On a (k, μ)-contact metric manifold admitting C-Bochner curvature tensor

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Abstract. In this paper we study (k, μ)-contact metric manifold endowed with a C-Bochner curvature tensor. Here we describe C-Bochner pseudosymmetric (k, μ)-contact metric manifold, then the manifold is η-Einstein manifold. Moreover, we also investigates (k, μ)-contact metric manifold satisfying the conditions $B(\xi, X).B = 0$ and $B(\xi, X).R = 0$.

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1 Introduction

A $(2n+1)$-dimensional Riemannian manifold M is said to be locally symmetric [7] if its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. As a generalization of locally symmetric manifolds, the semisymmetric manifolds [17] was introduced and defined by

$$(R(X, Y).R)(U, V)W = 0,$$

for all vector fields X, Y, U, V, $W \in \chi(M)$. Locally Symmetric manifolds were weakened by several authors, viz., [15, 17, 19]. In 1992, Deszcz [10] introduced the notion of pseudosymmetric manifolds which is defined by the condition

$$(R(X, Y).R)(U, V)W = L_R[((X \wedge Y).R)(U, V)W],$$

where L_R is some smooth function on M and $X \wedge Y$ is an endomorphism defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y.$$

In 1949, Bochner introduced a Kahler analogue of the Weyl conformal curvature tensor, which is known as the Bochner curvature tensor [4]. A geometric meaning of the Bochner curvature tensor is given by Blair [2]. By considering the Boothby-Wang's fibration [6], Matsumoto and Chuman defined C-Bochner curvature tensor [14] from the Bochner curvature tensor. The geometry of C-Bochner curvature tensor

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in a Riemannian manifold with different structures have been studied extensively by many geometers such as [8, 9, 12, 13]. The present paper is organized as follows:

In Section 2 we recall the notions and preliminary results of (k,μ)-contact metric manifolds and C-Bochner curvature tensor needed through the paper. In Section 3 we consider C-Bochner pseudosymmetric non-Sasakian (k,μ)-contact metric manifold. In this case the manifold is η-Einstein manifold. In fact, Section 4 is devoted to the study of non-Sasakian (k,μ)-contact metric manifold associated with the condition $B(\xi,X).B = 0$ and it is shown that the manifold is a η-Einstein manifold. Finally, in Section 5 we discuss non-Sasakian (k,μ)-contact metric manifold with $B(\xi,X).R = 0$ and obtained the existing result.

2 Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be contact manifold if it carries a global differentiable 1-form η which satisfies the condition $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Also a contact manifold admits an almost contact structure (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor field, ξ is a characteristic vector field and η is a global 1-form such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $M \times R$ defined by

$$J\left(X, \lambda\frac{d}{dt}\right) = \left(\phi X - \lambda\xi, \eta(X)\frac{d}{dt}\right),$$

is integrable, where X is tangent to M, t is the coordinate of R and λ a smooth function on $M \times R$. The condition of almost contact metric structure being normal is equivalent to vanishing of the torsion tensor $[\phi,\phi]+2d\eta\otimes\xi$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ. Let g be the compatible Riemannian metric with almost contact structure (ϕ,ξ,η), that is,

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad g(X,\xi) = \eta(X), \tag{2.2}$$

for all vector fields $X, Y \in \chi(M)$. A manifold M together with this almost contact metric structure is said to be almost contact metric manifold and it is denoted by $M(\phi,\xi,\eta,g)$. An almost contact metric structure reduces to a contact metric structure if $g(X,\phi Y) = d\eta(X,Y)$.

Blair, Koufogiorgos and Papantoniou [3] introduced the (k,μ)-nullity distribution of a contact metric manifold M and is defined by

$$N(k,\mu) : p \to N_p(k,\mu) = \{U \in T_pM \mid R(X,Y)U = (kI + \mu h)R_0(X,Y)U\},$$

for all $X, Y \in TM$, where $(k,\mu) \in R^2$. A contact metric manifold with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold. If $\mu = 0$, the (k,μ)-nullity distribution is reduced to the k-nullity distribution [18]. The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [18]

$$N(k) : p \to N_p(k) = \{U \in T_pM \mid R(X,Y)U = kR_0(X,Y)U\},$$

k being constant. If $\xi \in N(k)$, then we call a contact metric manifold M an $N(k)$-contact metric manifold.

