Real hypersurfaces of non-flat complex planes with generalized $\xi-$parallel Jacobi structure operator

Th. Theofanidis

Abstract. The aim of the present paper is the classification of real hypersurfaces M, whose Jacobi structure operator commutes with the shape operator, in a subspace of the tangent space T_pM of M at a point p. This class is large and difficult to classify, therefore a second condition is imposed: the Jacobi structure operator is *generalized* $\xi-$*parallel* in the same subspace of the first condition. The notion of generalized $\xi-$parallel Jacobi structure operator is introduced and studied for the first time and is weaker than $\xi-$ parallel Jacobi structure operator which has been studied so far.

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1 Introduction.

An n - dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a projective space $\mathbb{C}P^n$ if $c > 0$, a hyperbolic space $\mathbb{C}H^n$ if $c < 0$, or a Euclidean space $\mathbb{C}^n$ if $c = 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ will be denoted by (ϕ, ξ, η, g).

Real hypersurfaces in $\mathbb{C}P^n$ which are homogeneous, were classified by R. Takagi ([13]). The same author classified real hypersurfaces in $\mathbb{C}P^n$, with constant prinicipal curvatures in [14], but only when the number g of distinct principal curvatures satisfies $g = 3$. M. Kimura showed in [8] that if a Hopf real hypersurface M in $\mathbb{C}P^n$ has constant principal curvatures, then the number of distinct principal curvatures of M is 2, 3 or 5. J. Berndt gave the equivalent result for Hopf hypersurfaces in $\mathbb{C}H^n$ ([1]) where he divided real hypersurfaces into four model spaces, named A_0, A_1, A_2 and B. Analytic lists of constant principal curvatures can be found in the previously mentioned references as well as in [9], [11]. Real hypersurfaces of type A_1 and A_2 in $\mathbb{C}P^n$ and of type A_0, A_1 and A_2 in $\mathbb{C}H^n$ are said to be hypersurfaces of *type A* for simplicity and appear quite often in classification theorems. Real hypersurfaces of type A_1 in $\mathbb{C}H^n$ are divided into types $A_{1,0}$ and $A_{1,1}$ ([9]). For more information and examples on real hypersurfaces, we refer to [11].

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A Jacobi field along geodesics of a given Riemannian manifold (M, g) plays an important role in the study of differential geometry. It satisfies a well known differential equation which inspires Jacobi operators. For any vector field X, the Jacobi operator is defined by R_X: $R_X(Y) = R(Y, X)X$, where R denotes the curvature tensor and Y is a vector field on M. R_X is a self - adjoint endomorphism in the tangent space of M, and is related to the Jacobi differential equation, which is given by $\nabla_{\acute{\gamma}}(\nabla_{\acute{\gamma}}Y) + R(Y, \acute{\gamma})\acute{\gamma} = 0$ along a geodesic γ on M, where $\acute{\gamma}$ denotes the velocity vector along γ on M.

In a real hypersurface M of a complex space form $M_n(c)$, $c \neq 0$, the Jacobi operator on M with respect to the structure vector field ξ, is called the structure Jacobi operator and is denoted by $R_\xi(X) = R(X, \xi,)\xi = lX$. Conditions including this operator, generate larger classes than the conditions including the Riemannian tensor $R(X, Y)Z$. So operator l has been studied by quite a few authors and under several conditions.

In 2007, Ki, Perez, Santos and Suh ([6]) classified real hypersurfaces in complex space forms with ξ-parallel Ricci tensor and structure Jacobi operator. J. T. Cho and U - H Ki in [3] classified the real hypersurfaces whose structure Jacobi operator is symmetric along the Reeb flow ξ and commutes with the shape operator A.

In the present paper we classify real hypersurfaces M satisfying the condition

$$lA = Al, \tag{1.1}$$

restricted in the subspace $\mathbb{D} = ker(\eta)$ of T_pM for every point $p \in M$, where $\ker(\eta)$ consists of all vectors fields orthogonal to the Reeb flow ξ. This class is quite large and rather difficult to be classified, therefore a second condition had to be imposed:

$$(\nabla_\xi l)X = \omega(X)\xi, \tag{1.2}$$

where $\omega(X)$ is 1-form and $X \in ker(\eta) = \mathbb{D}$. This condition is much weaker than $\nabla_\xi l = 0$ that has been used so far ([3], [4], [5], [6]). Therefore a larger class is produced.

Finally, we mention that hypersurfaces in $M_2(c)$ have not been studied as thoroughly as the ones in $M_n(c)$, $n \geq 3$.

