Special odd-dimensional spaces with a symmetric affine connection

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Abstract. Odd-dimensional spaces endowed with a symmetric affine connection and an additional tensor structure are studied. The structure under consideration is defined by the directional vector fields of a given net. Equiaffine and Riemannian subspaces containing such structures are characterized. A connection with torsion which preserves by parallel translation the additional structure is defined and studied.

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1 Introduction

Spaces with symmetric affine connections, Weyl spaces and Riemannian spaces with additional tensor structures: almost product, almost paracontact and almost contact are studied in [1, 2, 3, 4, 5, 9, 10, 11, 12].

By help of n independent vector fields, in [13, 14, 15], an apparatus for the study of spaces with a symmetric affine connection and special compositions or nets is constructed. This apparatus is applied to the study of triples of compositions in [2] and almost paracontact and almost contact structures in [10].

In the present work, by help of the same apparatus, we study special odd-dimensional spaces endowed with a symmetric affine connection and a covariantly constant affinor. This affinor is defined by $2n$ independent vector fields and their reciprocal covectors. The obtained results are applied to equiaffine and Riemannian spaces. A non-symmetric affine connection which preserves by covariant differentiation the considered affinor structure is defined and studied.

2 Preliminaries

Let A_{2n+1} be a space endowed with a symmetric affine connection ∇. The Christoffel symbols of ∇ are denoted by $\Gamma^{\nu}_{\alpha\beta}$. Also, let us introduce the following notations

$$\begin{aligned} &\alpha,\beta,\gamma,\sigma,\nu,\delta,\tau = 1,2,...,2n+1, \quad \alpha_1,\alpha_2,...,\alpha_{2n+1} = 1,2,...,2n+1; \\ &\beta_1,\beta_2,...,\beta_{2n+1} = 1,2,...,2n+1, \quad i,j,k,s = 1,2,...,2n. \end{aligned} \tag{2.1}$$

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Let $\underset{\alpha}{v}^{\beta}$ be independent vector fields. The net defined by $\underset{\alpha}{v}^{\beta}$ is denoted by $\{\underset{\alpha}{v}\}$. The reciprocal covector fields $\overset{\alpha}{v}_{\beta}$ of the vector fields $\underset{\alpha}{v}^{\beta}$ are determined by

$$\underset{\alpha}{v}^{\beta}\,\overset{\nu}{v}_{\beta} = \delta^{\nu}_{\alpha} \quad \Longleftrightarrow \quad \underset{\beta}{v}^{\nu}\,\overset{\beta}{v}_{\alpha} = \delta^{\nu}_{\alpha}, \tag{2.2}$$

where δ^{ν}_{α} is the identity affinor.

Let $\left\{\underset{\alpha}{v}\right\}$ be the coordinate net. Then we have [13, 14]:

$$\begin{aligned} &\underset{1}{v}^{\alpha}\,(1,0,0,...,0)\,, \underset{2}{v}^{\alpha}\,(0,1,0,...,0)\,, ..., \underset{2n+1}{v^{\alpha}}\,(0,0,...,0,1)\,;\\ &\overset{1}{v}_{\alpha}\,(1,0,0,...,0)\,, \overset{2}{v}_{\alpha}\,(0,1,0,...,0)\,, ..., \overset{2n+1}{v_{\alpha}}\,(0,0,...,0,1)\,. \end{aligned} \tag{2.3}$$

In addition to the usual coordinates x^{α} $(\alpha = 1, 2, ..., 2n{+}1)$, in A_{2n+1} we introduce coordinates with respect to the coordinate net $\{\underset{\alpha}{v}\}$ which will be denoted by $\overset{\alpha}{u}$. Then, for an arbitrary vector field v^{α} we have $v^{\alpha}(\overset{1}{u}, \overset{2}{u}, ..., \overset{2n+1}{u})$.

