Characteristic Jacobi operator on contact Riemannian 3-manifolds

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Abstract. The Ricci tensor, φ-Ricci tensor and the characteristic Jacobi operator on contact Riemannian 3-manifolds are investigated.

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1 Introduction

In contact Riemannian geometry, the Jacobi operator ℓ along the Reeb vector field ξ plays an important role. The class of contact Riemannian manifolds with $\ell = 0$ is particularly large. For instance, Bang [1] showed that the normal bundle of a Legendre submanifold in a Sasakian manifold admits a contact Riemannian structure with $\ell = 0$ (See [4, Theorem 9.16]). Contact Riemannian 3-manifolds with vanishing ℓ were studeid by Gouli-Andreou [16] and Perrone [31] (and called M_ℓ-manifolds).

Kouforgiorgos and Tsichlias [25] showed that complete, simply connected, contact Riemannian 3-manifolds with vanishing ℓ and positive constant $|Q\xi|$ are Lie groups. Here Q is the Ricci operator.

In this paper, we study model spaces for the class of 3-dimensional M_ℓ-manifolds with constant $|Q\xi|$.

Ghosh and Sharma [15] introduced a new class of contact Riemannian manifolds. According to [15], a contact Riemannian manifold is said to be a *Jacobi* (κ, μ)-*contact space* if it satisfies

$$\ell = -\kappa\varphi^2 + \mu\, h$$

for some constants κ and μ. This class includes both the class of contact (κ, μ)-spaces and that of K-contact manifolds.

The only examples of Jacobi (κ, μ)-contact spaces which are neither contact (κ, μ) nor K-contact given in [15] are M_ℓ-manifolds, *i.e.*, Jacobi $(0, 0)$-contact spaces.

In the final section of this paper, we provide new examples of Jacobi (κ, μ)-contact 3-spaces which are neither contact (κ, μ)-spaces nor M_ℓ-manifolds.

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2 Preliminaries

2.1

Let M be a manifold and η a 1-form on M. Then the exterior derivative $d\eta$ is defined by

$$2d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]), \quad X, Y \in \mathfrak{X}(M).$$

Here $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M.

Now let (M,g) be a Riemannian manifold with its *Levi-Civita connection* ∇. Then the *Riemannian curvature* R of M is defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

On a Riemannian manifold (M,g), We define a curvaturelike tensor field $(X,Y,Z) \longmapsto (X \wedge Y)Z$ on M by

$$(X \wedge Y)Z = g(Y,Z)X - g(Z,X)Y.$$

Note that a Riemannian manifold (M,g) is of constant curvature c if and only if its Riemannian curvature R satisfies $R(X,Y) = c(X \wedge Y)$ for all X, $Y \in \mathfrak{X}(M)$.

2.2

Let (M,g) be a Riemannian manifold. For a nonzero tangent vector $v \in T_pM$, the *tidal force operator* F_v associated to v is a linear endomorphism on $(\mathbb{R}v)^\perp$ defined by $F_v(w) := -R(w,v)v$ for $w \perp v$ ([28, p. 219]). One can see that F_v is self-adjoint on $(\mathbb{R}v)^\perp$ and has the trace $\operatorname{tr} F_v = -\rho(v,v)$. Here ρ is the Ricci tensor field of (M,g). For a geodesic γ in (M,g), a vector field X along γ is said to be a *Jacobi field* along γ if it satisfies the *Jacobi equation*:

$$\nabla_{\gamma'}\nabla_{\gamma'}X = -F_{\gamma'}(X).$$

2.3

On a Riemannian 3-manifold (M,g), the Riemannian curvature R is described by the Ricci tensor field ρ and corresponding *Ricci operator* Q by

$$R(X,Y)Z = \rho(Y,Z)X - \rho(Z,X)Y + g(Y,Z)QX - g(Z,X)QY - \frac{\mathrm{s}}{2}(X \wedge Y)Z \tag{2.1}$$

for all vector fields X, Y and Z on M. Here s is the *scalar curvature*.

2.4

Let G be a Lie group equipped with a invariant Riemannian metric $\langle \cdot, \cdot \rangle$. Then the Levi-Civita connection ∇ of $(G, \langle \cdot, \cdot \rangle)$ is described by the *Koszul formula*:

$$2\langle \nabla_X Y, Z \rangle = -\langle X, [Y,Z] \rangle + \langle Y, [Z,X] \rangle + \langle Z, [X,Y] \rangle, \quad X, Y, Z \in \mathfrak{g}.$$

Here $\mathfrak{g}$ is the Lie algebra of G. Let us define a symmetric bilinear map $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$2\langle U(X,Y), Z \rangle = \langle X, [Z,Y] \rangle + \langle Y, [Z,X] \rangle \tag{2.2}$$

and call it the *natural-reducibility obstruction* of $(G, \langle\cdot,\cdot\rangle)$. One can see that the metric g is right-invariant if and only if $U = 0$.

A Lie group G is said to be *unimodular* if its left invariant Haar measure is right invariant. J. Milnor gave an infinitesimal reformulation of unimodularity for 3-dimensional Lie groups. We recall it briefly here.

