{q-1}g], ... \tag{2.1}$$

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which is called *descending central sequence*, then:

$$C_0 g = 0, \quad C_1 g = \text{centre}(g), ..., \quad C_q g = \text{centre}(g/C_{q-1}g), \dots \tag{2.2}$$

which is called *increasing central sequence* and

$$D^0 g = g, \quad D^1 g = [g, g], ..., \quad D^q g = [D^{q-1}g, D^{q-1}g], ... \tag{2.3}$$

which is called *derived sequence*.

If there exists an integer $q \geq 2$ such that $D^q g = \{0\}$, then the Lie algebra g is called *solvable*.

If there exists an integer $q \geq 2$ such that $C^q g = \{0\}$, then the Lie algebra g is called *Nilpotent* of nilpotency q.

A linear mapping f on g is called *derivation*, if it satisfies the relation:

$$f[x, y] = [fx, y] + [x, fy], \quad (\forall) x, y \in g.$$

The set of all derivations f on g is denoted by $D(g)$, that is

$$D(g) = \{f \ / \ f : g \to g, f \text{ linear and } [x, y] = [fx, y] + [x, fy].$$

The following mapping:

$$ad_x : g \to g, \quad ad_x : y \to ad_x y = [x, y]$$

is a derivation which is called *inner derivation.* The set of all inner derivations is denoted by $I(g)$, which is an ideal of the Lie algebra of derivations $D(g)$. The other derivations on g, which are not inner, are called *outer*, which are denoted by $D_i(g)$. It is known that

$$D(g) = I(g) \oplus D_i(g).$$

We must notice that $D_i(g)$ is an ideal of $D(g)$.

The Lie algebra g is called *characteristically Nilpotent* if the Lie algebra $D(g)$ is Nilpotent.

A Lie algebra g is said to be *simple* if it is non-abelian and has no proper ideals.

A Lie algebra g is said to be *semi-simple* if does not contain any non-zero abelian ideal.

If g is semi-simple then:

$$D(g) = I(g).$$

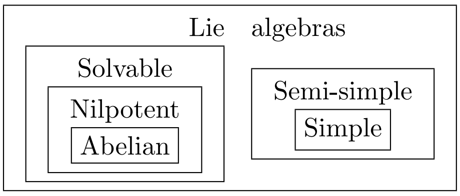
The maximal solvable ideal I of g is called *radical.*

The *center* $Z(g)$ of a Lie algebra g is an ideal of g, such that

$$Z(g) = \{z \in g, \ [x, z] = 0 \text{ for all } x \in g\}.$$

A Lie algebra g is called *abelian* if and only if $Z(g) = g$.

To make the thinks easier we can state the following diagram:



3. Main results

A very important problem is the classification of the Lie algebras, which means finding their Lie brackets.

Levi has proved the following theorem:

THEOREM 3.1 *Let g be a finite-dimensional Lie algebra and let R be a radical of g. Then there exists a semi-simple subalgebra S of g such that g is the direct sum of its linear subspaces R and S:*

$$g = R \oplus S.$$

The semi-simple Lie algebras have already been classified. It is known that the classification of solvable Lie algebras is related to the classification of Nilpotent Lie algebras.

The classification of nilpotent Lie algebras of dimension ≤ 7 has already be done by Dixmier [8], Morosov [9], Safiullina [10], Goze [11], Magnin [7].

The classification of Nilpotent Lie algebras of dimension 8 has been done recently by Tsagas and others.

PROBLEM 1. Another problem is to define the structure of the Lie algebra of derivations $D(g)$ and to find the relation between $D(g)$ and the Lie algebra g.

In [12] Tôgô has proved the following theorem:

THEOREM 3.2 *Let $D(g)$ be the derivation algebra of a Lie algebra g. Then:*
1) $D(g)$ is abelian if and only if g is one-dimensional Lie algebra.
2) $D(g)$ is non-abelian nilpotent if and only if g is characteristically nilpotent Lie algebra.
3) $D(g)$ is non-nilpotent solvable if and only if either g is characteristically solvable and not characteristically nilpotent, or g is the direct sum of a characteristically solvable ideal and an one-dimensional central ideal.
4) $D(g)$ is reductive (semi-simple) if and only if g is reductive (semi-simple respectively).
5) $D(g)$ is the direct sum of a semi-simple ideal and the non-abelian Nilpotent radical if and only if g is the direct sum of a semi-simple ideal and a characteristically Nilpotent ideal.
6) $D(g)$ is the direct sum of a semi-simple ideal and the non-nilpotent radical if and

only if either g is the direct sum of a semi-simple ideal and a characteristically solvable ideal which is not characteristically nilpotent or g is the direct sum of a semi-simple ideal, a characteristically solvable ideal and a one-dimensional central ideal.

PROBLEM 2. Another problem is to study the possibilities of Lie algebra g being radical of another Lie algebra L or derived from a Lie algebra L.

A Nilpotent Lie algebra g is called quasi-cyclic if and only if there is a subspace U of g such that:

$$g = U \oplus U^2 \oplus ... \oplus U^q,\ q \in R,$$

where $U^1 = U$, $U^i = [U, U^{i-1}]$ for $i \geq 2$.

We denote by $C(g)$ the set of all endomorphisms of g, which map g into the center Z of g and $[g, g]$ into $\{0\}$.

It can be proved that $C(g)$ is a subalgebra of $D(g)$. Each element of $C(g)$ is called central derivation.

The Lie algebra of derivations $D(g)$ of g can be written as:

$$D(g) = I(g) + C(g).$$

It is known that $I(g) \cap C(g)$ is not always empty.

A Lie algebra is called T when it is not the radical of any other Lie algebra L having the property

$$D(L) = I(L) + C(L).$$

Tôgô in [13] has proved that any non-abelian Nilpotent Lie algebra belongs to the class of the Lie algebras of type T, if it satisfies one of the following conditions:
(1) g is quasi-cyclic;
(2) the dimension of g is less than 6.

Barbari and Kobotis in [15] have proved the following theorems:

THEOREM 3.3 *The Nilpotent Lie algebras over a field K of characteristic zero of dimension 6 belong to the Lie algebras of type T.*

THEOREM 3.4 *The Nilpotent Lie algebras over a field K of characteristic zero of dimension 7 can be classified in two categories. The first consists of the Lie algebras of type T and the second one of the characteristically Nilpotent Lie algebras.*

OPEN PROBLEM 3. The classification of Nilpotent Lie algebras over a field K of characteristic zero in two categories:
I) Lie algebras of type T;
II) characteristically Nilpotent Lie algebras,
is an open problem.

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