The major and most difficult part, is to prove M is a Hopf hypersurface, that is ξ is a principal vector field and the classification follows right after that. In particular, the following theorem is proved:

Theorem 1.1. *Let M be a real hypersurface of a complex plane $M_2(c)$, $(c \neq 0)$, satisfying (1.1) and (1.2) for every vector field $X \in \mathbb{D}$. Then M is a Hopf hypersurface and satisfies $\nabla_\xi l = 0$. Furthermore, M is pseudo-Einstein, that is, there exist constants ρ and σ such that for any tangent vector X we have $QX = \rho X + \sigma g(X, \xi)\xi$, where Q is the Ricci tensor. Conversely, every pseudo-Einstein hypersurface in $M_2(c)$ satisfies (1.2) with $\omega = 0$.*

As shown in [7] the pseudo-Einstein hypersurfaces, are precisely those that are

• For $M_2(c) = \mathbb{C}P^2$: open subsets of geodesic spheres (type A_1);

• For $M_2(c) = \mathbb{C}H^2$: open subsets of
1. horospheres (type A_0);

2. geodesic spheres (type $A_{1,0}$);
3. tubes around totally geodesic complex hyperbolic lines $\mathbb{C}H^1$ (type $A_{1,1}$);

- Hopf hypersurfaces with $\eta(A\xi) = 0$.

An almost similar problem for $n \geq 3$ has been solved in [15]. In addition, the form ω has no restriction in its values, so it could vanish at some point. Therefore condition (1.2) could be called generalized $\xi-$parallel Jacobi structure operator, since it generalizes the notion of $\xi-$parallel Jacobi structure operator ($\nabla_\xi l = 0$).

2 Preliminaries

In this section, we explain explicitly the notions that were mentioned in section 0, as well as the notions that will appear in the paper. We also give a series of equations that will be our basic tools in our calculations and conclusions.

Let M_n be a Kaehlerian manifold of real dimension 2n, equipped with an almost complex structure J and a Hermitian metric tensor G. Then for any vector fields X and Y on $M_n(c)$, the following relations hold: $J^2X = -X$, $G(JX, JY) = G(X, Y)$, $\widetilde{\nabla}J = 0$, where $\widetilde{\nabla}$ denotes the Riemannian connection of G of M_n.

Let M_{2n-1} be a real $(2n-1)$-dimensional hypersurface of $M_n(c)$, and denote by N a unit normal vector field on a neighborhood of a point in M_{2n-1} (from now on we shall write M instead of M_{2n-1}). For any vector field X tangent to M we have $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX, $\eta(X)N$ is the normal component, and $\xi = -JN$, $\eta(X) = g(X, \xi)$, $g = G|_M$.

By properties of the almost complex structure J and the definitions of η and g, the following relations hold ([2]):

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \quad \eta(\xi) = 1 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \phi Y) = -g(\phi X, Y). \tag{2.2}$$

The above relations define an *almost contact metric structure* on M which is denoted by (ϕ, ξ, g, η). When an almost contact metric structure is defined on M, we can define a local orthonormal basis $\{e_1, e_2, ...e_{n-1}, \phi e_1, \phi e_2, ...\phi e_{n-1}, \xi\}$, called a $\phi-$*basis*. Furthermore, let A be the shape operator in the direction of N, and denote by ∇ the Riemannian connection of g on M. Then, A is symmetric and the following equations are satisfied:

$$\nabla_X \xi = \phi AX, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.3}$$

Since the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given by:

$$R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ -2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY, \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi]. \tag{2.5}$$

The tangent space T_pM, at every point $p \in M$, is decomposed as following:

$$T_pM = \mathbb{D}^{\perp} \oplus \mathbb{D},$$

where $\mathbb{D} = ker(\eta) = \{X \in T_pM : \eta(X) = 0\}$.

The subspace $ker(\eta)$ is more usually referred as $\mathbb{D}$ and called holomorphic distribution of M at p. Based on the decomposition of T_pM, by virtue of (2.3), we decompose the vector field $A\xi$ in the following way:

$$A\xi = \alpha\xi + \beta U, \tag{2.6}$$

where $\beta = |\phi\nabla_\xi\xi|$, α is a smooth function on M and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi \in ker(\eta)$, provided that $\beta \neq 0$.

If the vector field $A\xi$ is expressed as $A\xi = \alpha\xi$, then ξ is called *principal vector field.*

Finally differentiation of vector field X along a function f will be denoted by (Xf). All manifolds, vector fields, etc, of this paper are assumed to be connected and of class C^∞.