The following derivative equations are known to be valid [15]:

$$\nabla_{\sigma}\,\underset{\alpha}{v}^{\beta} = \underset{\alpha}{\overset{\nu}{T}}_{\sigma}\,\underset{\nu}{v}^{\beta}, \qquad \nabla_{\sigma}\,\overset{\alpha}{v}_{\beta} = -\underset{\nu}{\overset{\alpha}{T}}_{\sigma}\,\overset{\nu}{v}_{\beta}. \tag{2.4}$$

In accordance to [14, 15], in the parameters of the coordinate net we have

$$\underset{\alpha}{\overset{\beta}{T}}_{\sigma} = \Gamma^{\beta}_{\sigma\alpha}. \tag{2.5}$$

In the space A_{2n+1}, we consider a composition $X_p \times X_q$ of two basic manifolds X_p and X_q $(p + q = 2n + 1)$, i.e. their topological product. Two positions (tangent spaces) denoted by $P(X_p)$ and $P(X_q)$ pass thought each point of the space A_{2n+1} [6, 7]. According to [6], the coordinates $\overset{\alpha}{u}$ are adapted to the composition $X_p \times X_q$.

It is known that a composition is completely defined by the affinor field a^{β}_{α}, satisfying the condition $a^{\beta}_{\alpha}a^{\nu}_{\beta} = \delta^{\nu}_{\alpha}$ [6].

The integrability condition of a^{β}_{α} is given by $a^{\sigma}_{\beta}\nabla_{[\alpha}\; a^{\nu}_{\sigma]} - a^{\sigma}_{\alpha}\nabla_{[\beta}\; a^{\nu}_{\sigma]} = 0$ [6, 7].

Let us consider the following affinor

$$a^{\beta}_{\alpha} = \underset{i}{v}^{\beta}\overset{i}{v}_{\alpha} - \underset{2n+1}{v^{\beta}}\overset{2n+1}{v_{\alpha}}. \tag{2.6}$$

By (2.2) and (2.6) it follows that $a^{\beta}_{\alpha}a^{\nu}_{\beta} = \delta^{\nu}_{\alpha}$. Hence the affinor (2.6) defines a composition $X_{2n} \times X_1$. The projecting affinors are given by: $\overset{2n}{a}{}^{\beta}_{\alpha} = \underset{i}{v}^{\beta}\overset{i}{v}_{\alpha}$ and $\overset{1}{a}{}^{\beta}_{\alpha} = \underset{2n+1}{v^{\beta}}\overset{2n+1}{v_{\alpha}}$ [13, 14]. Obviously, $\underset{i}{v}^{\alpha} \in P(X_{2n})$ and $\underset{2n+1}{v^{\alpha}} \in P(X_1)$.

The composition $X_{2n} \times X_1$ is of the type (c,c) if the positions $P(X_{2n})$ and $P(X_1)$ are translated parallelly along any line in the space A_{2n+1} [7]. According to [7], the composition $X_{2n} \times X_1$ is of the type (c,c) if and only if in the adapted coordinates the following conditions are valid:

$$\Gamma^{2n+1}_{\sigma i} = \Gamma^{i}_{\sigma 2n+1} = 0. \tag{2.7}$$

3 Spaces A_{2n+1} with additional affinor structures

Let A_{2n+1} be a space with a asymmetric affine connection. Let us consider the following affinor

$$A_{\alpha}^{\beta} = \underset{1}{v}^{\beta}\overset{2}{v}_{\alpha} + \underset{2}{v}^{\beta}\overset{3}{v}_{\alpha} + ... + \underset{2n-1}{v}^{\beta}\overset{2n}{v}_{\alpha} + \underset{2n}{v}^{\beta}\overset{1}{v}_{\alpha}. \tag{3.1}$$

The equalities (2.1) and (3.1) yield

$$A_{\beta}^{\alpha_1}A_{\alpha_1}^{\alpha_2}A_{\alpha_2}^{\alpha_3}...A_{\alpha_{2n-1}}^{\nu} = \delta_{\beta}^{\nu} - \underset{2n+1}{v}^{\nu}\overset{2n+1}{v}_{\alpha}, \qquad A_{\alpha}^{\beta}\underset{2n+1}{v}^{\alpha} = 0, \qquad A_{\alpha}^{\beta}\overset{2n+1}{v}_{\beta} = 0.$$

According to (2.2), (2.3) and (3.1), in the parameters of the coordinate net $\{\underset{\alpha}{v}\}$, the $2n \times 2n$ real matrix (A_{α}^{β}) is given by

$$(A_{\alpha}^{\beta}) = \begin{pmatrix} 0 & 0 & 0 & ... & 0 & 1 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

