Complete totally real submanifolds of a complex projective space

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Abstract. The present paper deals with the classification of a complete totally submanifold of a complex projective space by applying Bochner formula.

M.S.C. 2010: 53C17, 53C40, 53C42.
Key words: complete submanifold; totally real submanifold; complex projective space, Bochner formula; Gauss and Weingarten formula.

1 Introduction

The study of submanifolds of a Riemannian space form (in particular complex space form) has been an area of interest for many differential geometers for many years. In [2], Barros studied the properties of compact minimal submanifolds of the Euclidean sphere $\mathbb{S}^n$ and obtained a characterization of $\mathbb{S}^n$. Moreover using Obata's theorem [9], Okumura [10] proved that an $(n-1)$-dimensional complete simply connected totally umbilical submanifold with non-zero constant mean curvature of an n-dimensional locally product Riemannian manifold is isometric to a sphere. In [6], Rio, Kupeli and Unal characterized Euclidean sphere using a standard differential equation which is the another version of Obata's differential equation.

On the other hand, Djoric and Okumura [5] discussed n -dimensional CR-submanifolds with $(n-1)$ as CR-dimension in a complex projective space and established an inequality between Ricci tensor, the scalar curvature and the mean curvature. Later, Pak and Kim [12] studied CR-submanifolds with $(n-1)$ as CR-dimension in a complex hyperbolic space.

Recently, we studied of the geometry of complete submanifolds of a Riemannian space form and proved the follwoing [8]; Let M^n be a complete submanifold of a Riemannian space form $\bar{M}^{n+p}(c)$, $(c \neq 0)$ with the Ricci curvature bounded from below and without boundary. If M admits a real valued non-constant function f such that $\Delta f + \lambda f = 0$ and $\lambda \leq nc$, then M^n is either isometric to a sphere $\mathbb{S}^n$ for $\lambda > 0$ or isometric to a warped product of the Euclidean line and a complete Riemannian manifold whose warping function ψ satisfies the equation $\dfrac{d^2\psi}{dt^2} + \dfrac{\lambda}{n}\psi = 0$. And, let M^n be a complete n-dimensional CR-submanifold without boundary and with the Ricci

Differential Geometry - Dynamical Systems, Vol.20, 2018, pp. 119-125.

curvature bounded from below and CR-dimension$(n-1)$ in the complex space form $\bar{M}^{\frac{(n+p)}{2}}(4)$. If $f : M^n \longrightarrow \mathbb{R}$ is any smooth function on M^n satisfying the conditions $\Delta f + \lambda f = 0$ and $\lambda \leq n$, then M^n is isometric to one of the following:

(a) connected component of the hyperbolic space,

(b) warped product of the Euclidean line and a complete Riemannian manifold, where the warping function ψ satisfies the equation $\dfrac{d^2\psi}{dt^2} + \dfrac{\lambda}{n}\psi = 0$,

(c) Euclidean sphere.

The purpose of the paper is devoted to study the geometry of a totally real submanifolds of a complex projective space. The main result of the paper is the following:

Theorem *Let M^n be a complete totally real submanifold of a complex projective space $\bar{M}^n$ with the Ricci curvature bounded from below and without boundary. If M admits a real valued non-constant function f such that $\Delta f + \lambda f = 0$ and $\lambda \leq n$, then M^n is isometric to one of the following:*

(a) connected component of the hyperbolic space,

(b) warped product of the Euclidean line and a complete Riemannian manifold, where the warping function ψ satisfies the equation $\dfrac{d^2\psi}{dt^2} + \dfrac{\lambda}{n}\psi = 0$,

(c) Euclidean sphere.

We remark in the future we want to apply these way of this paper to CR-submanifolds in quaternionic space forms which was defined by M. Barros, B-Y Chen and F. Urbano [1].

2 Preliminaries

Let $\bar{M}^n$ be the n-dimensional complex projective space with the Fubini-Study meric of constant holomorphic sectional curvature 4 and let M^n be a complete submanifold of $\bar{M}$. Let us consider an immersion $\psi : M^n \longrightarrow \bar{M}^n$ and let $\{e_1, e_2, \ldots, e_n, Je_1, \ldots, Je_n\}$ be an adapted orthonormal frame of $\bar{M}^n$ such that $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame to M^n and $\{Je_1, \ldots, Je_n\}$ is an orthonormal frame of the normal bundle $TM^\perp$ of M^n, where J is the complex sturucture of $\bar{M}^n$. We denote by $\bar{\nabla}$ and ∇ the Levi-Civita connection on $\bar{M}(n)$ and M^n, respectively. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\bar{\nabla}_X Je_i = -A_i X + \nabla^\perp_X Je_i, \quad i = 1, 2, \ldots, \tag{2.2}$$

for any vector X, Y tangent to M^n [4], where A_i is given by A_{Je_i}. Here $\nabla^\perp$ denotes the normal connection induced from $\bar{\nabla}$ in the normal bundle $TM^\perp$ of M^n, and h and A_α are the second fundamental form and the shape operator corresponding to Je_i, respectively. Further, h and A_i are related as

$$h(X, Y) = \sum_{i=1}^{n} g(A_i X, Y) Je_i. \tag{2.3}$$

Then we have the following equation

$$g(h(e_i, e_j), Je_k) = g(A_i e_j, e_k).$$

. The mean curvature vector H is given by $H = \frac{1}{n}\sum_{i=1}^{n}(trA_i)Je_i$. The equation of Gauss is given by

$$\begin{aligned} R(X,Y,Z,W) \;=\; & g(Y,Z)g(X,W) - g(Y,W)g(X,Z) + g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W) \\ & +2g(X,JY)g(JZ,W) + g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)). \end{aligned}$$