3 Auxiliary relations

We suppose there exists a point $p \in M$ such that $\beta \neq 0$ in a neighborhood $\mathcal{N}$ around p. We define the open subset $\mathcal{N}_1$ of $\mathcal{N}$ such that $\mathcal{N}_1 = \{q \in \mathcal{N} : \alpha \neq 0$ in a neighborhood around $q\}$.

Lemma 3.1. *Let M be a real hypersurface of a complex plane $M_2(c)$ satisfying (1.1) on $\mathbb{D}$. Then the following relations hold on $\mathcal{N}_1$.*

$$AU = \Big(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\Big)U + \beta\xi, \qquad A\phi U = (\frac{\gamma}{\alpha} - \frac{c}{4\alpha})\phi U. \tag{3.1}$$

$$\nabla_\xi\xi = \beta\phi U,\ \nabla_U\xi = \Big(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\Big)\phi U,\ \nabla_{\phi U}\xi = (\frac{c}{4\alpha} - \frac{\gamma}{\alpha})U. \tag{3.2}$$

$$\nabla_\xi U = \kappa_1\phi U, \quad \nabla_U U = \kappa_2\phi U, \quad \nabla_{\phi U}U = \kappa_3\phi U + (\frac{\gamma}{\alpha} - \frac{c}{4\alpha})\xi. \tag{3.3}$$

$$\nabla_\xi\phi U = -\kappa_1 U - \beta\xi,\ \ \nabla_U\phi U = -\kappa_2 U + \Big(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}\Big)\xi, \tag{3.4}$$

$$\nabla_{\phi U}\phi U = -\kappa_3 U.$$

where κ_1, κ_2, κ_3 are smooth functions on $\mathcal{N}_1$.

Proof.
In what follows we work on $\mathcal{N}_1$. By definition of the vector fields U, ϕU, ξ and due to (2.1), the set $\{U, \phi U, \xi\}$ is an orthonormal basis. From (2.4) we obtain

$$lU = \frac{c}{4}U + \alpha AU - \beta A\xi, \qquad l\phi U = \frac{c}{4}\phi U + \alpha A\phi U. \tag{3.5}$$

The inner products of lU with U and ϕU respectively yield

$$g(AU,U)=\frac{\epsilon}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha},\qquad g(AU,\phi U)=\frac{\delta}{\alpha} \tag{3.6}$$

where $\epsilon=g(lU,U)$ and $\delta=g(lU,\phi U)$.
So, (3.6) and $g(AU,\xi)=g(A\xi,U)=\beta$, yield

$$AU=\Big(\frac{\epsilon}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\Big)U+\frac{\delta}{\alpha}\phi U+\beta\xi. \tag{3.7}$$

Since l is symmetric with respect to metric g, the scalar products of the second of (3.5) with U and ϕU yield respectively

$$g(A\phi U,U)=\frac{\delta}{\alpha},\qquad g(A\phi U,\phi U)=\frac{\gamma}{\alpha}-\frac{c}{4\alpha},$$

where $\gamma=g(l\phi U,\phi U)$. So, the above equations and $g(A\phi U,\xi)=g(A\xi,\phi U)=0$, yield

$$A\phi U=(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})\phi U+\frac{\delta}{\alpha}U. \tag{3.8}$$

From (3.5), (3.7) and (3.8) we obtain $lU=\epsilon U+\delta\phi U$ and $l\phi U=\delta U+\gamma\phi U$. We make use of the last two equations, along with (1.6), (2.7), (2.8) and the symmetry of l, to analyze $g(lAU,\xi)=g(AlU,\xi)$ - which holds due to (1.1) - and obtain $\epsilon=0$. Similarly, from the same equations, $\epsilon=0$ and $g(lA\phi U,\xi)=g(Al\phi U,\xi)$ we take $\delta=0$. Therefore, from $\delta=\epsilon=0$ and (3.7), (3.8) we obtain (3.1). In addition we have shown

$$lU=0,\quad l\phi U=\gamma\phi U. \tag{3.9}$$

From equation (3.1) and relation (2.3) for $X=\xi$, $X=U$, $X=\phi U$, we obtain (3.2). Next we remind of the rule