Let $\mathfrak{g}$ be a 3-dimensional oriented Lie algebra with an inner product $\langle\cdot,\cdot\rangle$. Denote by $\times$ the *vector product operation* of the oriented inner product space $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. The vector product operation is a skew-symmetric bilinear map $\times : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is uniquely determined by the following conditions:

(i) $\langle X, X \times Y\rangle = \langle Y, X \times Y\rangle = 0$,

(ii) $|X \times Y|^2 = \langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2$,

(iii) if X and Y are linearly independent, then $\det(X, Y, X \times Y) > 0$,

for all $X, Y \in \mathfrak{g}$. On the other hand, the Lie-bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a skew-symmetric bilinear map. Comparing these two operations, we get a linear endomorphism $L_{\mathfrak{g}}$ which is uniquely determined by the formula

$$[X, Y] = L_{\mathfrak{g}}(X \times Y), \quad X, Y \in \mathfrak{g}.$$

Now let G be an oriented 3-dimensional Lie group equipped with a left invariant Riemannian metric. Then the metric induces an inner product on the Lie algebra $\mathfrak{g}$. With respect to the orientation on $\mathfrak{g}$ induced from G, the endomorphism field $L_{\mathfrak{g}}$ is uniquely determined. The unimodularity of G is characterized as follows.

Proposition 2.1. ([27]) *Let G be an oriented 3-dimensional Lie group with a left invariant Riemannian metric. Then G is unimodular if and only if the endomorphism $L_{\mathfrak{g}}$ is self-adjoint with respect to the metric.*

3 Contact 3-manifolds

3.1

Let M be a $(2n+1)$-dimensional manifold. A *contact form* is a 1-form η which satisfies $(d\eta)^n \wedge \eta \neq 0$ on M.

A plane field $\mathfrak{D} \subset TM$ with rank $2n$ is said to be a *contact structure* on M if for any point $p \in M$, there exists a contact form η defined on a neighbourhood U_p of p such that $\operatorname{Ker} \eta = \mathfrak{D}$ on U_p.

A $(2n+1)$-manifold M together with a contact structure is called a *contact manifold*.

In this note we assume that there exists a globally defined contact form η which annihilates $\mathfrak{D}$, *i.e.*, $\operatorname{Ker} \eta = \mathfrak{D}$. Moreover we fix a contact form η on M.

On a contact manifold (M, η) with a fixed contact form η, there exists a unique vector field ξ such that

$$\eta(\xi) = 1, \quad d\eta(\xi, \cdot) = 0.$$

The vector field ξ is called the *Reeb vector field* of (M,η). Note that ξ is traditionally called the *characteristic vector field* of M in analytical mechanics. Moreover, (M,η) admits a Riemannian metric g and an endomorphism field φ such that

$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi, \;\; \varphi\xi = 0, \tag{3.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{3.2}$$

and

$$d\eta = \Phi,$$

where Φ is a 2-form

$$\Phi(X,Y) = g(X, \varphi Y).$$

The structure (φ,ξ,η,g) is called a *contact Riemannian structure* of M associated to the contact form η. A contact manifold (M,η) together with its associated contact Riemannian structure is called a *contact Riemannian manifold* and denoted by (M,φ,ξ,η,g). Note that on a contact manifold M, the structure group $\mathrm{GL}_{2n+1}\mathbb{R}$ of the linear frame bundle $L(M)$ is reducibe to $\mathrm{U}_n \times \{1\}$.

3.2

More generally, an *almost contact Riemannian structure* of a $(2n+1)$-manifold M is a quartet (φ,ξ,η,g) of structure tensor fields which satisfies (3.1)–(3.2). A $(2n+1)$-manifold $M = (M,\varphi,\xi,\eta,g)$ equipped with an almost contact Riemannian structure is called an *almost contact Riemannian manifold*.

Definition 3.1. Let (M,φ,ξ,η,g) be an almost contact Riemannian manifold. A tangent plane Π_x at $x \in M$ is said to be *holomorphic* if it is invariant under φ_x.

It is easy to see that a tangent plane Π_x is holomorphic if and only if ξ_x is orthogonal to Π_x. The sectional curvature $H_x := K(\Pi_x)$ of a holomorphic plane Π_x is called the *holomorphic sectional curvature* of M at x.

3.3

Now we recall an endomorphism field h which is useful for the study of contact Riemannian manifolds.

$$hX = \frac{1}{2}(\pounds_\xi \varphi)X = \frac{1}{2}\{[\xi, \varphi X] - \varphi[\xi, X]\}.$$

Let $M = (M,\varphi,\xi,\eta,g)$ be a contact Riemannian 3-manifold, then M satisfies ([38]):

$$(\nabla_X \varphi)Y = (\xi \wedge (\mathrm{I}+h)X)Y, \quad X, Y \in \mathfrak{X}(M).$$

The *Webster curvature* W of a contact Riemannian 3-manifold M is defined by

$$W = \frac{1}{8}(\mathrm{s} - \rho(\xi,\xi) + 4).$$

Here ρ is the Ricci tensor and s is the scalar curvature of M, respectively. The *torsion invariant* of M introduced by Chern and Hamilton [7] is the square norm $|\tau|^2$ of $\tau = \pounds_\xi g$. The torsion invariant is computed as

$$|\tau|^2 = -2\rho(\xi,\xi) + 4.$$

3.4

Contact Riemannian 3-manifolds with $h = 0$ has been paid much attention from differential geometers.

