Estimates of potential functions and asymptotic volume ratio for the expanding Ricci soliton

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Abstract. In this paper, we deal with the complete non-compact expanding gradient Ricci soliton (M^n, g) with positive Ricci curvature. On the condition that the Ricci curvature is positive and the scalar curvature approaches 0 towards infinity, we firstly prove a useful estimate on the growth of potential functions. We then establish a useful ODE relationship between R(c) and V(c) on the expanding Ricci soliton. Based on these, we establish the volume growth estimate and prove the asymptotic volume ratio is 0 for some complete non-compact expanding gradient Ricci solitons.

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1 Introduction and main results

Ricci solitons are fixed points of the Ricci flow as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. From the equation point of view, they are natural generalizations of the Einstein metrics. In this paper, in particular we research the expanding gradient Ricci soliton and the definition is as follows:

Definition 1.1. A complete Riemannian manifold (M^n, g) is called an expanding gradient Ricci soliton if there exists a smooth function $f : M^n \to \mathbb{R}$, called the potential function such that

$$Rc + \nabla\nabla f = \lambda g, \tag{1.1}$$

where Rc is the Ricci curvature tensor and λ is a negative real number.

Recall that gradient Ricci solitons are the most widely studied Ricci solitons, and quite a few results on the classification of gradient Ricci solitons have appeared. In particular, if (M^n, g) is compact, then by the maximum principle, it is elementary to check that f in (1.1) has to be a constant so that the expanding gradient Ricci soliton M^n is actually an Einstein manifold with negative Ricci curvature. Hence in this paper, we are only interested in the non-compact case.

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In [3], it was proved by B.-L. Chen that the complete ancient solution to the Ricci flow, and in particular the shrinking and steady Ricci soliton, must have nonnegative scalar curvature. Then together with the fact

$$R + |\nabla f|^2 = C \tag{1.2}$$

for the steady Ricci soliton, the potential function f satisfies the following estimate

$$-\sqrt{C}r(x) + f(O) \le f(x) \le \sqrt{C}r(x) + f(O), \tag{1.3}$$

where $r(x)$ denotes the distance function from x to a fixed point O in M^n.

Moreover, if the Ricci curvature is positive and the scalar curvature R approaches 0 towards spatial infinity, then the following Lemma 1.2 proved by H. X. Guo in [6] implies that for the steady Ricci soliton, there is one point where R obtains its maximum, and the point of maximum is unique.

Lemma 1.1 (Guo)**.** *Let (M^n, g) be a steady gradient Ricci soliton with positive (or negative) Ricci curvature, then there is at most one critical point of R.*

Thus we can denote O as the unique point of maximum of R, called the origin, and assume $f(O) = 0$ by adding a constant. Calculating the constant in (1.2) at O we have

$$R + |\nabla f|^2 = R(O) = R_0. \tag{1.4}$$

Based on these, H. X. Guo also proved a more precise estimate for the potential function on the complete steady gradient Ricci soliton as follows [6]:

Theorem 1.2 (Guo)**.** *Let (M^n, g) be a complete steady gradient Ricci soliton with positive Ricci curvature, and the scalar curvature R approaches 0 towards infinity. Then for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that when $r(x) \ge r_\varepsilon$ we have*

$$\left(\sqrt{R_0} - \varepsilon\right) r(x) \le -f(x) \le \sqrt{R_0}r(x), \tag{1.5}$$

where $r(x) = d(x, O)$ and R_0 is the maximum of R.

In this paper, we research the potential function estimate for the complete expanding gradient Ricci soliton, and firstly establish the following similar result of Lemma 1.2, that if there is any maximum point of scalar curvature R, then it is unique.

Lemma 1.3. *Let (M^n, g) be a complete expanding gradient Ricci soliton with positive Ricci curvature, then there is at most one maximum point of the scalar curvature R.*

Remark 1. By Morse theory, Lemma 1.4 shows that M^n is diffeomorphic to the Euclidean space $\mathbb{R}^n$. Moreover if we assume that the Ricci curvature is positive and the scalar curvature R approaches 0 towards spatial infinity, then there must be at least one point where R obtains its maximum. Thus by Lemma 1.4, we have that the point of maximum is unique.

