On singular Lagrangians

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Abstract. The theory of singular (pseudo)Riemannian metrics is well known, according to Kupeli's monograph. A singular Lagrangian analogous to the (pseudo)Riemannian is defined and studied in our paper.

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1 Introduction

The theory of singular (pseudo)Riemannian metrics is well known, according to Kupeli's monograph. (see [2, 7] and the references therein). A singular Lagrangian analogous to the (pseudo)Riemannian one is intended to be given below.

We recall briefly the main facts involving (pseudo)Riemannian metrics.

A *singular (pseudo)Riemannian metric* is a bilinear map g (or a two-covariant tensor) in the fibers of TM, where $\pi_{TM} : TM \to M$ is the tangent bundle of a smooth manifold M. In order to involve a regularity condition (on foliations) we suppose that the nullity N_g of g has a constant rank $r > 1$. In [2, Definition 3.1.1] it is defined a natural Koszul derivative of g and according to [2, Lemma 3.1.2], a Koszul derivative of g exists iff (if and only if) $\mathcal{L}_U g = 0$ for every section $U \in \Gamma(N_g)$, where $\mathcal{L}$ denotes the Lie derivative; if it is the case, the distribution $N_g \subset TM$ is integrable ([2, Lemma 3.1.4, a)]), giving rise to a (pseudo)Riemannian foliation and a transverse and regular (pseudo)Riemannian metric $\bar{g} : N_{\mathcal{F}} \to \mathbb{R}$, where $\mathcal{F}$ is the regular foliation having $\tau\mathcal{F} = N_g$ as tangent bundle and $N_{\mathcal{F}}= \mathcal{TM}/\tau\mathcal{F}$ as its normal bundle.

Recall that if $\pi_E : E \to M$ is a vector bundle, then $\pi_{VE} : VE = \ker \tau\pi_E \to E$ is its vertical bundle, where $\tau\pi_E : TE \to TM$ denotes the differential map of π_E and $\ker \tau\pi_{VE}$ denotes the kernel of the epimorphism of vector bundles. We use local coordinates adapted to the vector bundle structure: (x^i) on M, (x^i, y^a) on E, (x^i, X^i) on TM, (x^i, y^a, Y^a) on VE, (x^i, y^a, X^i, Y^a) on TE, such that the vector bundle projections have natural local forms.

A Lagrangian is a smooth map $L : TM \to \mathbb{R}$, or, by extension, in a more general setting, $L : E \to \mathbb{R}$. Notice that we can consider $L : TM_* \to \mathbb{R}$, where the slashed bundle $TM_* \subset TM$ is obtained removing the image of the null section; but, for sake of simplicity we use $L : TM \to \mathbb{R}$.

Notice that every two-covariant tensor (called, by extension, a *metric tensor*) gives rise to a quadratic Lagrangian.

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2 Preliminaries on foliations

Let us consider an $(n+m)$-dimensional manifold M, connected and orientable. A foliation $\mathcal{F}$ of codimension n on M is defined by a foliated cocycle $\{U_i, \varphi_i, f_{i,j}\}$.

Every fibre of the local submersion φ_i is called a *plaque* of the foliation. The manifold M is decomposed into submanifolds, called *leafs* of $\mathcal{F}$. If $U \subset M$ is an open subset, then there is an *induced foliation* $\mathcal{F}_U$.

We denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$ and by $\Gamma(T\mathcal{F})$ the module of its global sections, i.e. the vector fields on M tangent to $\mathcal{F}$. The *normal bundle* of $\mathcal{F}$ is $N\mathcal{F} = TM/T\mathcal{F}$. A vector field on M is *transverse* if it locally projects on a local vector field of the transversal manifold.

A system of local coordinates adapted to $\mathcal{F}$ is given by coordinates $(x^u, x^{\bar{u}})$, $u = 1, \ldots, m$, $\bar{u} = 1, \ldots, n$ on an open subset U, where $\mathcal{F}_U$ is trivial and defined by the equations $dx^{\bar{u}} = 0$, $\bar{u} = 1, \ldots, n$.