We prove the following

Theorem 3.1. *The condition*

$$\nabla_{\sigma}A_{\alpha}^{\beta} = 0 \tag{3.3}$$

holds if and only if the coefficients of the derivative equations (2.4) satisfy

$$\begin{aligned}
&\overset{1}{\underset{1}{T}}_{\sigma} = \overset{2}{\underset{2}{T}}_{\sigma} = \overset{3}{\underset{3}{T}}_{\sigma} = ... = \overset{2n}{\underset{2n}{T}}_{\sigma}, \quad \overset{i}{\underset{2n+1}{T}}_{\sigma} = \overset{2n+1}{\underset{i}{T}}_{\sigma} = 0,\\
&\overset{1}{\underset{2n}{T}}_{\sigma} = \overset{2}{\underset{1}{T}}_{\sigma} = \overset{3}{\underset{2}{T}}_{\sigma} = ... = \overset{2n}{\underset{2n-1}{T}}_{\sigma},\\
&\overset{2}{\underset{2n}{T}}_{\sigma} = \overset{3}{\underset{1}{T}}_{\sigma} = \overset{4}{\underset{2}{T}}_{\sigma} = ... = \overset{2n}{\underset{2n-2}{T}}_{\sigma} = \overset{1}{\underset{2n-1}{T}}_{\sigma},\\
&\overset{3}{\underset{2n}{T}}_{\sigma} = \overset{4}{\underset{1}{T}}_{\sigma} = \overset{5}{\underset{2}{T}}_{\sigma} = ... = \overset{2n}{\underset{2n-3}{T}}_{\sigma} = \overset{2}{\underset{2n-1}{T}}_{\sigma} = \overset{1}{\underset{2n-2}{T}}_{\sigma},\\
&\overset{4}{\underset{2n}{T}}_{\sigma} = \overset{5}{\underset{1}{T}}_{\sigma} = \overset{6}{\underset{2}{T}}_{\sigma} = ... = \overset{2n}{\underset{2n-3}{T}}_{\sigma} = \overset{3}{\underset{2n-1}{T}}_{\sigma} = \overset{2}{\underset{2n-2}{T}}_{\sigma} = \overset{1}{\underset{2n-3}{T}}_{\sigma},\\
&\dots\dots\dots\dots\dots\dots\dots\\
&\overset{2n-1}{\underset{2n}{T}}_{\sigma} = \overset{2n}{\underset{1}{T}}_{\sigma} = \overset{1}{\underset{2}{T}}_{\sigma} = \overset{2}{\underset{3}{T}}_{\sigma} = ... = \overset{2n-2}{\underset{2n-1}{T}}_{\sigma}.
\end{aligned} \tag{3.4}$$

Proof. According to (2.4) and (3.1), equality (3.3) takes the form

$$\begin{aligned}
&\overset{\nu}{\underset{1}{T}}_{\sigma}\underset{\nu}{v}^{\beta}\overset{2}{v}_{\alpha} - \overset{2}{\underset{\nu}{T}}_{\sigma}\underset{1}{v}^{\beta}\overset{\nu}{v}_{\alpha} + \overset{\nu}{\underset{2}{T}}_{\sigma}\underset{\nu}{v}^{\beta}\overset{3}{v}_{\alpha} - \overset{3}{\underset{\nu}{T}}_{\sigma}\underset{2}{v}^{\beta}\overset{\nu}{v}_{\alpha} + ...+\\
&+ \overset{\nu}{\underset{2n-1}{T}}_{\sigma}\underset{\nu}{v}^{\beta}\overset{2n}{v}_{\alpha} - \overset{2n}{\underset{\nu}{T}}_{\sigma}\underset{2n-1}{v}^{\beta}\overset{\nu}{v}_{\alpha} + \overset{\nu}{\underset{2n}{T}}_{\sigma}\underset{\nu}{v}^{\beta}\overset{1}{v}_{\alpha} - \overset{1}{\underset{\nu}{T}}_{\sigma}\underset{2n}{v}^{\beta}\overset{\nu}{v}_{\alpha} = 0.
\end{aligned} \tag{3.5}$$