We denote O as the point of maximum of R, called the origin, and assume $f(O) = 0$ by adding a constant and $R(O) > 1$ by multiplying a constant. For the expanding gradient Ricci soliton, we also have

$$R + |\nabla f|^2 - 2\lambda f = C, \tag{1.6}$$

then calculating the constant in (1.6) at O we have

$$R + |\nabla f|^2 - 2\lambda f = R(O) = R_0. \tag{1.7}$$

For any $x \in M^n$, let $r(x) = dist(O, x)$ and $\gamma(s)$ denote the shortest geodesic from O to x, where s is the arclength, then we have the following potential function estimate for the complete expanding gradient Ricci soliton:

Theorem 1.4. *Let (M^n, g) be a complete expanding gradient Ricci soliton with positive Ricci curvature and the scalar curvature R approaches 0 towards infinity, then for any sufficient small $\varepsilon > 0$, there exists r_0 such that when $r(x) > r_0$, we have*

$$\begin{aligned}
&-\frac{1}{2\lambda}\left(\frac{\lambda}{1+\varepsilon}\left(r(x) - r_0\right) - \sqrt{R_0 + 2\lambda f(x_0)}\right)^2 + \frac{1}{2\lambda}R_0 \\
\leq &- f(x) \\
\leq &-\frac{1}{2\lambda}\left(\left(\lambda r(x) - \sqrt{R_0}\right)^2 - R_0\right),
\end{aligned}$$

where $r_0 = dist(O, x_0)$.

Then we define the functions

$$\mathrm{V}_1 : \mathbb{R} \to [0, \infty), \qquad \mathrm{R}_1 : \mathbb{R} \to [0, \infty)$$

by

$$\mathrm{V}_1(c) = \int_{\{x \in M^n | f(x) < c\}} d\mu, \qquad \mathrm{R}_1(c) = \int_{\{x \in M^n | f(x) < c\}} R\, d\mu.$$

In [1], H.-D. Cao and D.-T. Zhou established the following ODE relating $\mathrm{V}_1(c)$ and $\mathrm{R}_1(c)$ for the complete shrinking gradient Ricci soliton:

Theorem 1.5 (Cao-Zhou)**.** *Let (M^n, g) be a complete shrinking gradient Ricci soliton, then*

$$0 \leq \frac{n}{2}\mathrm{V}_1(c) - \mathrm{R}_1(c) = c\,\mathrm{V}_1'(c) - \mathrm{R}_1'(c). \tag{1.8}$$

In this paper, to prove our main result, we establish a similar result to (1.8) for the expanding Ricci soliton as follows:

Theorem 1.6. *Let (M^n, g) be a complete expanding gradient Ricci soliton, and define*

$$\mathrm{D}(c) = \{x \in M^n \,| -f(x) < c\},$$

$$\mathrm{V}(c) = \int_{\mathrm{D}(c)} d\mu = \int_{\{x \in M^n | -f(x) < c\}} d\mu$$

and

$$\mathrm{R}(c) = \int_{\mathrm{D}(c)} R\,d\mu = \int_{\{x\in M^n \mid -f(x)<c\}} R\,d\mu,$$

then

$$n\lambda\,\mathrm{V}(c) + (R_0 + 2\lambda c)\,\mathrm{V}'(c) = \mathrm{R}(c) + \mathrm{R}'(c)\,. \tag{1.9}$$

Recall that the asymptotic volume ratio (AVR) of a complete non-compact Riemannian manifold (N^n, h) is defined by

$$\mathrm{AVR}(h) = \lim_{r\to\infty} \frac{\mathrm{Vol}_h\, B(p,r)}{\omega_n r^n} \tag{1.10}$$

if the limit exists, where $B(p,r)$ denotes the geodesic ball in N^n with center p and radius r and ω_n is the volume of the unit Euclidean n-ball. It is easy to check that the $\mathrm{AVR}(h)$ is independent of the choice of p. Moreover, if (N^n, h) has nonnegative Ricci curvature, then this limit (1.10) exists by the Bishop-Gromov volume comparison theorem.

For the case of shrinking Ricci solitons, H.-D. Cao and D.-T. Zhou [1] proved the following result aided by an observation of Munteanu [7].

Theorem 1.7 (Cao-Zhou)**.** *Any complete non-compact shrinking gradient Ricci soliton must have at most Euclidean volume growth, i.e.,*

$$\limsup_{r\to\infty} \frac{\mathrm{Vol}\, B(O,r)}{\omega_n r^n} < \infty. \tag{1.11}$$

For the case of expanding Ricci solitons, by using Theorem 1.5 and 1.7, we can prove the following estimate.

Theorem 1.8. *Let (M^n, g), where $n = 2k$ and $k \geq 2$, be a complete non-compact expanding gradient Ricci soliton with positive Ricci curvature and the scalar curvature R approaches 0 towards infinity. Then there exist constants $C > 0$ and $\delta > 0$ such that for any $c \geq \delta$ we have*

$$\mathrm{Vol}_g\,(B(O,c)) \leq \frac{C}{c^n}, \tag{1.12}$$

furthermore, we have the asymptotic volume ratio $\mathrm{AVR}(g) = 0$.