$$Xg(Y,Z)=g(\nabla_X Y,Z)+g(Y,\nabla_X Z). \tag{3.10}$$

By virtue of (3.10) for $X=Z=\xi$, $Y=U$ and for $X=\xi$, $Y=Z=U$, it is shown respectively $\nabla_\xi U\perp\xi$ and $\nabla_\xi U\perp U$. So $\nabla_\xi U=\kappa_1\phi U$, where $\kappa_1=g(\nabla_\xi U,\phi U)$. In a similar way, (3.10) for $X=Y=Z=U$ and $X=Z=U$, $Y=\xi$ respectively yields $\nabla_U U\perp U$ and $\nabla_U U\perp\xi$. This means that $\nabla_U U=\kappa_2\phi U$, where $\kappa_2=g(\nabla_U U,\phi U)$. Finally, (3.10) for $X=\phi U$, $Y=Z=U$ and $X=\phi U$, $Y=U$, $Z=\xi$ (with the aid of (3.2)) yields respectively $\nabla_{\phi U}U\perp U$ and $g(\nabla_{\phi U}U,\xi)=\frac{\gamma}{\alpha}-\frac{c}{4\alpha}$. Therefore $\nabla_{\phi U}U=\kappa_3\phi U+(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})\xi$ where $\kappa_3=g(\nabla_{\phi U}U,\phi U)$ and (3.3) has been proved. In order to prove (3.4) we use the second of (2.3) with the following combinations: *i)* $X=\xi$, $Y=U$, *ii)* $X=Y=U$, *iii)* $X=\phi U$, $Y=U$, and make use of (2.6), (3.1), (3.3). □

By putting $X=U$, $Y=\xi$ in (2.5) we obtain $\nabla_U A\xi-A\nabla_U\xi-\nabla_\xi AU+A\nabla_\xi U=-\frac{c}{4}\phi U$, which is expanded by Lemma 3.1, to give

$$[(U\alpha)-(\xi\beta)]\xi+[(U\beta)-\big(\xi(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})\big)]U+$$

$$[\kappa_2\beta-(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})+(\frac{\gamma}{\alpha}-\frac{\beta^2}{\alpha})\kappa_1]\phi U.$$

Since the vector fields $U, \phi U$ and ξ are linearly independent, the above equations gives

$$(U\alpha) = (\xi\beta), \tag{3.11}$$

$$(U\beta) = \big(\xi(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})\big), \tag{3.12}$$

$$\kappa_2\beta - (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) + (\frac{\gamma}{\alpha} - \frac{\beta^2}{\alpha})\kappa_1 = 0. \tag{3.13}$$

In a similar way, from (2.5) we get $\nabla_{\phi U} A\xi - A\nabla_{\phi U}\xi - \nabla_\xi A\phi U + A\nabla_\xi \phi U = \frac{c}{4}U$, which is expanded by Lemma 3.1, to give

$$[(\phi U\alpha) + 3\beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) - \kappa_1\beta - \alpha\beta]\xi +$$
$$[\phi U\beta - \gamma + (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) + \kappa_1(\frac{\gamma}{\alpha} - \frac{\beta^2}{\alpha}) - \beta^2]U +$$
$$[\kappa_3\beta - \xi(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})]\phi U = 0,$$

which leads to

$$(\phi U\alpha) + 3\beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) - \kappa_1\beta - \alpha\beta = 0, \tag{3.14}$$

$$\phi U\beta - \gamma + (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) + \kappa_1(\frac{\gamma}{\alpha} - \frac{\beta^2}{\alpha}) - \beta^2 = 0, \tag{3.15}$$

$$\kappa_3\beta = \xi(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}). \tag{3.16}$$

Finally, (2.5) yields $\nabla_U A\phi U - A\nabla_U \phi U - \nabla_{\phi U} AU + A\nabla_{\phi U} U = -\frac{c}{2}\xi$, which is expanded by Lemma 3.1, to give

$$[-\phi U\beta + \gamma - 2(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) + \kappa_2\beta + \beta^2]\xi +$$
$$[\beta(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}) + 2\beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) + \kappa_2(\frac{\beta^2}{\alpha} - \frac{\gamma}{\alpha}) - \phi U(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})]U +$$
$$[U(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) + \kappa_3(\frac{\gamma}{\alpha} - \frac{\beta^2}{\alpha})]\phi U = 0.$$

The above relation leads to

$$\phi U\beta - \gamma + 2(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) - \kappa_2\beta - \beta^2 = 0, \tag{3.17}$$

$$\beta(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}) + 2\beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) + \kappa_2(\frac{\beta^2}{\alpha} - \frac{\gamma}{\alpha}) = \phi U(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}), \tag{3.18}$$

$$U(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})=-\kappa_3(\frac{\gamma}{\alpha}-\frac{\beta^2}{\alpha}). \tag{3.19}$$

We combine (3.15) and (3.17), by removing the term $\phi U\beta-\gamma-\beta^2$, to obtain

$$\kappa_2\beta+\kappa_1(\frac{\gamma}{\alpha}-\frac{\beta^2}{\alpha})=(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}). \tag{3.20}$$