The independence of the covector fields $\overset{\nu}{v}_\alpha$ implies that (3.5) is equivalent to the following equalities:

$$
\begin{aligned}
&\underset{2n}{\overset{\nu}{T}}{}_\sigma \underset{\nu}{v}^\beta - \underset{1}{\overset{1}{T}}{}_\sigma \underset{2n}{v}^\beta - \underset{1}{\overset{2}{T}}{}_\sigma \underset{1}{v}^\beta - \underset{1}{\overset{3}{T}}{}_\sigma \underset{2}{v}^\beta - \underset{1}{\overset{4}{T}}{}_\sigma \underset{3}{v}^\beta - \dots - \underset{1}{\overset{2n}{T}}{}_\sigma \underset{2n-1}{v}^\beta = 0,\\
&\underset{1}{\overset{\nu}{T}}{}_\sigma \underset{\nu}{v}^\beta - \underset{2}{\overset{1}{T}}{}_\sigma \underset{2n}{v}^\beta - \underset{2}{\overset{2}{T}}{}_\sigma \underset{1}{v}^\beta - \underset{2}{\overset{3}{T}}{}_\sigma \underset{2}{v}^\beta - \underset{2}{\overset{4}{T}}{}_\sigma \underset{3}{v}^\beta - \dots - \underset{2}{\overset{2n}{T}}{}_\sigma \underset{2n-1}{v}^\beta = 0,\\
&\underset{2}{\overset{\nu}{T}}{}_\sigma \underset{\nu}{v}^\beta - \underset{3}{\overset{1}{T}}{}_\sigma \underset{2n}{v}^\beta - \underset{3}{\overset{2}{T}}{}_\sigma \underset{1}{v}^\beta - \underset{3}{\overset{3}{T}}{}_\sigma \underset{2}{v}^\beta - \underset{3}{\overset{4}{T}}{}_\sigma \underset{3}{v}^\beta - \dots - \underset{3}{\overset{2n}{T}}{}_\sigma \underset{2n-1}{v}^\beta = 0,\\
&\dots\dots\dots\\
&\underset{2n-1}{\overset{\nu}{T}}{}_\sigma \underset{\nu}{v}^\beta - \underset{2n}{\overset{1}{T}}{}_\sigma \underset{2n}{v}^\beta - \underset{2n}{\overset{2}{T}}{}_\sigma \underset{1}{v}^\beta - \underset{2n}{\overset{3}{T}}{}_\sigma \underset{2}{v}^\beta - \underset{2n}{\overset{4}{T}}{}_\sigma \underset{3}{v}^\beta - \dots - \underset{2n}{\overset{2n}{T}}{}_\sigma \underset{2n-1}{v}^\beta = 0,\\
&\underset{2n+1}{\overset{2}{T}}{}_\sigma \underset{1}{v}^\beta + \underset{2n+1}{\overset{3}{T}}{}_\sigma \underset{2}{v}^\beta + \underset{2n+1}{\overset{4}{T}}{}_\sigma \underset{3}{v}^\beta + \dots + \underset{2n+1}{\overset{2n}{T}}{}_\sigma \underset{2n-1}{v}^\beta + \underset{2n+1}{\overset{1}{T}}{}_\sigma \underset{2n}{v}^\beta = 0.
\end{aligned}
\tag{3.6}
$$

Since the vector fields $\underset{\nu}{v}^\alpha$ are independent, equalities (3.6) are equivalent to conditions (3.4) which proves the statement. □

Let us introduce the following notations: $\Gamma^1_{\sigma 2n} = \overset{1}{\varphi}_\sigma$, $\Gamma^2_{\sigma 2n} = \overset{2}{\varphi}_\sigma, \dots, \Gamma^{2n-1}_{\sigma 2n} = \overset{2n-1}{\varphi_\sigma}$, $\Gamma^1_{\sigma 1} = \varphi_\sigma$.