In particular, more recently, observing the results in [1], [2], [5] and [9], B. Chow, P. Lu and B. Yang [4] derived a necessary and sufficient condition for non-compact shrinking Ricci solitons to have positive AVR as follows:

Theorem 1.9 (Chow-Lu-Yang)**.** *Let (M^n, g) be a complete non-compact shrinking gradient Ricci soliton, then $\mathrm{AVR}(g)$ exists (and is finite). Moreover, $\mathrm{AVR}(g) > 0$ if and only if*

$$\int_{n+2}^{\infty} \frac{\mathrm{R}(c)}{c\,\mathrm{V}(c)}\,dc < \infty. \tag{1.13}$$

Note that the method of Chow, Lu and Yang is interesting and motivate, in this paper, by using the approach in [4], we present another proof of the asymptotic volume ratio $\mathrm{AVR}(g) = 0$ for some complete non-compact expanding gradient Ricci solitons as follows.

Theorem 1.10. *Let (M^n, g), where $n = 2k$ and $k \geq 2$, be a complete non-compact expanding gradient Ricci soliton with positive Ricci curvature and the scalar curvature R approaches 0 towards infinity, then the asymptotic volume ratio $\mathrm{AVR}(g) = 0$.*

The paper is organized as follows. In section 2, we prove Theorem 1.5 and 1.7 by calculating, and we then obtain Theorem 1.9 as a corollary of Theorem 1.5 and 1.7. Based on these, in section 3, we prove Theorem 1.11.

2 Proof of Theorems 1.5 and 1.7

In this section, we firstly present the proof of Theorem 1.5.

Proof of Theorem 1.5. By (1.7) and the positivity of R we have

$$|\nabla f|^2 = R_0 - R + 2\lambda f < R_0 + 2\lambda f,$$

then

$$\left|\nabla \sqrt{R_0 + 2\lambda f}\right| = \frac{|\nabla (R_0 + 2\lambda f)|}{2\sqrt{R_0 + 2\lambda f}} < -\lambda.$$

Along any geodesic $\gamma(s)$ emanating from O we have

$$\begin{aligned} \left|\frac{d}{ds}\sqrt{R_0 + 2\lambda f}\,(\gamma\,(s))\right| &= \left|\left\langle \nabla \sqrt{R_0 + 2\lambda f}, \gamma'\,(s)\right\rangle\right| \\ &\leq \left|\nabla \sqrt{R_0 + 2\lambda f}\right| \\ &< -\lambda. \end{aligned}$$

Note the fact that the maximum of f is 0 so that $f < 0$, we deduce that

$$\left|\sqrt{R_0 + 2\lambda f\,(x)} - \sqrt{R_0}\right| \leq -\lambda r\,(x)\,.$$

By using $\lambda < 0$, we have

$$-f\,(x) \leq -\frac{1}{2\lambda}\left(\left(\lambda r\,(x) - \sqrt{R_0}\right)^2 - R_0\right). \tag{2.1}$$

For the lower bound of $f(x)$, we work on the integral curve of $-\frac{\nabla f}{|\nabla f|^2}$, which is denoted by $\beta\,(\sigma)$. Since

$$\frac{df\,(\beta\,(\sigma))}{d\sigma} = \langle \nabla f, \beta'\,(\sigma)\rangle = -\left\langle \nabla f, \frac{\nabla f}{|\nabla f|^2}\right\rangle = -1,$$

together with $f\,(O) = 0$ we have $f\,(\beta\,(\sigma)) = -\sigma$.

Since the scalar curvature R approaches 0 towards spatial infinity, we have

$$|\nabla f| \to \sqrt{R_0 + 2\lambda f}$$

as $x \to \infty$. Then given any sufficient small ε such that $0 < \varepsilon < \sqrt{R_0} - 1$, there exists σ_0 such that when $\sigma \geq \sigma_0$, we have

$$|\nabla f(\beta(\sigma))| > \sqrt{R_0 - 2\lambda\sigma} - \varepsilon$$

along the integral curve $\beta(\sigma)$.

