ON THE DPW METHOD FOR THE TANGENT GROUP

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Abstract

The present paper gives a brief description of the general DPW method for harmonic maps and presents aspects regarding its extension to the tangent group case. This is exemplified considering the harmonicity of maps from a Riemannian surface to the homogeneous space $TS^2 = TSU(2)/TU(1)$ via their lifts to the tangent space $TSU(2)$.

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1 The general DPW method

During the last decade, the methods which determine harmonic maps from Riemannian surfaces to symmetric spaces were enriched by the DPW method - a construction which proved its efficiency in various applicative areas: H-surfaces (CMC-surfaces), minimal surfaces, surfaces of constant Gaussian curvature, Willmore surfaces etc.

The prerequisites for the general DPW method are ([7]):

1. A Riemannian compact simply connected surface M of genus $g \geq 1$ and $\mathbb{D} \in \{\mathbb{C}, D^1\}$ its universal cover, where $D^1 = \{z \in \mathbb{C} | \mid z \mid < 1\}$.

2. A compact connected semisimple Lie group G with its Lie algebra $\mathcal{L}(G) = \mathfrak{g}$.

3. An automorphism $\sigma \in Aut(G)$ of order $m \geq 2$ with the fixed point set $G^\sigma \subset K \subset G_0^\sigma$, $\mathfrak{g}^\sigma = \mathcal{L}(K) \stackrel{n}{=} \mathfrak{k}$, where we denoted the induced map of σ on $\mathfrak{g}$ by the same symbol.

4. A solvable subgroup $B \subset K^{\mathbb{C}}$ which provides an Iwasawa decomposition for the complexified group $K^{\mathbb{C}}$

$$K^{\mathbb{C}} = K \cdot B, \qquad K \cap B = \{e\} \tag{1}$$

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and the corresponding splitting of the associated Lie algebras

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{g}, \qquad \mathfrak{g} = \mathcal{L}(B). \tag{2}$$

The goal of the general DPW method is to construct all the harmonic maps

$$f : M \to N = G/K = \pi(G),$$

where π is the projection $\pi : G \to G/K$.

The main features of this construction will be briefly described below. Considering the Cartan decomposition for the case $m = 2$,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad \mathfrak{p} = Ker(\sigma + Id), \mathfrak{k} = Ker(\sigma - Id), \tag{3}$$

the following commutation relations hold true

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{4}$$

The tangent bundle (TN, π, N) of the symmetric space N can be canonically identified with the subbundle $([\mathfrak{p}], pr_1, N) \subset (N \times \mathfrak{g}, pr_1, N)$ via the vector bundle *id*-isomorphism

$$\tau : TN \to [\mathfrak{p}], \tau(X_x) = (x, \beta(X_x)),$$

where for any $X_x = \frac{d}{dt}\,|_{t=0}\ exp\, t\xi.x \in T_xN \subset TN$, $\beta(X_x) = Adg \circ \pi_{\mathfrak{p}} \circ Adg^{-1}(\xi)$, $\pi_{\mathfrak{p}}$ being the canonical projection induced by (3). It can be shown ([3]) that the Levi-Civita connection $\overset{N}{\nabla}$ of TN is just the flat connection followed by projection on the fibers of $[\mathfrak{p}]$,

$$\beta(\overset{N}{\nabla}_{X_x} Y) = \pi_{[\mathfrak{p}]_x}(\overset{flat}{\nabla}_{X_x}(\beta(Y)))), \tag{5}$$

where $\pi_{[\mathfrak{p}]_x} = Adg \circ \pi_{\mathfrak{p}}$, $x = \hat{g}$, $X_x \in T_xN$, $Y \in \mathcal{X}(N)$. The relation (5) can be written briefly $\quad \beta \circ \overset{N}{\nabla} = \pi_{[\mathfrak{p}]} \circ d \circ \beta$.