Corollary 3.2. *Equality (3.3) holds if and only if in the parameters of the coordinate net* $\{\underset{\alpha}{v}\}$ *the Christoffel symbols satisfy*

$$
\begin{aligned}
&\Gamma^1_{\sigma 1} = \Gamma^2_{\sigma 2} = \dots = \Gamma^{2n}_{\sigma 2n} = \varphi_\sigma; \quad \Gamma^i_{\sigma 2n+1} = \Gamma^{2n+1}_{\sigma i} = 0,\\
&\Gamma^1_{\sigma 2n} = \Gamma^2_{\sigma 1} = \Gamma^3_{\sigma 2} = \dots = \Gamma^{2n}_{\sigma 2n-1} = \overset{1}{\varphi}_\sigma,\\
&\Gamma^2_{\sigma 2n} = \Gamma^3_{\sigma 1} = \Gamma^4_{\sigma 2} = \dots = \Gamma^{2n}_{\sigma 2n-2} = \Gamma^1_{\sigma 2n-1} = \overset{2}{\varphi}_\sigma,\\
&\Gamma^3_{\sigma 2n} = \Gamma^4_{\sigma 1} = \Gamma^5_{\sigma 2} = \dots = \Gamma^{2n}_{\sigma 2n-3} = \Gamma^2_{\sigma 2n-1} = \Gamma^1_{\sigma 2n-2} = \overset{3}{\varphi}_\sigma,\\
&\Gamma^4_{\sigma 2n} = \Gamma^5_{\sigma 1} = \Gamma^6_{\sigma 2} = \dots = \Gamma^{2n}_{\sigma 2n-4} = \Gamma^3_{\sigma 2n-1} = \Gamma^2_{\sigma 2n-2} = \Gamma^1_{\sigma 2n-3} = \overset{4}{\varphi}_\sigma,\\
&\dots\dots\dots\dots\dots\dots\\
&\Gamma^{2n-1}_{\sigma 2n} = \Gamma^{2n}_{\sigma 1} = \Gamma^1_{\sigma 2} = \Gamma^2_{\sigma 3} = \dots = \Gamma^{2n-2}_{\sigma 2n-1} = \overset{2n-1}{\varphi_\sigma}.
\end{aligned}
\tag{3.7}
$$

Proof. In the parameters of the coordinate net $\{\underset{\alpha}{v}\}$ equality (2.5) holds which implies the equivalence of (3.4) and (3.7). □

Corollary 3.3. *If (3.3) holds, the composition* $X_{2n} \times X_1$ *defined by the affinor (2.6) is of the type (c,c).*

Proof. By (3.3) and Corollary 3.2 it follows that in the parameters of the coordinate net $\{\underset{\alpha}{v}\}$ equalities (2.7) hold true. According to [7], equalities (2.7) imply that the composition $X_{2n} \times X_1$ is of the type (c,c). □

Example 1. Let A_{2n+1} be an equiaffine space with main n-vector $e_{\beta_1\beta_2\ldots\beta_{2n+1}}$. In the parameters of the coordinate net $\{\underset{\alpha}{v}\}$, the main density of A_{2n+1} is denoted by $e = e_{12\ldots 2n+1}$. According to [8], the space A_{2n+1} is equaffine if and only if

$$\Gamma^{\alpha}_{\sigma\alpha} = \partial_\sigma \ln e. \tag{3.8}$$

Let (3.3) be valid. Equalities (3.7) and (3.8) imply

$$\begin{aligned}
&\Gamma^1_{i1} = \Gamma^2_{i2} = \ldots = \Gamma^{2n}_{i2n} = \tfrac{1}{2n}\partial_i \ln e, \quad \Gamma^{2n+1}_{2n+1,2n+1} = \partial_{2n+1}\ln e,\\
&\Gamma^i_{\sigma 2n+1} = \Gamma^{2n+1}_{\sigma i} = 0,\\
&\varphi_1 = \overset{1}{\varphi}_2 = \overset{2}{\varphi}_3 = \overset{3}{\varphi}_4 = \ldots = \overset{2n-3}{\varphi}_{2n-2} = \overset{2n-2}{\varphi}_{2n-1} = \overset{2n-1}{\varphi}_{2n} = \tfrac{1}{2n}\partial_1 \ln e,\\
&\varphi_2 = \overset{1}{\varphi}_3 = \overset{2}{\varphi}_4 = \overset{3}{\varphi}_5 = \ldots = \overset{2n-3}{\varphi}_{2n-1} = \overset{2n-2}{\varphi}_{2n} = \overset{2n-1}{\varphi}_1 = \tfrac{1}{2n}\partial_2 \ln e,\\
&\varphi_3 = \overset{1}{\varphi}_4 = \overset{2}{\varphi}_5 = \overset{3}{\varphi}_6 = \ldots = \overset{2n-3}{\varphi}_{2n} = \overset{2n-2}{\varphi}_1 = \overset{2n-1}{\varphi}_2 = \tfrac{1}{2n}\partial_3 \ln e,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\varphi_{2n} = \overset{1}{\varphi}_1 = \overset{2}{\varphi}_2 = \overset{3}{\varphi}_3 = \ldots = \overset{2n-3}{\varphi}_{2n-3} = \overset{2n-2}{\varphi}_{2n-2} = \overset{2n-1}{\varphi}_{2n-1} = \tfrac{1}{2n}\partial_{2n} \ln e.
\end{aligned} \tag{3.9}$$