Furthermore, we modify equation (3.18) as following: we expand the term $\phi U(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})$ and then replace the terms $(\phi U\alpha)$, $(\phi U\beta)$ respectively from (3.14) and (3.15). The final relation is

$$\kappa_2(\beta^2-\gamma)-\beta c=\frac{\beta}{\alpha^2}(\gamma-\frac{c}{4})(\beta^2-\frac{c}{4})+\kappa_1\beta(\frac{\beta^2}{\alpha}-2\frac{\gamma}{\alpha}+\frac{c}{4\alpha}). \tag{3.21}$$

By virtue of (3.20), the term κ_2 is replaced in (3.21), and after calculations we result to

$$\kappa_1(\frac{\gamma^2}{\beta}-\frac{\beta c}{4})-\alpha\beta c-\frac{\gamma}{\beta}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})(\beta^2-\frac{c}{4})=0. \tag{3.22}$$

Lemma 3.2. *Let M be a real hypersurface of a complex plane $M_2(c)$ satisfying (1.1) and (1.2) on $\mathbb{D}$. Then, equations $\gamma=0$, $\kappa_1=-4\alpha$ and $\kappa_2=-4\beta-\frac{c}{4\alpha\beta}(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})$ hold on $\mathcal{N}_1$.*

Proof.
By making use of (1.2) for $X=U$ and with the aid of Lemma 3.1 and (3.9), we take

$$\kappa_1\gamma=0, \qquad (\xi\gamma)=0. \tag{3.23}$$

Let us assume there exists a point $p_1\in\mathcal{N}_1$ at which $\gamma\neq 0$. Then, there exists a neighborhood V_1 of p_1 such that $\gamma\neq 0$ in V_1. We are going to work in V_1 throughout the proof of this Lemma, in order to show $V_1=\varnothing$. Since $\gamma\neq 0$, (3.23) yields

$$\kappa_1=(\xi\gamma)=0. \tag{3.24}$$

From (2.4), (3.23) and Lemma 3.1 we obtain $R(U,\xi)U=0$. We also have $R(U,\xi)U=\nabla_U\nabla_\xi U-\nabla_\xi\nabla_U U-\nabla_{[U,\xi]}U$ which is analyzed with the aid of Lemma 3.1 and (3.23) giving $R(U,\xi)U=[-(\xi\kappa_2)-\kappa_3(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})]\phi U+[\kappa_2\beta-(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})]\xi$. The two expressions of $R(U,\xi)U$ give

$$(\xi\kappa_2)=-\kappa_3(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha}),\ \ \kappa_2\beta=(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}). \tag{3.25}$$

We differentiate the second of (3.25) along ξ and then replace the term $(\xi\kappa_2)$ from the first of (3.25), resulting to

$$\kappa_2(\xi\beta)=\frac{2\beta}{\alpha}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})(\xi\beta)-(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})(\frac{\beta^2}{\alpha^2}-\frac{c}{4\alpha^2})(\xi\alpha)+2\beta(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})\kappa_3. \tag{3.26}$$

Next we differentiate (3.22) along ξ, combined with (3.24), in order to obtain

$$\alpha\beta c(\xi\alpha)+\alpha^2c(\xi\beta)+\gamma(\gamma-\frac{c}{4})(\xi\beta)=0. \tag{3.27}$$

By making use of (3.25) and (3.27),we replace the terms κ_2 and $(\xi\alpha)$ respectively, in (3.26) and after calculations we obtain

$$[\frac{2\beta}{\alpha}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})+\frac{\gamma}{\alpha\beta c}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})](\xi\beta)=-2\beta(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})\kappa_3. \tag{3.28}$$

(3.22) is rewritten as $\frac{\gamma}{\alpha\beta c}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})=-\frac{\beta}{\alpha}$ which is used with (3.28) to obtain

$$(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})(\xi\beta)=-2(\beta^2-\frac{c}{4})\kappa_3. \tag{3.29}$$

We notice that $\gamma-\frac{c}{4}\neq 0$, otherwise (3.22) would yield $\alpha\beta c=0$ which is a contradiction. Therefore we combine (3.16), (3.24), (3.27) and (3.29), taking

$$[-\alpha^3\beta^2c-2\alpha^3(\beta^2-\frac{c}{4})-2\alpha\gamma(\gamma-\frac{c}{4})(\beta^2-\frac{c}{4})]\kappa_3.$$