For M simply connected, any map $f : M \to G/K$ lifts to $F : M \to G, f = \pi \circ F$, and the $\mathfrak{g}$-valued 1-form $\alpha = F^{-1}dF \in \Lambda^1(M, \mathfrak{g})$ splits relative to (3), $\alpha = \alpha_0 + \alpha_1$. Also, the splitting $TM^{\mathbb{C}} = T'M \oplus T''M$ of $TM^{\mathbb{C}}$ into its (1,0) and (0,1) tangent subspaces, induces the decompositions

$$\overset{N}{\nabla} = \nabla' + \nabla'', \ d = \partial + \bar{\partial}, \ \alpha_i = \alpha'_i + \alpha''_i, \ i = \overline{0, 1}, \tag{6}$$

and hence $\alpha = \alpha'_1 + \alpha_0 + \alpha''_1$.

It is non-trivial to show that the harmonicity of f rewrites $\nabla''\partial f = 0$, relation which using (5) becomes, in terms of α:

$$\partial\alpha'_1 + [\alpha''_0 \wedge \alpha'_1] = 0. \tag{7}$$

This condition, together with the Maurer-Cartan equations

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0, \tag{8}$$

which represent the integrability condition of the equation in F:

$$\alpha = F^{-1}dF, \tag{9}$$

provide the system (equivalent to (7) and (8)):

$$\begin{cases} d\alpha_0 + \frac{1}{2}[\alpha_0 \wedge \alpha_0] = -[\alpha_1' \wedge \alpha_1''], \\ \bar{\partial}\alpha_1' + \frac{1}{2}[\alpha_0 \wedge \alpha_1'] = 0. \end{cases} \tag{10}$$

These form a set of iff conditions for the existence and harmonicity (up to G-translations) of the function f, constructed ultimately as the projection of the lifted frame F produced by the pair of forms $\alpha_0 \in \Lambda^1(M, \mathfrak{k})$ and $\alpha_1 \in \Lambda^1(M, \mathfrak{p})$ via (9). Also, the system (10) turns out to be equivalent to the integrability conditions $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ of the "loopified" form

$$\alpha_\lambda = \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1'', \tag{11}$$

whose associated system

$$\tilde{F}^{-1}d\tilde{F} = \alpha_\lambda \tag{12}$$

can be integrated to produce "the extended lift"

$$\tilde{F} : M \to \Lambda G_\sigma,$$

unique up to a gauge transformation $H : M \to K$. We denoted by

$$\Lambda G_\sigma \overset{n}{=} \{h \mid h : S^1 \to G, h(e^{2\pi i/m}\lambda) = \sigma h(\lambda), \forall \lambda \in S^1 \equiv U(1)\},$$

the group of G-valued twisted loops.

In fact, for M simply connected (practically for $M = \mathbb{D}$), the harmonicity of f is equivalent to the existence of a holomorphic map $\tilde{f} : M \to \Lambda G_\sigma / K$ which is provided by a ΛG_σ-translation of a 1-form

$$\theta_\eta = \lambda^{-1}\eta + \lambda\bar{\eta} \in \Lambda\mathfrak{p}^{\mathbb{C}};$$

this map is related to f via $\tilde{f}\mid_{\lambda=1} = f$. The whole family of maps

$$\tilde{f}_\lambda \overset{n}{=} \tilde{f}(\cdot)(\lambda) : M \to G/K,\ \forall \lambda \in S^1$$

obtained from such holomorphic forms are harmonic [7].

The procedure which constructs (mod emerging singularities), the harmonic functions f from $\mathfrak{p}^{\mathbb{C}}$-valued holomorphic 1-forms, is called *the Weierstrass representation of harmonic maps* and is described below.