By (3.9) for the Ricci tensor we have $R_{2n+1,2n+1} = 0$, $R_{i2n+1} = -\partial_i(\partial_{2n+1} e)$.

Example 2. Let A_{2n+1} be a Riemannian space with metric tensor $g_{\alpha\beta}$ and Levi-Civita connection ∇ with Christoffel symbols $\Gamma^{\nu}_{\alpha\beta}$. Let (3.3) be valid. By (3.7) and $\nabla_{2n+1} g_{ij} = \nabla_i g_{2n+1,2n+1} = 0$ it follows that in parameters in the coordinate net $\{\underset{\alpha}{v}\}$

$$g_{ij} = g_{ij}(\overset{s}{u}), \quad g_{2n+1,2n+1} = g_{2n+1,2n+1}(\overset{2n+1}{u}). \tag{3.10}$$

According to [8], we have

$$\Gamma^{\alpha}_{\sigma\alpha} = \partial_\sigma \ln\sqrt{|g|} = \frac{\partial \ln\sqrt{|g|}}{\partial \overset{\sigma}{u}}, \tag{3.11}$$

where g is the determinant of $(g_{\alpha\beta})$.

Let in the parameters of the coordinate net

$$g_{\alpha\beta} = 0, \quad \alpha \neq \beta; \quad g_{ij} \neq 0, \quad i \neq j; \quad g_{2n+1,2n+1} > 0. \tag{3.12}$$

Because of (3.11) and (3.12) equalities (3.7) take the form

$$\begin{aligned}
&\Gamma^1_{11} = \tfrac{1}{2n}\partial_1 \ln\sqrt{|g|}, \quad \Gamma^2_{22} = \tfrac{1}{2n}\partial_2 \ln\sqrt{|g|}, \ldots, \Gamma^{2n}_{2n,2n} = \tfrac{1}{2n}\partial_{2n} \ln\sqrt{|g|},\\
&\Gamma^{2n+1}_{2n+1,2n+1} = \partial_{2n+1} \ln\sqrt{|g|}.
\end{aligned} \tag{3.13}$$

By (3.10), (3.12) and (3.13) we get

$$g_{ii} = e^{f(\overset{s}{u})}\varepsilon_{ii}, \quad g_{2n+1,2n+1} = F(\overset{2n+1}{u}), \tag{3.14}$$

where $\varepsilon_{ii} = \pm 1$, and $f(\overset{s}{u})$ and $F(\overset{2n+1}{u})$ are arbitrary functions. According to (3.12) and (3.14), the line element of the space is given by

$$\mathrm{d}s^2 = e^{f(\overset{s}{u})}(\varepsilon_{11}(\mathrm{d}\overset{1}{u})^2 + \varepsilon_{22}(\mathrm{d}\overset{2}{u})^2 + \ldots + \varepsilon_{2n,2n}(\mathrm{d}\overset{2n}{u})^2) + F(\overset{2n+1}{u})(\mathrm{d}\overset{2n+1}{u})^2.$$

4 Transformations of connections in A_{2n+1}

Let A_{2n+1} be a space with a symmetric affine connection endowed with the additional structure defined by (3.1) and satisfying (3.3). Let us consider the following covector field

$$w_\alpha = \sum_{\beta=1}^{2n+1} \overset{\beta}{v}_\alpha.$$

In the parameters of the coordinate net $\{\underset{\alpha}{v}\}$, according to (2.3), we have $w_\alpha(1, 1, ..., 1)$.