Let $x_0 = \beta(\sigma_0)$, then the length of β from x_0 to x can be estimated as

$$\begin{aligned}\int_{\sigma_0}^{\sigma} |\beta'(\sigma)| d\sigma &= \int_{\sigma_0}^{\sigma} \frac{1}{|\nabla f|} d\sigma \\ &< \int_{\sigma_0}^{\sigma} \frac{1}{\sqrt{R_0 - 2\lambda\sigma} - \varepsilon} d\sigma \\ &< \int_{\sigma_0}^{\sigma} \frac{1+\varepsilon}{\sqrt{R_0 - 2\lambda\sigma}} d\sigma,\end{aligned}$$

where we use the fact

$$0 < \varepsilon < \sqrt{R_0} - 1 \leq \sqrt{R_0 - 2\lambda\sigma} - 1.$$

Thus

$$\begin{aligned}\int_{\sigma_0}^{\sigma} |\beta'(\sigma)| d\sigma &< -\frac{1+\varepsilon}{\lambda}\left(\sqrt{R_0 - 2\lambda\sigma} - \sqrt{R_0 - 2\lambda\sigma_0}\right) \\ &= -\frac{1+\varepsilon}{\lambda}\left(\sqrt{R_0 + 2\lambda f(x)} - \sqrt{R_0 + 2\lambda f(x_0)}\right).\end{aligned}$$

On the other hand,

$$\int_{\sigma_0}^{\sigma} |\beta'(\sigma)| d\sigma \geq dist(x, x_0) = r(x) - r_0,$$

where $r_0 = dist(O, x_0)$, so that we have

$$-\frac{1+\varepsilon}{\lambda}\left(\sqrt{R_0 + 2\lambda f(x)} - \sqrt{R_0 + 2\lambda f(x_0)}\right) \geq r(x) - r_0.$$

Hence

$$-f(x) \geq -\frac{1}{2\lambda}\left(\sqrt{R_0 + 2\lambda f(x_0)} - \frac{\lambda}{1+\varepsilon}(r(x) - r_0)\right)^2 + \frac{1}{2\lambda} R_0. \tag{2.2}$$

□

Remark 2. Substituting (2.1) into (1.7), we can also get that:

Corollary 2.1. *Under the same assumptions of Theorem 1.5, we have*

$$R(x) \leq R_0 + 2\lambda f(x) \leq \left(\lambda r(x) - \sqrt{R_0}\right)^2. \tag{2.3}$$

Now we turn to prove Theorem 1.7.

Proof of Theorem 1.7. Firstly, by Theorem 1.5, for any sufficient small $\varepsilon > 0$, there exists r_0 such that when $r(x) > r_0$, we have

$$-\frac{\lambda}{2(1+\varepsilon)^2}(r(x)-C_1)^2 \leq -f(x) \leq -\frac{\lambda}{2}(r(x)+C_2)^2. \tag{2.4}$$

Then by the Co-Area formula (cf. [8]), we have

$$\mathrm{V}(c) = \int_0^c ds \int_{\partial\,\mathrm{D}(s)} \frac{1}{|\nabla(-f)|} dA, \tag{2.5}$$

hence

$$\mathrm{V}'(c) = \int_{\partial\,\mathrm{D}(c)} \frac{1}{|\nabla f|} dA. \tag{2.6}$$

Then taking the trace in (1.1), we have

$$R + \Delta f = n\lambda.$$

Thus by using the Divergence Theorem and (2.6) we have

$$\begin{aligned}
n\lambda\,\mathrm{V}(c) - \mathrm{R}(c) &= \int_{\mathrm{D}(c)} n\lambda d\mu - \int_{\mathrm{D}(c)} R d\mu \\
&= \int_{\mathrm{D}(c)} \Delta f d\mu \\
&= \int_{\partial\,\mathrm{D}(c)} \nabla f \cdot \frac{-\nabla f}{|\nabla f|} dA \\
&= -\int_{\partial\,\mathrm{D}(c)} |\nabla f| dA \\
&= \int_{\partial\,\mathrm{D}(c)} \frac{R - R_0 - 2\lambda f}{|\nabla f|} dA \\
&= \int_{\partial\,\mathrm{D}(c)} \frac{R}{|\nabla f|} dA - (R_0 + 2\lambda c)\,\mathrm{V}'(c)
\end{aligned}$$

On the other hand, by using the Co-Area formula again, we have

$$\mathrm{R}(c) = \int_{\mathrm{D}(c)} R\, d\mu = \int_0^c ds \int_{\partial\,\mathrm{D}(s)} \frac{R}{|\nabla(-f)|} dA,$$

which implies that

$$\mathrm{R}'(c) = \int_{\partial\,\mathrm{D}(c)} \frac{R}{|\nabla f|} dA. \tag{2.7}$$

Therefore, we have

$$n\lambda\,\mathrm{V}(c) - \mathrm{R}(c) = \mathrm{R}'(c) - (R_0 + 2\lambda c)\,\mathrm{V}'(c),$$

which completes the proof. □

Remark 3. By using the fact that

$$n\lambda\,\mathrm{V}(c) - \mathrm{R}(c) = -\int_{\partial\,\mathrm{D}(c)} |\nabla f| dA \leq 0,$$

we have also actually shown that

$$\frac{\int_{\mathrm{D}(c)} R\, d\mu}{\int_{\mathrm{D}(c)} d\mu} = \frac{\mathrm{R}(c)}{\mathrm{V}(c)} \geq n\lambda. \tag{2.8}$$

Namely, the average scalar curvature over $\mathrm{D}(c)$ is bounded from below by $n\lambda$.