1. For M simply connected domain, the space of harmonic maps

$$\mathcal{H} = \{f : M \to G/K \mid f \text{ harmonic},\ f(0) = eK\}$$

is in bijective correspondence with the factorized set of extended lifts

$$\mathcal{H}' = \{\tilde{F} : M \to \Lambda G_\sigma \mid \tilde{F}(0) = k \in K,\ \tilde{F}^{-1} d\tilde{F} = \alpha_\lambda \text{as in (11)}\}/\sim,$$

where

$$\tilde{F}_1 \sim \tilde{F}_2 \ \Leftrightarrow\ \tilde{F}_2 = \tilde{F}_1 \cdot H,\ H : M \to K. \tag{13}$$

2. The vector space of "holomorphic potentials"

$$\mathcal{P} = \{\mu \mid \mu = \begin{array}{l} \sum_{k \geq -1} \lambda^k \mu_k,\ \mu_{2i} \in \Lambda^1_{hol}(M, \mathfrak{k}^{\mathbb{C}}), \\ \mu_{2i-1} \in \Lambda^1_{hol}(M, \mathfrak{p}^{\mathbb{C}}),\ i \in \mathbb{N},\ \lambda \in S^1\} \end{array}$$

provides the harmonic maps of $\mathcal{H}'$ in the following way: the solution $g : M \to \Lambda {G_\sigma}^{\mathbb{C}}$ of the system

$$g^{-1} dg = \mu,\ g(0) = e \tag{14}$$

is spited by Iwasawa decomposition

$$g = \Phi b_+,\ \Phi(0) = e$$

providing the family of loops $\Phi : M \to \Lambda G_\sigma$, which prove to be - for each $\lambda \in S^1$, extended lifts $\Phi = \tilde{F}$ whose equivalence classes (mod (13)) lie in $\mathcal{H}'$ and provide thus corresponding harmonic functions in $\mathcal{H}$.

The map $\mu \in \mathcal{P} \stackrel{W}{\to} [\Phi] \in \mathcal{H}'$ is called *the Weierstrass representation for harmonic maps*, and any harmonic map in $\mathcal{H} \equiv \mathcal{H}'$ originates from some holomorphic potential.

Moreover, the loops of ΛG_σ are shown to be subject to a special "dressing" action (with positive-power holomorphic based loops - [7]) which induces a corresponding action on $\mathcal{H}$; as a consequence, the fibers of the Weierstrass map are the orbits of the action on $\mathcal{P}$ of the based holomorphic gauge group:

$$\mathcal{G}_0 = \{h : M \to \Lambda G_\sigma \mid h_{\bar{z}} = 0, h(0) = e,\ h \text{ extends holo to } D^1\}.$$

For non-based loops (free from the condition $h(0) = e$), the induced action on $\mathcal{H}$ produces new harmonic maps from a given one. As a co-result, it can be shown that any harmonic map is provided by holomorphic potentials $\mu \in \mathcal{P}$ which have no even powers in λ.

For example, the Weierstrass representation scheme can be applied to finite harmonic maps [7], which are shown to emerge from holomorphic potentials of the form

$$\mu_k = \sum_{|j+1| \leq 2k} \lambda^{2k+j} \xi_j \in \mathcal{P},\ k \in \mathbb{N}.$$

A remarkable fact is that the "meromorphic potentials"

$$\mathcal{P}_m = \{\xi = \lambda^{-1} \eta,\ \eta \in \Lambda^1_{mero}(M, \mathfrak{p}^{\mathbb{C}})\}$$

provide also by means of the Weierstrass representation *all* the harmonic maps $f : M \to G/K$.