Definition 3.2. A contact Riemannian 3-manifold is said to be a *Sasakian* 3*-manifold* if $h = 0$.

Proposition 3.1. *A contact Riemannian* 3*-manifold is Sasakian if and only if* $\tau = 0$.

Definition 3.3. A complete Sasakian 3-manifold M is said to be a *Sasakian space form* if it is of constant holomorphic sectional curvature.

3.5

On a $(2n+1)$-dimensional contact Riemannian manifold $(M, \varphi, \xi, \eta, g)$, we define a self-adjoint endomorphism field ℓ by

$$\ell(X) = R(X, \xi)\xi, \quad X \in \mathfrak{X}(M).$$

One can see that $\ell = -F_\xi$ on the contact distribution. Moreover ℓ and h satisfy the following relations;

$$h\xi = \ell(\xi) = 0, \ \eta \circ h = 0, \ \mathrm{tr}\, h = \mathrm{tr}\,(h\varphi) = 0, \ h\varphi + \varphi h = 0,$$

$$\nabla_\xi h = \varphi(I - \ell - h^2), \ \mathrm{tr}\, \ell = 2n - \mathrm{tr}\,(h^2),$$

The self adjoint operator ℓ is called the *characteristic Jacobi operator* of M. In case $\dim M = 3$, we have the following equations:

$$\tau(X, Y) = 2g(\varphi X, hY), \quad (\nabla_\xi \tau)(X, Y) = 2g(\varphi X, (\nabla_\xi h)Y).$$

Lemma 3.2. (*cf.* [6], [19]) *Let M be a contact Riemannian* 3*-manifold. Then there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ such that* :

$$he_1 = \lambda e_1, \ e_2 = \varphi e_1, \ e_3 = \xi, \quad \lambda \geq 0.$$

With respect to $\mathcal{E}$, the Levi-Civita connection ∇ is given by

$$\begin{array}{ccc} \nabla_{e_1} e_1 = be_2, & \nabla_{e_1} e_2 = -be_1 + (1+\lambda)\xi, & \nabla_{e_1}\xi = -(1+\lambda)e_2, \\ \nabla_{e_2} e_1 = -ce_2 + (\lambda - 1)e_3, & \nabla_{e_2} e_2 = ce_1, & \nabla_{e_2}\xi = (1-\lambda)e_1, \\ \nabla_\xi e_1 = \alpha e_2, & \nabla_\xi e_2 = -\alpha e_1, & \nabla_\xi \xi = 0. \end{array}$$

The Ricci operator Q is given by

$$\begin{aligned} Qe_1 &= \rho_{11} e_1 + \xi(\lambda) e_2 + (2b\lambda - e_2(\lambda))\xi, \\ Qe_2 &= \xi(\lambda) e_1 + \rho_{22} e_2 + (2c\lambda - e_1(\lambda))\xi, \\ Q\xi &= (2b\lambda - e_2(\lambda)) e_1 + (2c\lambda - e_1(\lambda)) e_2 + 2(1-\lambda^2)\xi, \end{aligned}$$

where

$$\rho_{11} = \frac{\mathrm{s}}{2} + \lambda^2 - 2\alpha\lambda - 1, \quad \rho_{22} = \frac{\mathrm{s}}{2} + \lambda^2 + 2\alpha\lambda - 1.$$

The covariant derivative $\nabla_\xi h$ *of* h *by* ξ *is given by*

$$\nabla_\xi h = -2\alpha h\varphi + \nu h.$$

The function ν *is given by* $\nu = 0$ *when* M *is Sasakian and* $\nu = \xi(\lambda)/\lambda$ *when* M *is non-Sasakian.*

Remark 3.4. Perrone obtained the following result.

Proposition 3.3. ([31, Proposition 2.1]) *On a contact Riemannian metric manifold* M*, the following four conditions are mutually equivalent.*

- $\nabla_\xi h = 0$,
- $\nabla_\xi \tau = 0$,
- $\nabla_\xi \ell = 0$,
- $\ell\varphi = \varphi\ell$.

3.6

Let M be an almost contact Riemannian manifold. We define a tensor field ρ^* on M by

$$\rho^*(X,Y) := \frac{1}{2}\,\mathrm{trace}\, R(X,\varphi Y)\varphi.$$

One can see that $\rho^*(X,\xi) = 0$ for all $X \in \mathfrak{X}(M)$. Next we denote by ρ^φ the symmetric part of ρ^*, that is,

$$\rho^\varphi(X,Y) = \frac{1}{2}\{\rho^*(X,Y) + \rho^*(Y,X)\}.$$

We call ρ^φ the *φ-Ricci tensor field* of M [11].