Now we turn to prove Theorem 1.9 by using Theorem 1.7.

Proof of Theorem 1.9. Multiplying

$$n\lambda\,\mathrm{V}(c) + (R_0 + 2\lambda c)\,\mathrm{V}'(c) = \mathrm{R}(c) + \mathrm{R}'(c)$$

by $\frac{1}{2}\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}}$ we have

$$\begin{aligned}\frac{d}{dc}\left(\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n}{2}} \mathrm{V}(c)\right) &= \frac{n\lambda}{2}\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \mathrm{V}(c) + \left(\frac{R_0}{2} + \lambda c\right)^{\frac{n}{2}} \mathrm{V}'(c)\\ &= \frac{1}{2}\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \left(\mathrm{R}(c) + \mathrm{R}'(c)\right),\end{aligned}$$

then integrate from c_0 to c it follows that

$$\begin{aligned}&\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n}{2}} \mathrm{V}(c) - \left(\frac{R_0}{2} + \lambda c_0\right)^{\frac{n}{2}} \mathrm{V}(c_0)\\ =&\int_{c_0}^{c} \frac{d}{ds}\left(\left(\frac{R_0}{2} + \lambda s\right)^{\frac{n}{2}} \mathrm{V}(s)\right) ds\\ =&\int_{c_0}^{c} \frac{1}{2}\left(\frac{R_0}{2} + \lambda s\right)^{\frac{n-2}{2}} \left(\mathrm{R}(s) + \mathrm{R}'(s)\right) ds\\ =&\frac{1}{2}\left(\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \mathrm{R}(c) - \left(\frac{R_0}{2} + \lambda c_0\right)^{\frac{n-2}{2}} \mathrm{R}(c_0)\right)\\ &+ \frac{1}{4}\int_{c_0}^{c} \left(\frac{R_0}{2} + \lambda s\right)^{\frac{n-4}{2}} \mathrm{R}(s)\left(R_0 + 2\lambda s - (n-2)\lambda\right) ds.\end{aligned}$$

When $n = 4(k+1)$, then we have $\left(\frac{R_0}{2} + \lambda s\right)^{\frac{n-4}{2}} \geq 0$. Then when $c_0 \geq \frac{n-2}{2} - \frac{R_0}{2\lambda}$, it follows that $R_0 + 2\lambda s - (n-2)\lambda \leq 0$, thus

$$\begin{aligned}&\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n}{2}} \mathrm{V}(c) - \left(\frac{R_0}{2} + \lambda c_0\right)^{\frac{n}{2}} \mathrm{V}(c_0)\\ \leq&\frac{1}{2}\left(\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \mathrm{R}(c) - \left(\frac{R_0}{2} + \lambda c_0\right)^{\frac{n-2}{2}} \mathrm{R}(c_0)\right)\end{aligned} \tag{2.9}$$

follows from the observation that $\mathrm{R}(c)$ is nonnegative, because the scalar curvature $R \geq 0$.

Moreover, by using (2.8) and $n = 4(k+1)$ we also have

$$\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \mathrm{R}(c) \leq n\lambda \left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \mathrm{V}(c), \tag{2.10}$$

for any $c \geq -\frac{R_0}{2\lambda}$. Then substituting (2.10) into (2.9), we can also get that

$$\begin{aligned} &\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \mathrm{V}(c)\left(R_0 + 2\lambda c - n\lambda\right) \\ \leq &\left(\frac{R_0}{2} + \lambda c_0\right)^{\frac{n-2}{2}} \left(\left(R_0 + 2\lambda c_0\right)\mathrm{V}(c_0) - \mathrm{R}(c_0)\right), \end{aligned}$$

which implies that

$$\mathrm{V}(c) \leq \frac{\left(\frac{R_0}{2} + \lambda c_0\right)^{\frac{n-2}{2}} \left(\left(R_0 + 2\lambda c_0\right)\mathrm{V}(c_0) - \mathrm{R}(c_0)\right)}{\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \left(R_0 + 2\lambda c - n\lambda\right)} \tag{2.11}$$