A central question is to obtain the exact form of meromorphic potentials which provide the extended frames $\tilde{F}(z, \bar{z}, \lambda)$. This goal is accomplished by solving "the $\bar{\partial}$-problem" [2] and applying a generalization of the Grauert theorem, to obtain the global holomorphic loop $\tilde{g}$ given by the relation:

$$\tilde{F} = \tilde{g} w_+^{-1}.$$

Then the holomorphic potential is $\xi = \tilde{g}^{-1} d\tilde{g} \in \mathcal{P}$ and the meromorphic potential is $\xi = \tilde{g}_-^{-1} d\tilde{g}_- \in \mathcal{P}_m$ - obtained from the negative loop $\tilde{g}_-$ given by the further Birkhoff decomposition of $\tilde{g}$:

$$\tilde{g} = \tilde{g}_- \tilde{g}_+,$$

where $\tilde{g}_\pm \in \Lambda^\pm G_\sigma^{\mathbb{C}}$ and

$$\Lambda^+ {G_\sigma}^{\mathbb{C}} = \{g \in \Lambda {G_\sigma}^{\mathbb{C}} \mid g(0) = e,\ g \text{ extends holomorphically to } D^1\},$$
$$\Lambda^- {G_\sigma}^{\mathbb{C}} = \{g \in \Lambda {G_\sigma}^{\mathbb{C}} \mid g(\infty) = e,\ g \text{ extends holomorphically to } \mathbb{C} \setminus D^1\}.$$

2 The tangent group case

The main decompositions of the DPW procedure described in the first section can be extended to general connected Lie groups which admit a faithful representation and the DPW procedure itself - for homogeneous locally symmetric spaces [4]. As a particular case, we shall discuss the homogeneous reducible space $N = T(G/K) = TG/TK$, where G is a connected Lie group, $K \subset G$ is a closed subgroup and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a splitting which satisfies (4). Then one has $T\mathfrak{g} = T\mathfrak{k} \oplus T\mathfrak{p}$.

A function $f : \mathbb{D} \to N = TG/TK$ can be lifted to a frame $F : \mathbb{D} \to TG$, which is harmonic if it minimizes the energy action:

$$\delta \int \mid dF \mid dV = 0,$$

with the norm provided by the metric on $T^*\mathbb{D} \otimes F^{-1}TG$. In the given case, for TG endowed with a metric produced by a scalar product on its semidirect sum Lie algebra $\mathcal{L}(TG) = \mathfrak{g} \circledS \mathfrak{g}$ ([12]), the Euler-Lagrange equations write:

$$[ad(F^{-1}F_x)]^*(F^{-1}F_x) + [ad(F^{-1}F_y)]^*(F^{-1}F_y) = $$
$$= (F^{-1}F_x)_x + (F^{-1}F_y)_y,\ \forall (x, y) \in \mathbb{D}. \qquad (15)$$

For an invariant metric on TG, the relations (15) become

$$\partial(F^{-1}F_{\bar{z}}) + \bar{\partial}(F^{-1}F_z) = 0, \qquad (16)$$

where $\partial = \frac{\partial}{\partial z}, z = x + iy$ and coincide with the ones derived in [14]. The bi-invariant metrics have on the Lie algebra $\mathcal{L}(TG) = \mathfrak{g} \circledS \mathfrak{g}$ (called also "the inflation of $\mathfrak{g}$", [1]) the form:

$$\Gamma = \begin{pmatrix} P & Q \\ Q & 0 \end{pmatrix}, \qquad Q, P : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

with Q, P invariant bilinear forms on $\mathfrak{g}$ and Q non-degenerate. If G admits an invariant metric, TG belongs to the general family of groups whose Lie algebras form the smallest class of Lie algebras which contain the simple and abelian groups and which is stable under direct sum and double extension, i.e., the family of groups which posses an invariant metric ([10]). In particular, $\mathcal{L}(TG)$ is a trivial double extension by an algebra endowed with an invariant metric Q; namely, we have an isomorphism of Lie algebras

$$\mathcal{L}(TG) = \mathfrak{g} \circledS \mathfrak{g} \stackrel{\phi}{\approx} \mathfrak{g} \oplus \{0\} \oplus \mathfrak{g}^*, \text{ where :}$$

$$\phi(\xi, \eta) = (\xi, \eta^*), \eta^*(a) = Q(\eta, a), \forall a \in \mathfrak{g}.$$