Definition 3.5. An almost contact Riemannian manifold M is said to be a *weakly φ-Einstein manifold* if

$$\rho^\varphi(X,Y) = \lambda g^\varphi(X,Y), \quad X,Y \in (M)$$

for some function λ. Here the symmetric tensor field g^φ is defined by

$$g^\varphi(X,Y) = g(\varphi X, \varphi Y), \quad X,Y \in \mathfrak{X}(M).$$

When λ is a constant, then M is said to be a *φ-Einstein manifold*. The function $\mathrm{s}^\varphi = \mathrm{trace}\,\rho^\varphi$ is called the *φ-scalar curvature* of M.

Remark 3.6. An almost contact Riemannian manifold M is said to be *weakly $*$-Einstein* if

$$\rho^*(X,Y) = \lambda g(X,Y), \quad X,Y \in \mathcal{D}$$

for some function λ. The function $\mathrm{s}^* = \mathrm{trace}\,\rho^*$ is called the *$*$-scalar curvature* of M. A weakly $*$-Einstein manifold of constant $*$-scalar curvature is called a *$*$-Einstein manifold*. Clearly $\mathrm{s}^\varphi = \mathrm{s}^*$.

4 H-contact manifolds

4.1

In this section we study Ricci tensor and φ-Ricci tensor of H-contact 3-manifolds.

First we recall the notion of H-contact manifold. Let (M, g) be a Riemannian manifold with unit tangnet sphere bundle T_1M. We equip the Sasaki-lift metric on T_1M. Denote by $\mathfrak{X}_1(M)$ the space of all smooth unit vector fields on M. A unit vector field $V \in \mathfrak{X}_1(M)$ is said to be *harmonic* if it is a critical point of the energy functional restricted to $\mathfrak{X}_1(M)$.

In particular, a contact Riemannian manifold M is said to be an *H-contact manifold* if its Reeb vector field is harmonic in above sense.

The H-contact property is characterized in terms of Ricci opeartor as follows.

Theorem 4.1 ([35])**.** *A contact Riemannian 3-manifold M is H-contact if and only if ξ is an eigenvector field of Q, that is, $Q\xi = \sigma\xi$ for some function σ.*

To characterize the class of H-contact manifolds, we here recall the following definition.

Definition 4.1. A contact Riemannian manifold M is said to be a *contact generalized (κ, μ, ν)-space* if its Riemannian curvature R satisfies

$$R(X, Y)\xi = (\kappa\,\mathrm{I} + \mu h + \nu\varphi h)\{\eta(Y)X - \eta(X)Y\} \tag{4.1}$$

for all X, $Y \in \mathfrak{X}(M)$. Here κ, μ and ν are smooth functions. In particular, A contact generalized $(\kappa, \mu, 0)$-space is called a *contact generalized (κ, μ)-space.*

Definition 4.2. Let M be a contact generalized (κ, μ)-space. If both the functions κ and μ are constants, then M is called a *contact (κ, μ)-space* (see [18], [23], [24]). A generalized (κ, μ)-space is said to be proper if $|d\kappa|^2 + |d\mu|^2 \neq 0$.

One can see that Sasakian manifolds are contact (κ, μ)-spaces with $\kappa = 1$ and $h = 0$.

Remark 4.3. Contact generalized (κ, μ, ν)-spaces are of particular interest in dimension 3. In fact the following results are known.

Theorem 4.2 ([22])**.** *Let M be a non-Sasakian contact generalized (κ, μ, ν)-space. If* $\dim M > 3$*, then κ and μ are constants and $\nu = 0$.*

Corollary 4.3 ([23])**.** *Let M be a non-Sasakian contact generalized (κ, μ)-space. If* $\dim M > 3$*, then κ and μ are constants.*

Perrone gave the following variational characterization of generalized (κ, μ)-property.

Theorem 4.4 ([34])**.** *On a contact Riemannian 3-manifold M, its Reeb vector field ξ is a harmonic map from (M, g) into the unit tangent sphere bundle T_1M equipped with the Sasaki-lift metric of g if and only if M satisfies the generalized (κ, μ)-condition on an open dense subset of M.*

Kouforgiorgos, Markellos and Papantoniou generalized Perrone's theorem as follows.

Theorem 4.5 ([22]). *Let M be a contact Riemannian 3-manifold. If M is a contact generalized (κ, μ, ν)-space then M is an H-contact manifold, that is, its Reeb vector field ξ is a critical point of the energy functional on the space of all unit vector fields on M. Conversely, if M is H-contact, then M satisfies the generalized (κ, μ, ν)-condition on an open dense subset of M.*

Proposition 4.6 ([22]). *Let M be a non-Sasakian 3-dimensional contact generalized (κ, μ, ν)-space. Then there exits a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3 = \xi\}$ such that*

$$he_1 = \lambda e_1, \quad he_2 = -\lambda e_2, \quad e_2 = \varphi e_1,$$

where $\lambda = \sqrt{1-\kappa} > 0$. The Ricci operator Q is given by

$$Q = \alpha \mathrm{I} + \beta \eta \otimes \xi + \mu h + \nu \varphi h$$

with

$$\begin{aligned}
\alpha &= \frac{1}{2}(\mathrm{s} - 2\kappa) = \frac{\mathrm{s}}{2} - (1-\lambda^2) = g(\nabla_\xi e_1, e_2), \\
\beta &= \frac{1}{2}(6\kappa - \mathrm{s}) = -\frac{\mathrm{s}}{2} + 3(1-\lambda^2), \\
\mu &= -2\alpha, \quad \nu = \frac{\xi(\lambda)}{\lambda}.
\end{aligned}$$