If we had $\kappa_3\neq 0$ in a neighborhood of V_1 then the above relation and (3.22) would give $\beta^2=\frac{c}{2}\Rightarrow(\xi\beta)=0\Rightarrow\beta^2=\frac{c}{4}$ (due to (3.29)) which is a contradiction. Therefore $\kappa_3=0$. Since $\kappa_3=0$, (3.11), (3.12), (3.27) and (3.29) imply $([U,\xi]\alpha)=([U,\xi]\beta)=0$. However, these Lie brackets are also estimated from Lemma 3.1, (3.14), (3.15), (3.24), which means we have the following:

$$(\beta^2-\frac{c}{4})\big(3(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})-\alpha\big)=0, \tag{3.30}$$

$$(\beta^2-\frac{c}{4})\big(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})-\beta^2-\gamma\big)=0.$$

Due to (3.22) it must be $\beta^2-\frac{c}{4}\neq 0$ of V_1. Then from (3.30) we acquire

$$\frac{\gamma}{\alpha}-\frac{c}{4\alpha}=\frac{\alpha}{3}\Leftrightarrow(\phi U\alpha)=0,\quad \beta^2-\frac{c}{4}=3(\beta^2+\gamma)\Leftrightarrow(\phi U\beta)=0. \tag{3.31}$$

From (3.31) we modify (3.20):

$$\kappa_2=\frac{1}{3\beta}(\beta^2-\frac{c}{4}) \tag{3.32}$$

We make use of the last relation, (3.24), (3.31), $\kappa_3=0$ and Lemma 3.1 to show $R(\phi U,U)U=\Big(-\kappa_2^2-(\frac{\beta^2}{\alpha}-\frac{c}{4\alpha})(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})\Big)\phi U$. The same vector field is calculated from (2.4), (3.31), (3.32) and Lemma 3.1 and then we equalize the two expressions of $R(\phi U,U)U$, resulting to

$$-\frac{1}{9\beta^2}(\beta^2-\frac{c}{4})^2=c+\frac{2}{3}(\beta^2-\frac{c}{4}) \tag{3.33}$$

In the same way, by calculating $R(\phi U,\xi)U$ with the aid of Lemma 3.1, $\kappa_1=\kappa_3=0$, (3.16), (3.31) we obtain $\kappa_2=2\beta$, which is combined with (3.32) giving

$$6\beta^2=\beta^2-\frac{c}{4}\Leftrightarrow\beta^2=-\frac{c}{20}.$$

The above result and (3.33) lead to $\beta^2=-\frac{c}{8}$, which is a contradiction in V_1, meaning $V_1=\varnothing$. The rest of the proof follows from (3.21), (3.22), (3.23). □

4 The set $\mathcal{N}_1$ is the empty set.

We start with the following result:

Lemma 4.1. *Let M be a real hypersurface of a complex plane $M_2(c)$ satisfying (1.1) and (1.2) on $\mathbb{D}$. Then $\kappa_3 = 0$ holds on $\mathcal{N}_1$.*

Proof.
Lemma 3.2, (3.11), (3.12), (3.16) and (3.19) yield

$$(4.1) \qquad (U\alpha) = (\xi\beta) = \frac{4\alpha\beta^2}{c}\kappa_3, \quad (\xi\alpha) = \frac{4\alpha^2\beta}{c}\kappa_3, \quad (U\beta) = \beta(\frac{4\beta^2}{c} + 1)\kappa_3.$$

We use Lemmas 3.1, 3.2 and relations (3.14), (4.1) to calculate

$$[\phi U, U]\alpha = (\nabla_{\phi U}U - \nabla_U \phi U)\alpha = \kappa_3(\frac{\beta c}{\alpha} - 5\alpha\beta - \frac{12\alpha\beta^3}{c} - \frac{\beta^3}{\alpha}).$$

On the other hand, from Lemmas 3.1, 3.2 and equations (3.14), (4.1), we obtain

$$[\phi U, U]\alpha = \phi U(U\alpha) - U(\phi U\alpha) = \phi U(U\alpha) + (3\alpha\beta + \frac{24\alpha\beta^3}{c} - \frac{3\beta c}{4\alpha})\kappa_3.$$

Equalizing the two expressions of $[\phi U, U]\alpha$ we result to

$$(4.2) \qquad \phi U(U\alpha) = (\frac{7\beta c}{4\alpha} - \frac{36\alpha\beta^3}{c} - \frac{\beta^3}{\alpha} - 8\alpha\beta)\kappa_3.$$