The double extension is endowed with the corresponding metric:

$$\Gamma^*((x, a^*), (y, b^*)) = a^*(y) + b^*(x) + P(x, y).$$

As a particular case, for Q invariant metric on G, TG admits also the metric:

$$\Gamma_{r,s} = \begin{pmatrix} rQ & sQ \\ sQ & 0 \end{pmatrix}, \qquad r \notin \{0, -s\},\ r, s \in \mathbb{R}$$

and we have the following result:
Proposition. The harmonicity equations of a mapping

$$F : \mathbb{D} \to TG \equiv G \ltimes \mathfrak{g}$$

have the form:

$$\left\{ \begin{array}{lcl} \bar{\partial} A_z + \partial A_{\bar{z}} &=& 0 \\ \bar{\partial} a_z + \partial a_{\bar{z}} &=& 0 \end{array} \right., \tag{17}$$

where TG is identified with the semidirect product group $G \ltimes \mathfrak{g}$ via the isomorphism $\phi : TG \to G \ltimes \mathfrak{g}$,

$$\phi(X_g) = (g, v),\ v = L_{g^{-1}\,*} X_g, \forall X_g \in T_g G,$$

where $(A_z, a_z) = (g^{-1} g_z, v_z + [g^{-1} g_z, v])$.
Remarks. 1. The equations (17) have the detailed form:

$$\left\{ \begin{array}{lcl} \Delta g &=& \frac{1}{2}(g_{\bar{z}} g^{-1} g + g_z g^{-1} g_{\bar{z}}) \\ \Delta v &=& \frac{1}{2}\{[v_{\bar{z}}, g^{-1} g_z] + [v_z, g^{-1} g_{\bar{z}}]\}. \end{array} \right. \tag{18}$$

2. The first set of equations in (17) represent exactly the harmonicity condition for the map $pr_1 \circ \phi \circ F$; in particular, for G compact and semisimple, all the harmonic projections of this form can be determined by the DPW method [7].

For the general group case (e.g. G non-compact) certain results were obtained ([4]) towards an extended DPW method, namely:
• the Birkhoff and Iwasawa decompositions for loop groups - when the map takes values into the coset manifold of connected Lie group which admits a faithful representation and
• the equivalent characterization of harmonicity in terms of loopified forms, for maps into homogeneous reductive spaces.

In the tangent group case, a faithful representation of G induces canonically one for TG; taking as example a semisimple group G, for the tangent-group complex loops the Iwasawa-type decomposition is given by the following result:
Theorem 1. Any element $\tilde{g} \in \Lambda {G_\sigma}^{\mathbb{C}}$ can be decomposed

$$\tilde{g} = gmg_+,$$

where

$$\begin{cases} g \in & \Lambda TG_\sigma = \{\phi \in \Lambda {G_\sigma}^C \mid \phi(\lambda) \in G, \forall \lambda \in S^1\}, \\ g_+ \in & \Lambda^+ TG_\sigma^{\mathbb{C}}, \\ m \in & \underset{m}{\Lambda} G_\sigma^{\mathbb{C}} \subset \bigcup_{\mu \in \underset{m}{\Lambda} G_\sigma^C} (\Lambda^-_{+(\mu)} G_\sigma^{\mathbb{C}}) \cdot \mu \end{cases}$$

with the following notations:

$$\begin{array}{rcl} \Lambda^- TG_\sigma^{\mathbb{C}} & = & \{\phi \in \Lambda TG_\sigma^{\mathbb{C}} \mid \phi(\infty) = e,\ \phi \text{ extends holo to } \mathbb{C} \setminus D^1\}, \\ \Lambda^-_{+(\mu)} G_\sigma^{\mathbb{C}} & = & \{\phi \in \Lambda^+ G_\sigma^{\mathbb{C}} \mid \mu \phi \mu^{-1} \in \Lambda^- G_\sigma^{\mathbb{C}}\} \end{array}$$

and where $\underset{m}{\Lambda} G_\sigma^{\mathbb{C}}$ is the middle-term loop-space of the loop-Iwasawa decomposition for the semisimple group case [9].