A particular example of a foliation is a *fibered manifold*, called a *simple foliation*. In particular, a *locally trivial fibration*.

There are elementary examples of simple foliations that come from no trivial fibrations and the spaces of leaves are not Hausdorff separated.

3 Tangent bundle geometry

Let us briefly recall now some constructions from the tangent space geometry [3, 8]. Any vector field $X \in \mathcal{X}(M)$ can be lifted to a *vertical lift* $X^v \in \Gamma(VTM) \subset \mathcal{X}(TM)$ and to a *complete lift* $X^c \in \mathcal{X}(TM)$. Using local coordinates, if $X = X^i\left(x^j\right)\dfrac{\partial}{\partial x^i}$, then

$$X^v = X^i \frac{\partial}{\partial y^i},\ X^c = X^i \frac{\partial}{\partial x^i} + y^j \frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial y^i}.$$

We use further the following formulas related to these two lifts:

$$(3.1)\quad (fX)^v = fX^v,\ (X+Y)^v = X^v + Y^v,\ [X^v, Y^v] = 0,\ [X^v, Y^c] = [X.Y]^v,$$
$$(3.2)\quad (fX)^c = fX^c + df(\Delta)X^v,\ (X+Y)^c = X^c + Y^c,\ [X^c, Y^c] = [X, Y]^c,$$

where $X, Y \in \mathcal{X}(M)$, $f \in \mathcal{F}(M)$, $[\cdot,\cdot]$ denotes the Lie bracket and $df(\Delta) = \dfrac{\partial f}{\partial x^i}y^i$ denotes the evaluation of $df = \dfrac{\partial f}{\partial x^i}dx^i \in \mathcal{X}^*(M)$ (the differential of f, lifted to $\mathcal{F}(TM)$) and $\Delta = y^i\dfrac{\partial}{\partial y^i} \in \Gamma(VTM) \subset \mathcal{X}(TM)$ (the Liouville vector field).

4 The extended distribution

Let us suppose now that a regular k-dimensional distribution $\mathcal{D} \subset TM$ is given.

Proposition 4.1. *The $\mathcal{F}(TM)$-linear spans of the vertical lifts and respectively vertical and complete lifts of vector fields from $\mathcal{D}$ give rise to two regular distributions $\mathcal{D}^v \subset \Gamma(VTM)$ and $\mathcal{D}^{cv} \subset TTM$ of dimensions k and $2k$ respectively. Also we have:*

1. *$\pi_{TTM}(\mathcal{D}^v) = \bar{0}_{TM}$ (where $\bar{0}_{TM}$ is the image of the null section $M \to TM$) and $\pi_{TTM}(\mathcal{D}^c) = \mathcal{D}$.*

2. *The distribution $\mathcal{D}^v$ is always integrable, but $\mathcal{D}^{cv}$ is integrable iff $\mathcal{D}$ is integrable.*

Thus if a (regular) foliation $\mathcal{F}$ is given on M, then, by Proposition 4.1, the vertical and complete lifts from $\tau\mathcal{F}$ give rise together to a foliation $\mathcal{F}^{cv}$ on the manifold TM. Notice that $(\tau\mathcal{F})^v$ gives rise to a foliation with leaves in the fibers of $\pi_{TM} : TM \to M$, thus it projects to the points of M.

We use now adapted coordinates to the foliation $\mathcal{F}$. Consider $(x^u, x^{\bar{u}})$ coordinates on M, where $(x^{\bar{u}})$ are transverse coordinates. Then $(x^u, x^{\bar{u}}, y^u, y^{\bar{u}})$ are coordinates on TM, where $(x^{\bar{u}}, y^{\bar{u}})$ are transverse coordinates, and $(x^u, x^{\bar{u}}, y^{\bar{u}})$ are coordinates on $N_{\mathcal{F}}$. A foliation $\mathcal{F}_{N_{\mathcal{F}}} = \Pi_{N_{\mathcal{F}}}(\mathcal{F}^{cv})$ is induced on $N\mathcal{F}$ by the canonical projection $\Pi_{N_{\mathcal{F}}} : TM \to N_{\mathcal{F}} = \mathcal{TM}/\tau\mathcal{F}$.