for any $c > \frac{n}{2} - \frac{R_0}{2\lambda}$. Furthermore, by Theorem 1.5, for any sufficient small $\varepsilon > 0$, there exists r_0 such that when $r(x) > r_0$, we have

$$-\frac{\lambda}{2(1+\varepsilon)^2}\left(r(x) - C_1\right)^2 \leq -f(x) \leq -\frac{\lambda}{2}\left(r(x) + C_2\right)^2,$$

thus

$$\mathrm{Vol}_g\left(B(O,c)\right) = \{x \in M^n \,|r(x) \leq c\} = \left\{x \in M^n \,\middle|-f(x) \leq C_3 c^2\right\} = \mathrm{V}\left(C_3 c^2\right).$$

Together with (2.11), we have

$$\mathrm{Vol}_g\left(B(O,c)\right) = \mathrm{V}\left(C_3 c^2\right) \leq \frac{C_4}{\left(C_3 c^2\right)^{\frac{n}{2}}} = \frac{C}{c^n}.$$

On the other hand, when $n = 4(k+1) + 2$, then for $s \geq \frac{n-2}{2} - \frac{R_0}{2\lambda}$ we have $\left(\frac{R_0}{2} + \lambda s\right)^{\frac{n-4}{2}} \leq 0$ and $R_0 + 2\lambda s - (n-2)\lambda \leq 0$, then as the proof in the case $n = 4k$, we have

$$\begin{aligned} &\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n}{2}} \mathrm{V}(c) - \left(\frac{R_0}{2} + \lambda c_0\right)^{\frac{n}{2}} \mathrm{V}(c_0) \\ \geq &\frac{1}{2}\left(\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \mathrm{R}(c) - \left(\frac{R_0}{2} + \lambda c_0\right)^{\frac{n-2}{2}} \mathrm{R}(c_0)\right) \end{aligned} \tag{2.12}$$

for $c_0 \geq \frac{n-2}{2} - \frac{R_0}{2\lambda}$. Moreover, by using (2.8) and $n = 4(k+1) + 2$ we also have

$$\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \mathrm{R}(c) \geq n\lambda \left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \mathrm{V}(c), \tag{2.13}$$

for any $c \geq -\frac{R_0}{2\lambda}$. Then substituting (2.13) into (2.12), we can also get that

$$\begin{aligned} & \left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} \mathrm{V}(c)(R_0 + 2\lambda c - n\lambda) \\ \geq & \left(\frac{R_0}{2} + \lambda c_0\right)^{\frac{n-2}{2}} \left((R_0 + 2\lambda c_0)\mathrm{V}(c_0) - \mathrm{R}(c_0)\right), \end{aligned}$$

which implies that

$$\mathrm{V}(c) \leq \frac{\left(\frac{R_0}{2} + \lambda c_0\right)^{\frac{n-2}{2}} \left((R_0 + 2\lambda c_0)\mathrm{V}(c_0) - \mathrm{R}(c_0)\right)}{\left(\frac{R_0}{2} + \lambda c\right)^{\frac{n-2}{2}} (R_0 + 2\lambda c - n\lambda)}$$

for any $c > \frac{n}{2} - \frac{R_0}{2\lambda}$. Then as the proof in the case $n = 4k$, we also have

$$\mathrm{Vol}_g\left(B(O, c)\right) \leq \frac{C}{c^n}.$$

Thus the asymptotic volume ratio (AVR) can be calculated by

$$0 \leq \mathrm{AVR}(g) = \lim_{c \to \infty} \frac{\mathrm{Vol}_g B(O, c)}{\omega_n c^n} \leq \lim_{c \to \infty} \frac{C}{\omega_n c^{2n}} = 0.$$

□

3 Proof of Theorem 1.11

In this section, we prove Theorem 1.11 by using Theorem 1.5 and 1.7.

Proof of Theorem 1.11. Let

$$\mathrm{P}(c) = (R_0 + 2\lambda c)^{\frac{n}{2}} \mathrm{V}(c) - (R_0 + 2\lambda c)^{\frac{n-2}{2}} \mathrm{R}(c)$$

and

$$\mathrm{N}(c) = \frac{\mathrm{R}(c)}{(R_0 + 2\lambda c)\mathrm{V}(c)},$$

then

$$\begin{aligned} \frac{\mathrm{N}(c)}{\mathrm{N}(c) - 1} &= \frac{\frac{\mathrm{R}(c)}{(R_0+2\lambda c)\mathrm{V}(c)}}{\frac{\mathrm{R}(c)}{(R_0+2\lambda c)\mathrm{V}(c)} - 1} \\ &= \frac{(R_0 + 2\lambda c)^{\frac{n-2}{2}} \mathrm{R}(c)}{(R_0 + 2\lambda c)^{\frac{n-2}{2}} \mathrm{R}(c) - (R_0 + 2\lambda c)^{\frac{n}{2}} \mathrm{V}(c)} \\ &= -\frac{(R_0 + 2\lambda c)^{\frac{n-2}{2}} \mathrm{R}(c)}{\mathrm{P}(c)}. \end{aligned} \tag{3.1}$$