Also, under the same assumptions, holds the Birkhoff-type decomposition given by the following theorem:
Theorem 2. Any element $\tilde{g} \in \Lambda {G_\sigma}^{\mathbb{C}}$ can be decomposed as

$$\tilde{g} = g_- D g_+,$$

where

$$\begin{cases} g_\pm \in \Lambda^\pm TG_\sigma^{\mathbb{C}}, \\ D \in \Lambda \mathfrak{g}_\sigma^{\mathbb{C}} = \{\phi \in \Lambda TG_\sigma^{\mathbb{C}} \mid \phi(\lambda) \in \mathfrak{g}^{\mathbb{C}} \equiv \{e\} \ltimes \mathfrak{g}^{\mathbb{C}} \triangleleft TG^{\mathbb{C}}\}. \end{cases}$$

3 Application - the case $N = TS^2$

Remark firstly that TS^2 is a symmetric space, since $S^2 = G/K$ with $G = SU(2), K = G^\sigma = U(1) \equiv S^1$, for $\sigma = Ad[\mathrm{diag}(i, -i)]$. Then $N \equiv TS^2 = TG/TK$ [1] is a symmetric space and hence homogeneous, so that the harmonic maps $f : \mathbb{D} \to TS^2$ can be characterized in terms of "loopified forms" [4]; any such harmonic map admits a lift to $TG = TSU(2)$.
The lift $F : \mathbb{D} \to TSU(2) \equiv SU(2) \ltimes su(2)$ splits locally into two maps: the first map

$$F_1 = g : \mathbb{D} \to SU(2)$$

projects to S^2 and is the lift of the normal Gauss field of a CMC-surface; the second one,

$$F_2 = v : \mathbb{D} \to su(2),$$

where $su(2) \equiv \mathbb{R}^3$ ([5]), is subject to a system of linear PDE's of second order

$$\Delta v = [v_z, B] + [v_{\bar{z}}, A]$$

with the constraints $A_{\bar{z}} - B_z = [A, B]$, which are exactly the integrability conditions of the system:

$$\begin{cases} A = g^{-1} g_z \ , \\ B = g^{-1} g_{\bar{z}} \ . \end{cases}$$

The group TG is endowed with a metric of the form $\Gamma = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$, where K is the unique metric (up to a multiplicative constant), produced by the nondegenerate Killing form on $\mathfrak{g} = su(2)$; the Euler-Lagrange equations of the lift split into two parts (18).

The DPW method determines completely the first component F_1 of the lift of the harmonic maps $f : \mathbb{D} \rightarrow N$, from the meromorphic and the holomorphic potentials [6]. Namely, the meromorphic potentials which determine $F_1 = g$ have the form:

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & a(z) \\ \mathfrak{g}(z)0 & \end{pmatrix} dz \in \mathcal{P}_m, \ \lambda \in S^1,$$

with $a, b : \mathbb{D} \rightarrow \mathbb{C}$ special meromorphic functions ([5]).

Also, remark that since the Lie group $SU(2)$ is connected and semisimple, the extended Iwasawa and Birkhoff decompositions are applicable and that the decomposed complex loops take practically values into the semidirect product $TG^{\mathbb{C}} = (TSU(2))^{\mathbb{C}} \equiv SL(2, \mathbb{C}) \ltimes sl(2, \mathbb{C})$.

4 Conclusions

In the first section was briefly presented the DPW method, which determines all the harmonic maps from Riemann surfaces to symmetric spaces ([7]); in the second section, the main results which extend the DPW method ([4, 9]) specialize to homogeneous reducible spaces corresponding to the tangent group case. In the third section the specific case TS^2 is discussed in relation with the basic DPW method ([5, 6, 7]).

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