By making use of Lemmas 3.1, 3.2 and relation (4.1), it is proved that

$$[\phi U, \xi]\beta = (\nabla_{\phi U}\xi - \nabla_\xi \phi U)\beta = [\frac{\beta c}{4\alpha} - \frac{12\alpha\beta^3}{c} + \frac{\beta^3}{\alpha} - 4\alpha\beta]\kappa_3.$$

However, the same differentiation is calculated with aid of Lemma 3.2 and equations (3.15), (3.1):

$$[\phi U, \xi]\beta = \phi U(\xi\beta) - \xi(\phi U\beta) = \phi U(\xi\beta) + [-\frac{\beta c}{2\alpha} + \frac{24\alpha\beta^3}{c}]\kappa_3.$$

Comparing the two expressions of $[\phi U, \xi]\beta$ we are led to

$$(4.3) \qquad \phi U(\xi\beta) = [\frac{\beta^3}{\alpha} + \frac{3\beta c}{4\alpha} - \frac{36\alpha\beta^3}{c} - 4\alpha\beta]\kappa_3.$$

From (3.11), (4.2) and (4.3) we acquire

$$(4.4) \qquad (2\alpha^2 + \beta^2 - \frac{c}{2})\kappa_3 = 0.$$

Let us assume there exists a point $p_2 \in \mathcal{N}_1$ at which $\kappa_3 \neq 0$. Then, there exists a neighborhood V_2 of p_2 such that $\kappa_3 \neq 0$ in V_2. Therefore (3.4) yields $\alpha^2 + \beta^2 = \frac{c}{2}$, which is differentiated along ξ, with the aid of (4.1) and $\kappa_3 \neq 0$, giving $2\alpha^2 + \beta^2 = 0$ which is a contradiction. This means there are no points of $\mathcal{N}_1$ where $\kappa_3 \neq 0$ and so $\kappa_3 = 0$ holds on $\mathcal{N}_1$. □

Now that $\kappa_3 = 0$, from (4.1) we have $[U,\xi]\alpha = U(\xi\alpha) - \xi(U\alpha) = 0$. Furthermore, from Lemmas 3.1, 3.2 and (3.14) we have $[U,\xi]\alpha = (\nabla_U\xi - \nabla_\xi U)\alpha = \beta(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + 4\alpha)(\frac{3c}{4\alpha} - 3\alpha)$. So we conclude

$$3\beta(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + 4\alpha)(\frac{c}{4\alpha} - \alpha) = 0. \tag{4.5}$$

Similarly, from (4.1) and Lemma 4.1 we have $[U,\xi]\beta = 0$, while from Lemmas 3.1, 3.2 and (3.15) we have $[U,\xi]\beta = (\nabla_U\xi - \nabla_\xi U)\beta = (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + 4\alpha)[(\frac{c}{4\alpha})(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}) - 3\beta^2]$. So we have shown

$$(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + 4\alpha)[(\frac{c}{4\alpha})(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}) - 3\beta^2] = 0. \tag{4.6}$$

Let us assume there exists a point $p_3 \in \mathcal{N}_1$ at which $\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + 4\alpha \neq 0$. Then, there exists a neighborhood V_3 of p_3 such that $\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + 4\alpha \neq 0$ in V_3. In this case (4.5) and (4.6) yield respectively $\frac{c}{4} = \alpha^2 > 0$ and $\frac{c}{4\alpha}(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}) - 3\beta^2 = 0$. We combine the last two relations by removing the term α^2 and obtain $c = -8\beta^2 < 0$ which is a contradiction to $\frac{c}{4} = \alpha^2 > 0$. So there exists no point of $\mathcal{N}_1$ where $\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + 4\alpha \neq 0$, hence it must be $\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} + 4\alpha = 0 \Rightarrow$

$$\beta^2 + 4\alpha^2 = \frac{c}{4} > 0. \tag{4.7}$$

Differentiating (4.7) along ϕU we acquire $\beta(\phi U\beta) + 4\alpha(\phi U\alpha) = 0$ which is expanded by (3.14), (3.15) and Lemma 3.2, giving

$$\frac{c}{4\alpha^2}(\beta^2 - \frac{c}{4}) - 3\beta^2 - 12\alpha^2 + 3c = 0.$$

Replacing the term $\beta^2 - \frac{c}{4}$ with $-4\alpha^2$ - due to (4.7) - we get $\beta^2 + 4\alpha^2 = \frac{2c}{3} < 0$ which is a contradiction to (4.7). So $\mathcal{N}_1 = \varnothing$.