More specifically we have

$$\begin{aligned}
Q\epsilon_1 &= \frac{1}{2}(\mathrm{s} - 2\kappa + 2\mu\sqrt{1-\kappa})e_1 + \nu\sqrt{1-\kappa}e_2, \\
Q\epsilon_2 &= \nu\sqrt{1-\kappa}\epsilon_1 + \frac{1}{2}(\mathrm{s} - 2\kappa - 2\mu\sqrt{1-\kappa})e_2, \\
Qe_3 &= 2\kappa e_3.
\end{aligned}$$

The covariant derivative $\nabla_\xi h$ of h is given by $\nabla_\xi h = \mu h\varphi + \nu h$.

4.2

Now we study pseudo-symmetry of 3-dimensional contact generalized (κ, μ, ν)-spaces.

Let M be a 3-dimensional contact generalized (κ, μ, ν)-spaces. Then the characteristic polynomial $\Psi(t) = \det(t\delta_{ij} - \rho_{ij})$ for Q is given by

$$\Psi(t) = (t - 2\kappa)F(t),$$

where

$$F(t) = t^2 - (\mathrm{s} - 2\kappa)t + \frac{1}{4}(\mathrm{s} - 2\kappa)^2 - (1-\kappa)(\mu^2 + \nu^2).$$

Now we investigate pseudo-symmetry of generalized (κ, μ, ν)-spaces.

- Case 1: $\rho_{33} = 2\kappa$ solves $F(t) = 0$: Direct computation shows that $F(2\kappa) = 0$ if and only if
$$(1-\kappa)(\mu^2 + \nu^2) = \left(3\kappa - \frac{\mathrm{s}}{2}\right)^2$$

In this case we have

$$F(t) = t^2 - (\mathrm{s} - 2\kappa)t + 2\kappa\mathrm{s} - 8\kappa^2 = (t - 2\kappa)(t + 4\kappa - \mathrm{s}).$$

Thus the principal Ricci curvatures are

$$2\kappa,\ 2\kappa,\ -4\kappa + \mathrm{s}.$$

- Case 2: $F(t) = 0$ has double roots: The discriminant $\mathcal{D}$ of $F(t) = 0$ is

$$\mathcal{D} = 4(1 - \kappa)(\mu^2 + \nu^2).$$

Hence $F(t) = 0$ has double roots if and only if $\mu = \nu = 0$. Since $\mu = -2\alpha$ and $\nu = \xi(\lambda)/\lambda$, M is a generalized $(\kappa, 0)$-space with constant ξ-sectional curvature. Hence λ is constant and so is κ. The principal Ricci curvatures are

$$\mathrm{s}/2 - \kappa,\ \mathrm{s}/2 - \kappa,\ 2\kappa.$$

Theorem 4.7. *Let M be a non-Sasakian 3-dimensional contact generalized (κ, μ, ν)-space. Then M is pseudo-symmetric if and only if $\mu = 0$ or $\mu^2+\nu^2 = \left|(3\kappa - \mathrm{s}/2)/\sqrt{1-\kappa}\right|$. In the former case, M is a contact $(\kappa, 0)$-space.*

4.3

Next let us consider Ricci $*$-tensor of contact generalized (κ, μ, ν)-spaces.

Take a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ as before. Then we have

$$R(e_i, \varphi e_i)\xi = 0, \quad i = 1, 2, 3.$$

Thus, for any tangent vector field Y, we have

$$\begin{aligned} \rho^*(\xi, Y) &= \frac{1}{2}\sum_{i=1}^{2} g(R(\xi, \varphi Y)\varphi e_i), e_i) \\ &= g(R(\varphi e_1, e_1)\xi, \varphi Y) = 0. \end{aligned}$$

Proposition 4.8. *Let M be a 3-dimensional contact generalized (κ, μ, ν)-space. Then $\rho^*(\xi, \cdot) = 0$. Hence M is weakly $*$-Einstein.*

Corollary 4.9. *Let M be a 3-dimensional contact generalized (κ, μ, ν)-space. If its $*$-scalar curvature is constant, then M is φ-Einstein.*

5 M_ℓ-manifolds

5.1

Let M be a contact Riemannian manifold. Then M is said to be an M_ℓ-*manifold* if its characteristic Jacobi field ℓ vanishes. We study pseudo-symmetry of 3-dimensional M_ℓ-manifolds.