The class of (k,μ)-contact metric manifolds contains both the class of Sasakian ($k=1$ and $h=0$) and non-Sasakian ($k \neq 1$ and $h \neq 0$) manifolds. Throughout this paper we study $(2n+1)$-dimensional non-Sasakian (k,μ)-contact metric manifolds. In a (k,μ)-contact metric manifold the following relations hold [3]:

$$h^2 = (k-1)\phi^2, \tag{2.3}$$

$$R_0(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{2.4}$$

$$R(X,Y)\xi = (kI + \mu h)R_0(X,Y)\xi, \tag{2.5}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.6}$$

$$\begin{aligned} S(X,Y) &= [2(n-1) - n\mu]g(X,Y) + [2(n-1)+\mu]g(hX,Y) \\ &+ [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \end{aligned} \tag{2.7}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n-2+\mu)g(hX,Y). \tag{2.8}$$

The C-Bochner curvature tensor on a (k,μ)-contact metric manifold M is given by [14]

$$\begin{aligned} B(X,Y)Z &= R(X,Y)Z + \frac{1}{n+3}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY \\ &- g(Y,Z)QX + S(\phi X,Z)Y - S(\phi Y,Z)X + g(\phi X,Z)Q\phi Y \\ &- g(\phi Y,Z)Q\phi X + 2S(\phi X,Y)\phi Z + 2g(\phi X,\phi Y)Q\phi Z \\ &- S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY \\ &+ \eta(Y)\eta(Z)QX] - \frac{p+n-1}{n+3}[g(\phi X,Y)Z - g(\phi Y,Z)\phi X \\ &+ 2g(\phi X,Y)\phi Z] - \frac{p-4}{n+3}[g(X,Z)Y - g(Y,Z)X] \\ &+ \frac{p}{n+3}[g(X,Z)\eta(Y)\xi + \eta(X)\eta(Z)Y - g(Y,Z)\eta(X)\xi \\ &- \eta(Y)\eta(Z)X], \end{aligned} \tag{2.9}$$

where $p = \frac{r+n-1}{n+1}$, $S(X,Y) = g(QX,Y)$ and $r = (n-1)(n-3+k)$.
In a (k,μ)-contact metric manifold M, the C-Bochner curvature tensor satisfies the following relations:

$$\begin{aligned} B(X,Y)\xi &= \frac{-k(n-3)-4}{n+3}[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX \\ &- \eta(X)hY], \end{aligned} \tag{2.10}$$

$$\begin{aligned} \tilde{B}(X,Y,\xi,Z) &= \frac{-k(n-3)-4}{n+3}[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)] \\ &+ \mu[\eta(Y)g(hX,Z) - \eta(X)g(hY,Z)], \end{aligned} \tag{2.11}$$

$$\begin{aligned} \tilde{B}(X,\xi,Y,Z) &= \tilde{B}(X,\xi,Y,Z) = \frac{k(n-3)+4}{n+3}[\eta(Z)g(X,Y) \\ &- \eta(Y)g(X,Z)] + \mu h[\eta(Z)g(X,Y) - \eta(Y)g(X,Z)], \end{aligned} \tag{2.12}$$

$$B(X,\xi)\xi = \frac{-k(n-3)-4}{n+3}[X - \eta(X)\xi] + \mu[hX], \tag{2.13}$$

$$B(\xi,\xi)X = 0. \tag{2.14}$$

By virtue of (2.9), let $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and using (2.2), (2.6) and (2.8), we get

$$\sum_{i=1}^{n} g(B(e_i,Y)Z,e_i) = l\eta(Y)\eta(Z) + mg(hY,Z), \tag{2.15}$$

where $l = \frac{16kn(n+1)-2n(n+3)-6r+8}{(n+1)(n+3)}$ and $m = \frac{12}{n+3}(2n-2+\mu)$.

Definition 2.1. A $(2n+1)$-dimensional (k,μ)-contact metric manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for any vector fields X and Y, where a and b are constants.

3 C-Bochner pseudosymmetric (k,μ)-contact metric manifold

A $(2n+1)$-dimensional (k,μ)-contact metric manifold M is said to be C-Bochner pseudosymmetric if

$$(R(X,Y).B)(U,V)W = L_B[((X\wedge Y).B)(U,V)W], \tag{3.1}$$

holds on the set $U_B = \{x\in M : B\neq 0\}$ at x, where L_B is some function on U_B and B is the C-Bochner curvature tensor.

Theorem 3.1. *Let M be a $(2n+1)$-dimensional non-Sasakian (k,μ)-contact metric manifold. If M is C-Bochner pseudosymmetric, then M is an η-Einstein manifold.*

Proof. Let M be a $(2n+1)$-dimensional C-Bochner pseudosymmetric (k,μ)-contact metric manifold. Substituting $Y=\xi$ in (3.1), we get

$$\begin{aligned}(R(X,\xi).B)(U,V)W &= L_B[((X\wedge\xi)(B(U,V)W) - B((X\wedge\xi)U,V)W \\ &- B(U,(X\wedge\xi))W - B(U,V)(X\wedge\xi)W].\end{aligned} \tag{3.2}$$

Now the left hand side of (3.2) gives

$$\begin{aligned}R(X,\xi)B(U,V)W - B(R(X,\xi)U,V)W &- B(U,R(X,\xi)V)W \\ &- B(U,V)R(X\wedge\xi)W.\end{aligned} \tag{3.3}$$