5 Proof of Theorem 0.1

Because of Section 4, and by definition of the sets $\mathcal{N}$, $\mathcal{N}_1$ in the beginning of section 3, in the set $\mathcal{N}$, equation (2.6) takes the form $A\xi = \beta U$. This means that the vector fields AU and $A\phi U$ are decomposed with respect to the $\phi-$basis $\{U, \phi U, \xi\}$ as:

$$AU = \mu_1 U + \mu_2\phi U + \beta\xi, \quad A\phi U = \mu_2 U + \mu_3\phi U, \tag{5.1}$$

for some functions μ_1, μ_2, μ_3. In addition, from (2.4) and $A\xi = \beta U$ we obtain $lU = \frac{c}{4}U$ and $l\phi U = \frac{c}{4}\phi U$. Combining the previous two equations with (5.1) and (1.1), we analyze $lAU = AlU$ to take $\beta = 0$ which is a contradiction in $\mathcal{N}$. So $\mathcal{N} = \varnothing$ and the real hypersurface M consists of points where $\beta = 0$, i.e, M is a Hopf hypersurface.

Since M is Hopf, we have $A\xi = \alpha\xi$ and α is constant ([11]). The inner product of $(\nabla_\xi l)X = \omega(X)\xi$ with ξ (because of (2.3), (3.10) and $A\xi = \alpha\xi$) yields $\omega(X) = 0$. This means that $\nabla_\xi l = 0$.

It is easy to check that $(\nabla_\xi l)\xi = 0$ for any Hopf hypersurface. Now consider a vector field $X \in \mathbb{D}$. From the Gauss equation we have $lX = (\alpha A + \frac{c}{4})X$, so that

$$(\nabla_\xi l)X = \nabla_\xi lX - l\nabla_\xi X$$
$$= \nabla_\xi(\alpha A + \frac{c}{4})X - (\alpha A + \frac{c}{4})\nabla_\xi X,$$

since $\nabla_\xi X$ is also in $\mathbb{D}$. We can simplify this, using the Codazzi equation, to get

$$(\nabla_\xi l)X = \alpha(\nabla_\xi A)X$$
$$= \alpha((\nabla_X A)\xi + \frac{c}{4}\phi X)$$
$$= \alpha((\alpha - A)\phi AX + \frac{c}{4}\phi X).$$

In particular, If X is chosen to be a principal vector field, such that $AX = \lambda_1 X$ and $A\phi X = \lambda_2 \phi X$, then the condition $\nabla_\xi l = 0$ implies that

$$\alpha(\lambda_1 - \lambda_2) = 0$$

where we have used the well known relation for Hopf hypersurfaces

$$\lambda_1\lambda_2 = \frac{\lambda_1 + \lambda_2}{2}\alpha + \frac{c}{4}.$$

If $\alpha \neq 0$ then $\lambda_1 = \lambda_2$ is locally constant since it satisfies $\lambda_1^2 = \alpha\lambda_1 + \frac{c}{4}$. Therefore, M is an open subset of type A hypersurface, based on the theorems of Kimura and Berndt and the lists of principal curvatures in [13] and [9]. In case $\alpha = 0$, we have $\lambda_1 \neq \lambda_2$ or $\lambda_1 = \lambda_2$ with $\lambda_1^2 = \frac{c}{4}$ and the classification follows from [7].

Conversely, let M be of type A_1 in $\mathbb{C}P^2$ or type A_0, $A_{1,0}$, $A_{1,1}$ in $\mathbb{C}H^2$. Take $X \in \mathbb{D}$ a principal vector field with principal curvature λ, and α the principal curvature of ξ. (2.4) yields $lX = (\alpha A + \frac{c}{4})X$, $\forall X \in \mathbb{D}$. Furthermore, in a real hypersurface of the previously mentioned types, we have $\lambda^2 = \alpha\lambda + \frac{c}{4}$, thus from the last two equations we have $lX = \lambda^2 X$, which is used to show $(\nabla_\xi l)X = 0$. The last equation and $(\nabla_\xi l)\xi = \nabla_\xi l\xi - l\nabla_\xi \xi = 0$ show that real hypersurfaces of type A satisfy (1.2) with $\omega = 0$.

If M is Hopf with $\alpha = 0$ then (2.4) yields $lX = \frac{c}{4}X$ for every $X \in D$. Therefore $(\nabla_\xi l)X = 0$ holds. In addition we have $(\nabla_\xi l)\xi = 0$, thus $(\nabla_\xi l)X = 0$ holds for every X, which means $\omega = 0$. □

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Theoharis Theofanidis,
Aristotle University of Thessaloniki,
Department of Mathematics, Thessaloniki, Greece.
E-mail onslaught5000@hotmail.com , theoftheo@math.auth.gr