Then we have

(2.4)

$$\mathrm{Ric}(e_i, e_j) = (n-1)g(e_i, e_j) + \sum_{k=1}^{n}(\mathrm{tr}\ A_k)g(A_k e_i, e_j) - \sum_{k=1}^{n} g(h(e_k, e_i), h(e_j, e_k)).$$

The following generalized maximum principle due to Omori [11] and Yau [13] will be used in order to prove our theorems.

Theorem 2.1. *Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below and $f \in C^2(M)$ a function bounded from above on M^n. Then, for any $\epsilon > 0$, there exists a point $p \in M^n$ such that*

$$f(p) \geq \mathrm{sup} f - \epsilon, ||\mathrm{grad} f|| < \epsilon, \Delta f(p) < \epsilon.$$

For a function $f : M^n \longrightarrow \mathbb{R}$, Bochner formula is given by [2]

$$\frac{1}{2}\Delta \|\nabla f\|^2 = \|\mathrm{Hess}\ f\|^2 + \mathrm{Ric}(\nabla f, \nabla f) + g(\nabla f, \nabla(\Delta f)) \tag{2.5}$$

where Hess, Ric and Δ stand for the Hessian form, Ricci tensor and the Laplacian, respectively, and the square of the norm of an operator A is given by $\|A\|^2 = tr(AA^*)$.

3 Application of Bochner formula in space forms

The results of the paper will be proved by appying Bochner formula. To prove theorem, we need the following lemma which we will state and prove first.

Lemma 3.1 *Let M^n be a submanifold without boundary of a complex projective space $\bar{M}^n$, Let $f : M^n \longrightarrow \mathbb{R}$ be any function on M^n and λ be the first eigenvalue of the Laplacian of M^n, i.e. $\Delta f + \lambda f = 0$. Then for any $t \in \mathbb{R}$ we have*

$$\|\mathrm{Hess}\ f\|^2 = \|\mathrm{Hess}\ f - tfI\|^2 - (2t + \frac{nt}{\lambda})(\|\nabla f\|^2 - \frac{1}{2}\Delta f^2),$$

where Hess f *and I denote the Hessian operator of f and the identity operator, respectively. The norm of any operator A is Euclidean, i.e. $\|A\| = \mathrm{tr}(AA^*)$.*

Proof. We have

$$\|\mathrm{Hess}\ f - tfI\|^2 = \|\mathrm{Hess}\ f\|^2 + t^2 f^2 \|I\|^2 - 2tf\mathrm{I}\mathrm{Hess}f.$$

for any $t \in \mathbb{R}$. It is clear that $\|I\|^2 = \mathrm{tr}(II^*) = n$ and $I\mathrm{Hess}\ f = \mathrm{trHess}\ f$. Now

$$\Delta f = g^{ij}\nabla_j\nabla_i f = \nabla^i\nabla_i f = \mathrm{trHess} f.$$

Therefore

$$\|\mathrm{Hess}\ f - tfI\|^2 = \|\mathrm{Hess}\ f\|^2 + nt^2f^2 + 2t\lambda f^2,$$

which implies that

$$\|\mathrm{Hess}\ f - tfI\|^2 = \|\mathrm{Hess}\ f\|^2 + (2t + \frac{nt^2}{\lambda})\lambda f^2. \tag{3.1}$$

Also we know that

$$\Delta f^2 = 2f\Delta f + 2\|\nabla f\|^2.$$

This gives

$$\lambda f^2 = \|\nabla f\|^2 - \frac{1}{2}\Delta f^2. \tag{3.2}$$

From equations (3.1) and (3.2) we get

$$\|\mathrm{Hess}\ f - tfI\|^2 = \|\mathrm{Hess}\ f\|^2 + (2t + \frac{nt^2}{\lambda})(\|\nabla f\|^2 - \frac{1}{2}\Delta f^2),$$

which implies that

$$\|\mathrm{Hess}\ f\|^2 = \|\mathrm{Hess}\ f - tfI\|^2 - (2t + \frac{nt^2}{\lambda})(\|\nabla f\|^2 - \frac{1}{2}\Delta f^2). \tag{3.3}$$