Note that $\frac{\mathrm{R}(c)}{\mathrm{V}(c)}$ is the average scalar curvature over the set $\mathrm{D}(c)$, and (3.1) and the ODE (1.9) implies that

$$\begin{aligned}
\mathrm{P}'(c) &= \frac{d}{dc}\left((R_0+2\lambda c)^{\frac{n}{2}}\mathrm{V}(c)-(R_0+2\lambda c)^{\frac{n-2}{2}}\mathrm{R}(c)\right)\\
&= \mathrm{V}'(c)(R_0+2\lambda c)^{\frac{n}{2}}+n\lambda\mathrm{V}(c)(R_0+2\lambda c)^{\frac{n-2}{2}}-(R_0+2\lambda c)^{\frac{n-2}{2}}\mathrm{R}'(c)\\
&\quad -(n-2)\lambda(R_0+2\lambda c)^{\frac{n-4}{2}}\mathrm{R}(c)\\
&= (R_0+2\lambda c)^{\frac{n-2}{2}}\left(n\lambda\mathrm{V}(c)+(R_0+2\lambda c)\mathrm{V}'(c)-\mathrm{R}'(c)-\mathrm{R}(c)\right)\\
&\quad +(R_0+2\lambda c)^{\frac{n-4}{2}}\mathrm{R}(c)\left(R_0+2\lambda c-(n-2)\lambda\right)\\
&= (R_0+2\lambda c)^{\frac{n-4}{2}}\mathrm{R}(c)\left(R_0+2\lambda c-(n-2)\lambda\right)\\
&= -\frac{\mathrm{N}(c)}{\mathrm{N}(c)-1}\frac{R_0+2\lambda c-(n-2)\lambda}{R_0+2\lambda c}\mathrm{P}(c)
\end{aligned}$$

Then we choose c_0 such that $\mathrm{P}(c_0)\neq 0$, and integrate

$$\mathrm{P}'(c) = -\frac{\mathrm{N}(c)}{\mathrm{N}(c)-1}\frac{R_0+2\lambda c-(n-2)\lambda}{R_0+2\lambda c}\mathrm{P}(c). \tag{3.2}$$

from c_0 to c we have

$$\mathrm{P}(c) = \mathrm{P}(c_0)\exp\left(-\int_{c_0}^{c}\frac{\mathrm{N}(s)}{\mathrm{N}(s)-1}\frac{R_0+2\lambda s-(n-2)\lambda}{R_0+2\lambda s}ds\right). \tag{3.3}$$

Note that when $s>\frac{n-2}{2}-\frac{R_0}{2\lambda}$, we have

$$\max\left\{R_0+2\lambda s, R_0+2\lambda s-(n-2)\lambda\right\}<0,$$

then

$$\begin{aligned}
&|\mathrm{P}(c)|\\
&=|\mathrm{P}(c_0)|\exp\left(-\int_{c_0}^{\frac{n-2}{2}-\frac{R_0}{2\lambda}}\frac{\mathrm{N}(s)}{\mathrm{N}(s)-1}\frac{R_0+2\lambda s-(n-2)\lambda}{R_0+2\lambda s}ds\right)\\
&\quad\exp\left(-\int_{\frac{n-2}{2}-\frac{R_0}{2\lambda}}^{c}\frac{\mathrm{N}(s)}{\mathrm{N}(s)-1}\frac{R_0+2\lambda s-(n-2)\lambda}{R_0+2\lambda s}ds\right)\\
&\leq|\mathrm{P}(c_0)|\exp\left(-\int_{c_0}^{\frac{n-2}{2}-\frac{R_0}{2\lambda}}\frac{\mathrm{N}(s)}{\mathrm{N}(s)-1}\frac{R_0+2\lambda s-(n-2)\lambda}{R_0+2\lambda s}ds\right)
\end{aligned} \tag{3.4}$$

for any $c>\frac{n-2}{2}-\frac{R_0}{2\lambda}$.