We consider the connection ${}^1\nabla$ with Christoffel symbols ${}^1\Gamma^\nu_{\alpha\beta}$ defined by

$${}^1\Gamma^\nu_{\alpha\beta} = \Gamma^\nu_{\alpha\beta} + T^\nu_{\alpha\beta},$$

where the deformation tensor $T^\nu_{\alpha\beta}$ is given by

$$T^\nu_{\alpha\beta} = w_\alpha A^\nu_\beta. \tag{4.1}$$

Let the indices $i, j \in \{1, 2, ..., 2n\}$ whenever $\bar{i}, \bar{j} \in \{2, 3, ..., 2n, 1\}$. Because of (3.2) and (3.7), in the parameters of the coordinate net we have

$$A^i_{\bar{i}} = 1; \quad A^i_j = 0, \quad \bar{i} \neq j; \quad A^{2n+1}_\alpha = A^\alpha_{2n+1} = 0;$$
$$T^i_{\alpha\bar{i}} = {}^1\Gamma^i_{\alpha\bar{i}} - \Gamma^i_{\alpha\bar{i}} = w_\alpha; \quad T^i_{\alpha j} = {}^1\Gamma^i_{\alpha j} - \Gamma^i_{\alpha j} = 0, \quad j \neq \bar{i};$$
$$T^{2n+1}_{\alpha\beta} = {}^1\Gamma^{2n+1}_{\alpha\beta} = 0; \quad T^\beta_{\alpha,2n+1} = {}^1\Gamma^\beta_{\alpha,2n+1} = 0.$$

According to (4.1) and ${}^1\nabla_\sigma A^\nu_\alpha = \nabla_\sigma A^\nu_\alpha + T^\nu_{\sigma\rho}A^\rho_\alpha - T^\rho_{\sigma\alpha}A^\nu_\rho$, we obtain ${}^1\nabla_\sigma A^\nu_\alpha = \nabla_\sigma A^\nu_\alpha$, i.e. the affinor $\nabla_\sigma A^\nu_\alpha$ is invariant under the transformation of ∇ into ${}^1\nabla$. Thus, ${}^1\nabla_\sigma A^\nu_\alpha = 0$ if and only if $\nabla_\sigma A^\nu_\alpha = 0$.

Let $R_{\alpha\beta\sigma.}{}^\nu$ and ${}^1R_{\alpha\beta\sigma.}{}^\nu$ be the curvature tensors of ∇ and ${}^1\nabla$, respectively. Then the following relation is well-known

$${}^1R_{\alpha\beta\sigma.}{}^\nu = R_{\alpha\beta\sigma.}{}^\nu + \nabla_\alpha T^\nu_{\beta\sigma} - \nabla_\beta T^\nu_{\alpha\sigma} + T^\nu_{\alpha\delta}T^\delta_{\beta\sigma} - T^\nu_{\beta\delta}T^\delta_{\alpha\sigma}. \tag{4.2}$$

By (4.2) and (4.1) we obtain

$${}^1R_{\alpha\beta\sigma.}{}^\nu = R_{\alpha\beta\sigma.}{}^\nu + 2\nabla_{[\alpha} w_{\beta]} A^\nu_\sigma. \tag{4.3}$$

In the parameters of the coordinate net, by (4.3) we have

$${}^1R_{\alpha\beta j.}{}^i = R_{\alpha\beta j.}{}^i, \quad j \neq \bar{i}, \quad {}^1R_{\alpha\beta\bar{i}.}{}^i = R_{\alpha\beta\bar{i}.}{}^i + \nabla_{[\alpha} w_{\beta]}. \tag{4.4}$$

The second of the equalities (4.4) implies

$${}^1R_{\alpha\beta\bar{i}.}{}^i - {}^1R_{\alpha\beta\bar{j}.}{}^j = R_{\alpha\beta\bar{i}.}{}^i - R_{\alpha\beta\bar{j}.}{}^j.$$

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