We say that a Lagrangian $L : TM \to \mathbb{R}$ is *basic* according to the foliation $\mathcal{F}$ on M, if L is a basic function of the foliation $\mathcal{F}^{cv}$. Using local coordinates, L has the form $L = L(x^{\bar{u}}, y^{\bar{u}})$. For such an L, the Hessian is obviously singular, but we can consider the *basic Hessian* H_L as a bilinear form in the fibers of $VN_{\mathcal{F}}$. We say that L is *transversally regular* if its basic Hessian is nondegenerate. If it is the case, the normal bundle $N_{\mathcal{F}^{cv}}$ has a Whitney decomposition $N_{\mathcal{F}^{cv}} = VN_{\mathcal{F}} \oplus HN_{\mathcal{F}}$ and an isomorphism of $VN_{\mathcal{F}}$ and $HN_{\mathcal{F}}$. It allows to extend the nondegenerate basic Hessian H_L (on $VN_{\mathcal{F}}$) to a nondegenerate basic bilinear form H'_L (on $HN_{\mathcal{F}}$), giving together a nondegenerate basic bilinear form H''_L on $N_{\mathcal{F}^{cv}}$. Using the general arguments in [4, 3.2], one can obtain the following result:

Proposition 4.2. *The transverse metric H''_L lifts to a singular metric H'''_L in the fibers of $\pi_{TTM} : TTM \to TM$ that projects to H''_L on $N_{\mathcal{F}^{cv}} = TTM/\tau\mathcal{F}^{cv}$.*

Consider a Lagrangian L and denote by H_L its Hessian. We have:

5 The Main result

Theorem 5.1. *Let $L : TM \to \mathbb{R}$ be a Lagrangian. The following conditions A, B and C are equivalent:*

A *The following conditions $A1 - A3$ hold:*

- **A1** *– the nullity bundle $\mathcal{N}_{H_L} \subset VTN$ has a constant rank $r > 0$ and there is a distribution $\mathcal{D} \subset TM$ such that $\mathcal{N}_{H_L} = \mathcal{D}^v$;*
- **A2** *– there is a singular metric H'''_L in the fibers of $TTM \to TM$, that restricts to the Hessian H_L on VTM, $\mathcal{N}_{H'''_L} = \mathcal{D}^{cv}$ and $X(L) = 0$, $(\forall)X \in \Gamma(\mathcal{D}^{cv})$;*
- **A3** *– $\mathcal{L}_U H'''_L = 0$, $(\forall)U \in \Gamma\left(\mathcal{N}_{H'''_L}\right)$;*

B *The following conditions $B1 - B3$ hold:*

- **B1** *=A1;*
- **B2** *– the distribution $\mathcal{D}^{cv}$ is integrable giving a foliation $\mathcal{F}^{cv}$;*
- **B3** *– L is a basic function according to the foliation $\mathcal{F}^{cv}$.*

C *The Lagrangian L is transversally regular according to a regular foliation $\mathcal{F}$ on M.*

6 Further developments

A systematic study of projectable Lagrangians and Hamiltonians was performed in separate papers [5, 6, 7], but not a monograph like D.N. Kupeli's work [2].

Most of physical and/or mathematical settings on Lagrangians and Hamiltonians can be translated into a foliated language, using an appropriate approach. Singular Lagrangians, as considered here, can be used for, where a series of papers of O. C. Stoica (as, for example [7]) are useful.

In order to handle the singular cases, one can use *algebroids*, or *Lie algebroids*, where our constructions can be extended.

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