In view of (2.5), the above equation reduces to

$$\begin{aligned}&k[\tilde{B}(U,V,W,\xi)X - \tilde{B}(U,V,W,X)\xi - \eta(U)B(X,V)W + g(X,U)B(\xi,V)W \\ &-\eta(V)B(U,X)W + g(X,V)B(U,\xi)W - \eta(W)B(U,V)X + g(X,W)B(U,V)\xi] \\ &+\mu h[\tilde{B}(U,V,W,\xi)X - \tilde{B}(U,V,W,X)\xi - \eta(U)B(X,V)W + g(X,U)B(\xi,V)W \\ &-\eta(V)B(U,X)W + g(X,V)B(U,\xi)W - \eta(W)B(U,V)X + g(X,W)B(U,V)\xi].\end{aligned} \tag{3.4}$$

Similarly right hand side of (3.2) gives

$$L_B[\tilde{B}(U,V,W,\xi)X - \tilde{B}(U,V,W,X)\xi - \eta(U)B(X,V)W + g(X,U)B(\xi,V)W - \eta(V)B(U,X)W + g(X,V)B(U,\xi)W - \eta(W)B(U,V)X + g(X,W)B(U,V)\xi]. \tag{3.5}$$

Substituting (3.4) and (3.5) in (3.2) with $V = \xi$, we get

$$(L_B - (k+\mu h))[\tilde{B}(U,\xi,W,\xi)X - \tilde{B}(U,\xi,W,X)\xi - \eta(U)B(X,\xi)W + g(X,U)B(\xi,\xi)W - \eta(\xi)B(U,X)W + g(X,\xi)B(U,\xi)W - \eta(W)B(U,\xi)X + g(X,W)B(U,\xi)\xi] = 0. \tag{3.6}$$

Using (2.10)-(2.14) in the above relation we have either $L_B = (k + \mu h)$ or

$$\begin{aligned} B(U,X)W &= \frac{k(n-3)+4}{n+3}[g(U,W)X - g(X,W)U] \\ &+ \mu h[g(X,W)U - g(U,W)X]. \end{aligned} \tag{3.7}$$

The above equation can be written as

$$\begin{aligned} \tilde{B}(U,X,W,T) &= \frac{k(n-3)+4}{n+3}[g(U,W)g(X,T) - g(X,W)g(U,T)] \\ &+ \mu h[g(X,W)g(U,T) - g(U,W)g(X,T)]. \end{aligned} \tag{3.8}$$

Putting $U = T = e_i$ in (3.8), and by virtue of (2.15) we have

$$\frac{k(n-3)+4}{n+3}(1-n)g(X,W) = l\eta(X)\eta(W) + (\mu(1-n)+m)g(hX,W). \tag{3.9}$$

Using (2.7) in (3.9), we get

$$S(X,W) = Ag(X,W) + B\eta(X)\eta(W), \tag{3.10}$$

where

$$\begin{aligned} A &= \frac{(2(n-1)-n\mu)[(n+3)(\mu(n-1)-m) + (n-1)(k(n-3)+4)]}{(n+3)(\mu(n-1)-m)}, \\ \textit{and} \quad B &= \frac{l(2(n-1)+\mu) + (2(1-n)+n(2k+\mu))(\mu(n-1)-m)}{(\mu(n-1)-m)}. \end{aligned}$$

Thus M is an η-Einstein manifold. □

Also in a (k,μ)-contact metric manifold Blair, Koufogiorgos and Papantoniou [3] proved the following:

Lemma 3.2. *A $(2n+1)$-dimensional non-Sasakian (k,μ)-contact metric manifold which is η-Einstein manifold is an $N(k)$-contact metric manifold.*

Now from above result and Theorem 3.1, we can obtain the following corollary:

Corollary 3.3. *If a $(2n+1)$-dimensional non-Sasakian (k,μ)-contact metric manifold is C-Bochner pseudosymmetric, then the manifold is an $N(k)$-contact metric manifold.*

4 (k,μ)-contact metric manifold with $B(\xi,X).B=0$

Theorem 4.1. *If M is a $(2n+1)$-dimensional non-Sasakian (k,μ)-contact metric manifold which satisfies the condition $B(\xi,X).B=0$, then M is an η-Einstein manifold.*

Proof. Let M be a $(2n+1)$-dimensional (k,μ)-contact metric manifold.
The condition $B(\xi,X).B=0$ turns into

$$B(\xi,X)B(Y,Z)\xi - B(B(\xi,X)Y,Z)\xi - B(Y,B(\xi,X)Z)\xi - B(Y,Z)B(\xi,X)\xi = 0. \tag{4.1}$$