Proof of Theorem:
Equation (2.4) yields

$$\begin{aligned}\sum_{i,j}\mathrm{Ric}(f_ie_i, f_je_j) &= \sum_{i,j}(n-1)f_if_jg(e_i,e_j) + \sum_{i,j}f_if_jg(h(e_i,e_i),h(e_j,e_j)) \\ &\quad - \sum_{i,j,k}f_if_jg(h(e_i,e_k),h(e_j,e_k)),\end{aligned}$$

where $\nabla f = \sum_i f_ie_i$. This gives us

$$\begin{aligned}\sum_{i,j}\mathrm{Ric}(f_ie_i, f_je_j) &= (n-1)\|\nabla f\|^2 + \sum_{i,j}f_if_jg(h(e_i,e_i),h(e_j,e_j)) \\ &\quad - \sum_{i,j,k}f_if_jg(h(e_{i,}e_k),h(e_j,e_k))\end{aligned}$$

$$= (n-1)\|\nabla f\|^2 + + \sum_{i,j}f_if_jg(h(e_i,e_i),h(e_j,e_j)) - \sum_i g(h(\nabla f,e_i),h(\nabla f,e_i))$$

$$= (n-1)\|\nabla f\|^2 + \sum_{i,j}f_if_jg(h(e_i,e_i),h(e_j,e_j)) - \sum_i\|h(\nabla f,e_i)\|^2. \tag{3.4}$$

It reminds Bochner formula (2.4)

$$\frac{1}{2}\Delta \|\nabla f\|^2 = \|\text{Hess } f\|^2 + \text{Ric}(\nabla f, \nabla f) + g(\nabla f, \nabla(\Delta f)).$$

Now plugging the values of $\|\text{Hess } f\|^2$ and $\text{Ric}(\nabla f, \nabla f)$ from equations (3.3) and (3.4) into equation (2.4), we get

$$\frac{1}{2}\Delta \|\nabla f\|^2 = \|\text{Hess } f - tfI\|^2 - (2t + \frac{nt^2}{\lambda})(\|\nabla f\|^2 - \frac{1}{2}\Delta f^2) + (n-1)\|\nabla f\|^2 + \sum_{i,j} f_i f_j g(h(e_i, e_i), h(e_j, e_j)) - \sum_i \|h(\nabla f, e_i)\|^2 - \lambda \|\nabla f\|^2.$$

Also according to the definition of the first eigenvalue λ we must have $\frac{\text{Ric}(\nabla f, \nabla f)}{(n-1)||\nabla f||^2} \geq \frac{\lambda}{n}$ [3], [9] and the assumption of $\Delta f + \lambda f = 0$ and hence

$$\frac{1}{2}\Delta \|\nabla f\|^2 = \|\text{Hess } f - tfI\|^2 + \frac{1}{2}(2t + \frac{nt^2}{\lambda})\Delta f^2 + (n-1)\|\nabla f\|^2 + \sum_{i,j} f_i f_j g(h(e_i, e_i), h(e_j, e_j)) - \sum_i \|h(\nabla f, e_i)\|^2 - (n-1)\frac{\lambda}{n}\|\nabla f\|^2 - (2t + \frac{nt^2}{\lambda} + \lambda - (n-1)\frac{\lambda}{n})\|\nabla f\|^2.$$

If $t = -\frac{\lambda}{n}$ then the R.H.S. of the above equation reduces to

$$\frac{1}{2}\Delta \|\nabla f\|^2 + \frac{\lambda}{2n}\Delta f^2 - \left\|\text{Hess } f + \frac{\lambda}{n} fI\right\|^2 \geq 0. \tag{3.5}$$

It is easy to see that

$$\left\|\text{Hess } f + \frac{\lambda}{n} fI\right\|^2 \geq 0. \tag{3.6}$$

From the assumption of the Ricci curvature bounded from below and equations (3.5), (3.6) we conclude that

$$\left\|\text{Hess} f + \frac{\lambda}{n} fI\right\|^2 = 0,$$

which implies that $\text{Hess} f + \frac{\lambda}{n} fI = 0$. The above result for $\lambda \leq 0$ breaks up into two possible isometries of M^n given by

(i) M^n is isometric to a connected component of the hyperbolic space if $(\nabla f)_p = 0$ at some $p \in M^n$ [6].

(ii) M^n is isometric to the warped product of the Euclidean line and a complete Riemannian manifold if ∇f is non-vanishing, where warping function ψ on $\mathbb{R}$ satisfies the equation [6]

$$\frac{d^2\psi}{dt^2} + \lambda\psi = 0, \psi > 0.$$

Further if λ satisfies the inequality $0 < \lambda \leq n$, then from equation (3.5) we have

$$\frac{1}{2}\Delta \|\nabla f\|^2 + \frac{\lambda}{2n}\Delta f^2 - \left\|\mathrm{Hess}f + \frac{\lambda}{n} fI\right\|^2 \geq 0. \tag{3.7}$$

But we clearly have

$$\left\|\mathrm{Hess}f + \frac{\lambda}{n} fI\right\|^2 \geq 0. \tag{3.8}$$

Combining the assumption of the Ricci curvature bounded from below and the inequalities (3.7), (3.8), we obtain

$$\left\|\mathrm{Hess}f + \frac{\lambda}{n} fI\right\|^2 = 0,$$

which gives

$$\mathrm{Hess}f + \frac{\lambda}{n} fI = 0 \quad \text{for } 0 < \lambda \leq n.$$

Hence M^n is isometric to a sphere [9]. This completes the proof of the theorem.

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