Moreover, (2.8) implies that $\mathrm{R}(c)\geq n\lambda\mathrm{V}(c)$, then when $c>\frac{n}{2}-\frac{R_0}{2\lambda}$ we have

$$\mathrm{V}(c)-\frac{1}{R_0+2\lambda c}\mathrm{R}(c)\geq\left(1-\frac{n\lambda}{R_0+2\lambda c}\right)\mathrm{V}(c). \tag{3.5}$$

Then when $n = 4k$, we have

$$\begin{aligned}
\mathrm{P}(c) &= (R_0 + 2\lambda c)^{\frac{n}{2}} \left(\mathrm{V}(c) - \frac{1}{R_0 + 2\lambda c} \mathrm{R}(c) \right) \\
&\geq (R_0 + 2\lambda c)^{\frac{n}{2}} \left(1 - \frac{n\lambda}{R_0 + 2\lambda c} \right) \mathrm{V}(c),
\end{aligned}$$

which implies that

$$0 \leq \frac{\mathrm{V}(c)}{c^{\frac{n}{2}}} \leq \frac{\mathrm{P}(c)}{c^n \left(\frac{R_0 + 2\lambda c}{c} \right)^{\frac{n}{2}} \left(1 - \frac{n\lambda}{R_0 + 2\lambda c} \right)} \tag{3.6}$$

for c is large enough.

Furthermore, by Theorem 1.5, for any sufficient small $\varepsilon > 0$, there exists r_0 such that when $r(x) > r_0$, we have

$$-\frac{\lambda}{2(1+\varepsilon)^2} (r(x) - C_1)^2 \leq -f(x) \leq -\frac{\lambda}{2} (r(x) + C_2)^2 .$$

Thus

$$\begin{aligned}
\mathrm{AVR}(g) &= \lim_{c \to \infty} \frac{\mathrm{Vol}_g B(O,c)}{\omega_n c^n} \\
&= \lim_{c \to \infty} \frac{\{x \in M^n \,| r(x) \leq c\}}{\omega_n c^n} \\
&= \lim_{c \to \infty} \frac{\left\{x \in M^n \,\middle| -f(x) \leq -\frac{\lambda}{2} c^2 \right\}}{\omega_n c^n} \\
&= \lim_{c \to \infty} \frac{\left(-\frac{\lambda}{2}\right)^{\frac{n}{2}} \mathrm{V}\left(-\frac{\lambda}{2} c^2\right)}{\omega_n \left(-\frac{\lambda}{2} c^2\right)^{\frac{n}{2}}} \\
&= \frac{\left(-\frac{\lambda}{2}\right)^{\frac{n}{2}}}{\omega_n} \lim_{c \to \infty} \frac{\mathrm{V}(c)}{c^{\frac{n}{2}}},
\end{aligned}$$

together with (3.4) and (3.6) we have

$$\begin{aligned}
\mathrm{AVR}(g) &= \frac{\left(-\frac{\lambda}{2}\right)^{\frac{n}{2}}}{\omega_n} \lim_{c \to \infty} \frac{\mathrm{V}(c)}{c^{\frac{n}{2}}} \\
&\leq \frac{\left(-\frac{\lambda}{2}\right)^{\frac{n}{2}}}{\omega_n} \lim_{c \to \infty} \frac{\mathrm{P}(c)}{c^n \left(\frac{R_0 + 2\lambda c}{c} \right)^{\frac{n}{2}} \left(1 - \frac{n\lambda}{R_0 + 2\lambda c} \right)} \\
&= \frac{1}{\omega_n 2^n} \lim_{c \to \infty} \frac{\mathrm{P}(c)}{c^n} \\
&= 0.
\end{aligned}$$

On the other hand, when $n = 4k + 2$, (3.5) implies that

$$\begin{aligned} \mathrm{P}(c) &= (R_0 + 2\lambda c)^{\frac{n}{2}} \left(\mathrm{V}(c) - \frac{1}{R_0 + 2\lambda c} \mathrm{R}(c) \right) \\ &\leq (R_0 + 2\lambda c)^{\frac{n}{2}} \left(1 - \frac{n\lambda}{R_0 + 2\lambda c} \right) \mathrm{V}(c), \end{aligned}$$

which also implies that

$$0 \leq \frac{\mathrm{V}(c)}{c^{\frac{n}{2}}} \leq \frac{\mathrm{P}(c)}{c^n \left(\frac{R_0 + 2\lambda c}{c} \right)^{\frac{n}{2}} \left(1 - \frac{n\lambda}{R_0 + 2\lambda c} \right)}$$

for c is large enough. Then as the proof in the case that $n = 4k$, we also have $\mathrm{AVR}(g) = 0$. □

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