Using (2.10) in (4.1), we have

$$\left(\frac{k(n-3)+4}{n+3}I + \mu h\right)[R_0(\xi,X)B(Y,Z)\xi - B(R_0(\xi,X)Y,Z)\xi - B(Y,R_0(\xi,X)Z) - B(Y,Z)R_0(\xi,X)\xi] = 0. \tag{4.2}$$

In view of (2.4), the above equation becomes

$$\begin{aligned} 0 &= \left(\frac{k(n-3)+4}{n+3}I + \mu h\right)[g(X,B(Y,Z)\xi) - \eta(B(Y,Z)\xi)X \\ &- g(X,Y)B(\xi,Z)\xi + \eta(Y)B(X,Z)\xi - g(X,Z)B(Y,\xi)\xi \\ &+ \eta(Z)B(Y,X)\xi - \eta(X)B(Y,Z)\xi + B(Y,Z)X]. \end{aligned} \tag{4.3}$$

Now using (2.10) in (4.3), we obtain

$$\left(\tfrac{k(n-3)+4}{n+3} + \mu h\right)\{B(Y,Z)X + \left(\tfrac{k(n-3)+4}{n+3}\right)[g(X,Z)Y - g(X,Y)Z] + \mu h[g(X,Y)Z - g(X,Z)Y]\} = 0. \tag{4.4}$$

Therefore, either $\frac{k(n-3)+4}{n+3} + \mu h = 0$ or

$$\begin{aligned} B(Y,Z)X &= \frac{k(n-3)+4}{n+3}[g(X,Y)Z - g(X,Z)Y] \\ &+ \mu h[g(X,Z)Y - g(X,Y)Z]. \end{aligned} \tag{4.5}$$

On contracting (4.5), and using (2.15) we get

$$S(X,Z) = A'g(X,Z) + B'\eta(X)\eta(Z), \tag{4.6}$$

where

$$\begin{aligned} A' &= \frac{(2(n-1)-n\mu)[(n+3)(2n\mu-m)+2n(k(n-3)+4)]}{(n+3)(2n\mu-m)}, \\ \text{and} \quad B' &= \frac{(2(n-1)+n(2k+\mu))(2n\mu-m)+l(2(n-1)+\mu)}{(n+1)(n+3)}. \end{aligned}$$

Hence M is an η-Einstein manifold. □

From Lemma 3.2 and Theorem 4.1, we have the following corollary:

Corollary 4.2. *Let M be a $(2n+1)$-dimensional non-Sasakian (k,μ)-contact metric manifold with $B(\xi,X).B=0$, then the manifold is an $N(k)$-contact metric manifold.*

5 (k,μ)-contact metric manifolds with $B(\xi,X).R=0$

Let us consider the (k,μ)-contact metric manifold satisfying condition $B(\xi,X).R=0$. Then

$$\begin{aligned}(5.1)\qquad & B(\xi,X)R(Y,Z)\xi - R(B(\xi,X)Y,Z)\xi - R(Y,B(\xi,X)Z)\xi \\ & -R(Y,Z)B(\xi,X)\xi = 0,\end{aligned}$$

which in view of (2.10) gives

$$\begin{aligned}(5.2)\qquad & \left(\frac{k(n-3)+4}{n+3}I+\mu h\right)[R_0(\xi,X)R(Y,Z)\xi - R(R_0(\xi,X)Y,Z)\xi \\ & -R(Y,R_0(\xi,X)Z) - R(Y,Z)R_0(\xi,X)\xi] = 0.\end{aligned}$$

Using (2.4), the above equation reduces to

$$\begin{aligned}(5.3)\qquad 0 &= \left(\frac{k(n-3)+4}{n+3}I+\mu h\right)[g(X,R(Y,Z)\xi) - \eta(R(Y,Z)\xi)X \\ &- g(X,Y)R(\xi,Z)\xi + \eta(Y)R(X,Z)\xi - g(X,Z)R(Y,\xi)\xi \\ &+ \eta(Z)R(Y,X)\xi - \eta(X)R(Y,Z)\xi + R(Y,Z)X].\end{aligned}$$

In view of (2.5), equation (5.3) gives

$$(5.4)\qquad \left(\frac{k(n-3)+4}{n+3}+\mu h\right)\{R(Y,Z)X + (kI+\mu h)[g(X,Z)Y - g(X,Y)Z]\} = 0.$$

Therefore, either $\frac{k(n-3)+4}{n+3}+\mu h=0$ or

$$(5.5)\qquad R(Y,Z)X = (kI+\mu h)[g(X,Z)Y - g(X,Y)Z].$$

Hence we can state the following result:

Theorem 5.1. *Let M be a $(2n+1)$-dimensional (k,μ)-contact metric manifold satisfying the condition $B(\xi,X).R=0$. Then either $\frac{k(n-3)+4}{n+3}+\mu h=0$ or equation (5.5) is holds true.*

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