Assume that M is a 3-dimensional non-Sasakian contact Riemannian manifold. Take a local orthonormal frame field $\mathcal{E} = (e_1, e_2, e_3)$ as in Lemma 3.2, then we have

$$\ell(e_1) = -(\lambda^2 + 2\alpha\lambda - 1)e_1 + d\lambda(\xi)e_2,$$

$$\ell(e_2) = d\lambda(\xi)e_1 - (\lambda^2 - 2\alpha\lambda - 1)e_2.$$

Thus $\ell = 0$ if and only if $a = 0$ and $\lambda^2 = 1$. In such a case, the Ricci operator Q is given by

$$Qe_1 = \frac{\mathrm{s}}{2}e_1 + 2b\xi, \quad Qe_2 = \frac{\mathrm{s}}{2}e_2 + 2c\xi, \quad Q\xi = 2be_1 + 2ce_2.$$

Proposition 5.1. *A non-Sasakian contact Riemannian* 3*-manifold is a* M_ℓ*-manifold if and only if* $\eta(Q\xi) = 0$ *and* $\rho_{11} = \rho_{22} = \mathrm{s}/2$.

Now let M be a 3-dimensional M_ℓ-manifold. Then the characteristic polynomial for Q is

$$\det(t\mathrm{I} - \rho) = (t - \mathrm{s}/2)F(t), \quad F(t) = t^2 - \frac{s}{2}t - 4(b^2 + c^2).$$

Thus Ricci eigenvalues are $\{\rho_1, \rho_+, \rho_-\}$,

$$\rho_1 = \rho_{11} = \rho_{22} = \frac{s}{2}, \quad \rho_\pm = \frac{\mathrm{s}}{4} \pm \frac{1}{4}\sqrt{\mathrm{s}^2 + 16|Q\xi|^2}.$$

- ρ_1 solves $F(t)$ if and only if $Q\xi = 0$, *i.e.*, $b = c = 0$. Thus we have Ricci eigenvalues $\mathrm{s}/2$, $\mathrm{s}/2$ and 0. On the other hand, since $Q\xi = 0$, we have $\mathrm{s} = 0$. Thus $Q = 0$ so M is flat.

- $\rho_+ = \rho_-$ if and only if $\mathrm{s} = 0$ and $Q\xi = 0$, so M is flat.

Proposition 5.2. *A* 3*-dimensional* M_ℓ *manifold is pseudo-symmetric if and only if it is flat.*

5.2

Koufogiorgos and Tsichlias obtained the following result.

Theorem 5.3 ([25])**.** *Let* M *be a complete, simply connected* 3*-dimesional contact Riemannian manifold with* $\ell = 0$*. Assume that the norm* $|Q\xi|$ *is a constant, say* q *on* M*.*

- *If* $q = 0$ *then* M *is flat.*

- *If* $q > 0$*, then for each point* $p \in M$*, there exits a unique Lie group structure such that* p *is the unit element, the orthonomal frame field* $\{Q\xi/q, -\varphi Q\xi/q, \xi\}$ *and the metric* g *are left invariant with respect to it. Moreover* M *has constant negative scalar curvature.*

This theorem motives us to study homogeneous contact Riemannian 3-manifolds satisfying $\ell = 0$.

6 Homogeneous contact Riemannian 3-manifolds

6.1

In this section we collect some fundamental facts on contact Riemannian 3-manifolds which has isometric actions of Lie groups which preserve contact structure.

Definition 6.1. A diffeomorphism f on a contact 3-manifold (M, η) is said to be a *contact transformation* if f preserves the contact structure $\mathfrak{D} = \mathrm{Ker}\,\eta$. In particular, a contact transformation f is said to be a *strictly contact transformation* if f preserves η, *i.e.*, $f^*\eta = \eta$.

Definition 6.2. A contact Riemannian 3-manifold $M = (M, \varphi, \xi, \eta, g)$ is said to be a *homogeneous contact Riemannian* 3*-manifold* if there exists a Lie group H of isometries which acts transitively on M such that every element of H is a strictly contact transformation.

Here we recall the following result due to Tanno [37].

Lemma 6.1. *Let M be a contact Riemannian* 3*-manifold and f a diffeomorphism on M. If f is φ-holomorphic,* i.e., *$df \circ \varphi = \varphi \circ df$, then there exists a positive constant a such that*

$$f_*\xi = a\xi, \; f^*\eta = a\eta, \; f^*g = ag + a(a-1)\eta \otimes \eta.$$

This Lemma implies that every φ-holomorphic isometry is a strict contact transformation.

By virtue of a result of Sekigawa [36], Perrone obtained the following classification.

Theorem 6.2 ([32])**.** *Let M be a simply connected homogeneous contact Riemannian* 3*-manifold, then M is a Lie group equipped with left invariant contact Riemannian structure.*

7 Unimodular Lie groups

7.1

Let G be a 3-dimensional unimodular Lie group with a left invariant metric $\langle \cdot, \cdot \rangle$. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1, e_2] = c_3 e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \qquad c_i \in \mathbb{R}. \tag{7.1}$$

Three-dimensional unimodular Lie groups are classified by Milnor as follows:

Signature of (c_1, c_2, c_3)	Simply connected Lie group	Property
$(+,+,+)$	$\mathrm{SU}(2)$	compact and simple
$(-,-,+)$	$\widetilde{\mathrm{SL}}_2\mathbb{R}$	non-compact and simple
$(0,+,+)$	$\widetilde{E}(2)$	solvable
$(0,-,+)$	$E(1,1)$	solvable
$(0,0,+)$	Heisenberg group Nil_3	nilpotent
$(0,0,0)$	$(\mathbb{R}^3,+)$	Abelian

To describe the Levi-Civita connection ∇ of G, we introduce the following constants:

$$\mu_i = \frac{1}{2}(c_1 + c_2 + c_3) - c_i.$$

Proposition 7.1. *The Levi-Civita connection is given by*

$$\begin{array}{ccc} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = \mu_1 e_3, & \nabla_{e_1} e_3 = -\mu_1 e_2 \\ \nabla_{e_2} e_1 = -\mu_2 e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = \mu_2 e_1 \\ \nabla_{e_3} e_1 = \mu_3 e_2, & \nabla_{e_3} e_2 = -\mu_3 e_1 & \nabla_{e_3} e_3 = 0. \end{array}$$

The Riemannian curvature R is given by

$$R(e_1, e_2)e_1 = (\mu_1\mu_2 - c_3\mu_3)e_2, \quad R(e_1, e_2)e_2 = -(\mu_1\mu_2 - c_3\mu_3)e_1,$$

$$R(e_2, e_3)e_2 = (\mu_2\mu_3 - c_1\mu_1)e_3, \quad R(e_2, e_3)e_3 = -(\mu_2\mu_3 - c_1\mu_1)e_2,$$

$$R(e_1, e_3)e_1 = (\mu_3\mu_1 - c_2\mu_2)e_3, \quad R(e_1, e_3)e_3 = -(\mu_3\mu_1 - c_2\mu_2)e_1.$$

The basis $\{e_1, e_2, e_3\}$ diagonalises the Ricci tensor. The principal Ricci curvatures are given by

$$\rho_1 = 2\mu_2\mu_3, \quad \rho_2 = 2\mu_1\mu_3, \quad \rho_3 = 2\mu_1\mu_2.$$

The natural-reducibility obstruction U is given by

$$U(e_1, e_2) = \frac{1}{2}(-c_1 + c_2)e_3, \quad U(e_1, e_3) = \frac{1}{2}(c_1 - c_3)e_2, \quad U(e_2, e_3) = \frac{1}{2}(-c_2 + c_3)e_1.$$

7.2

According to a result due to Perrone, simply connected homogeneous contact Riemannian 3-manifolds are classified by the Webster scalar curvature W and the torsion invariant $|\tau|^2$ as follows:

Theorem 7.2. *Let $(M^3, \varphi, \xi, \eta, g)$ be a simply connected homogeneous contact Riemannian 3-manifold. Then M is a Lie group G together with a left invariant contact Riemannian structure (φ, ξ, η, g). If G is unimodular, then G is one of the following;*

1. *the Heisenberg group* Nil_3 *if* $W = |\tau| = 0$.
2. $\mathrm{SU}(2)$ *if* $4\sqrt{2}W > |\tau|$.
3. $\widetilde{E}(2)$ *if* $4\sqrt{2}W = |\tau| > 0$.
4. $\widetilde{\mathrm{SL}}_2\mathbb{R}$ *if* $-|\tau| \neq 4\sqrt{2}W < |\tau|$.
5. $E(1,1)$ *if* $4\sqrt{2}W = -|\tau| < 0$.

The Lie algebra $\mathfrak{g}$ of G is generated by an orthonormal basis $\{e_1, e_2, e_3\}$ as in (7.1) *with $c_3 = 2$. The left invariant contact Riemannian structure is determined by*

$$\xi = e_3, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \ \varphi\xi = 0.$$

Proposition 7.3. *The endomorphism field h, the Webster scalar curvature and the torsion invariant of a unimodular Lie group G equipped with a left invariant homogeneous contact Riemannian structure are given by*

$$he_1 = -\frac{1}{2}(c_1 - c_2)e_1, \quad he_2 = \frac{1}{2}(c_1 - c_2)e_2.$$

$$W = \frac{1}{4}(c_1 + c_2), \quad |\tau|^2 = (c_1 - c_2)^2.$$

The holomorphic sectional curvature of G is

$$H = -3 + \frac{1}{4}(c_1 - c_2)^2 + c_1 + c_2.$$

Corollary 7.4. *If a unimodular Lie group G is non-Sasakian,* i.e., $c_1 \neq c_2$, *then G is a (κ, μ)-space with*

$$\kappa = 1 - \frac{1}{4}(c_1 - c_2)^2, \quad \mu = 2 - (c_1 + c_2).$$

Proposition 7.5. *The φ-Ricci tensor field of a unimodular Lie group G is given by*

$$\rho^{\varphi}_{11} = \rho^{\varphi}_{22} = H, \quad \rho^{\varphi}_{ij} = 0 \text{ for other } i, j.$$

Hence G is φ-Einstein.

7.3

The characteristic Jacobi operator ℓ of a unimodular Lie group G is computed as

$$\ell(e_1) = \ell_1 e_1, \quad \ell(e_2) = \ell_2 e_2,$$

where

$$\ell_1 = \frac{1}{4}(c_1 - c_2)(c_1 + 3c_2 - 4) + 1, \quad \ell_2 = -\frac{1}{4}(c_1 - c_2)(3c_1 + c_2 - 4) + 1.$$

Hence $\ell = 0$ if and only if $c_1 + c_2 = 0$ and $c_1 - c_2 = \pm 2$.

$$\begin{cases} (c_1, c_2) = (0, 2) & \lambda = 1, \\ (c_1, c_2) = (2, 0) & \lambda = -1, \end{cases}$$

Thus the only possible Lie algebra is $\mathfrak{e}(2)$ and the metric is flat.

Remark 7.1. Let us denote by $\widetilde{E}(2)$ the universal covering of $E(2)$ equipped with the flat associated metric. Then $\widetilde{E}(2)$ is realised as $\mathbb{R}^3(x, y, z)$ with contact form $\eta = (dz - y dx)/2$ with

$$g = \begin{pmatrix} 1 + y^2 + z^2 & z & -y \\ z & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}.$$

This example can be generalizedto arbitrary odd-dimension as follows (*cf.* [4, p. 121]):

$$g = \frac{1}{4}\begin{pmatrix} \delta_{ij} + y_i y_j + \delta_{ij} z & \delta_{ij} z & -y_i \\ \delta_{ij} z & \delta_{ij} & 0 \\ -y_j & 0 & 1 \end{pmatrix}, \quad \eta = \frac{1}{2}\left(dz - \sum_{i=1}^{n} y_i dx_i\right)$$

on $\mathbb{R}^{2n+1}$. The resulting contact Riemannian manifold satisfies $\ell = 0$ but not contact $(0,0)$-space for $n > 1$. In particular this contact Riemannian manifold is non-flat for $n > 1$.

8 Non-unimodular Lie groups

8.1

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Denote by ad the *adjoint representation* of $\mathfrak{g}$,

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}); \quad \mathrm{ad}(X)Y = [X, Y].$$

Then one can see that tr ad;

$$X \longmapsto \mathrm{tr}\,\mathrm{ad}(X)$$

is a Lie algebra homomorphism into the commutative Lie algebra $\mathbb{R}$. The kernel

$$\mathfrak{u} = \{X \in \mathfrak{g} \mid \mathrm{tr}\,\mathrm{ad}(X) = 0\}$$

of tr ad is an ideal of $\mathfrak{g}$ which contains the ideal $[\mathfrak{g}, \mathfrak{g}]$.

Now we equip a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on G. Denote by $\mathfrak{u}$ the orthogonal complement of $\mathfrak{u}$ in $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$. Then the homomorphism theorem implies that $\dim \mathfrak{u}^{\perp} = \dim \mathfrak{g}/\mathfrak{u} \leq 1$.

The following criterion for unimodularity is known (see [27, p. 317]).

Lemma 8.1. *A Lie group G with a left invariant metric is unimodular if and only if $\mathfrak{u} = \mathfrak{g}$.*

Based on this criterion, the ideal $\mathfrak{u}$ is called the *unimodular kernel* of $\mathfrak{g}$. In particular, for a 3-dimensional non-unimodular Lie group G, its unimodular kernel $\mathfrak{u}$ is commutative and of 2-dimension.

8.2

Now let us consider 3-dimensional non-unimodular Lie groups equipped with left invariant contact Riemannian structure. Here we recall Perrone's construction [32].

Let G be a 3-dimensional non-unimodular homogeneous contact Riemannian manifold. Then one can easily check that $\xi \in \mathfrak{u}$. We take an orthonormal basis $\{e_2, e_3 = \xi\}$ of $\mathfrak{u}$. Then $e_1 = -\varphi e_2 \in \mathfrak{u}^{\perp}$ and hence $\mathrm{ad}(e_1)$ preserves $\mathfrak{u}$. Express $\mathrm{ad}(e_1)$ as

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3$$

over $\mathfrak{u}$. The compatibility condition $d\eta = \Phi$ implies that $\beta = 2$. Next, $\nabla_{\xi}\xi = 0$ implies that $\delta = 0$. Moreover one can deduce that $[e_2, e_3] = 0$ from the Jacobi identity.

Remark 8.1. Milnor [27] chose the following orthonormal basis $\{u_1, u_2, u_3\}$ for a non-unimodular Lie group G with left invariant Riemannian metric.

$$u_1 \in \mathfrak{u}^\perp, \quad \langle \mathrm{ad}(u_1)u_2, \mathrm{ad}(u_1)u_3\rangle = 0.$$

This orthonormal basis $\{u_1, u_2, u_3\}$ satisfies

$$[u_1, u_2] = \alpha u_2 + \beta u_3, \ [u_2, u_3] = 0, \ [u_1, u_3] = \gamma u_2 + \delta u_3$$

with $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$. Moreover $\{u_1, u_2, u_3\}$ diagonalises the Ricci tensor. On the other hand, the basis $\{e_1, e_2, e_3\}$ constructed for a non-unimodular homogeneous contact Riemannian 3-manifold G does not satisfy the orthogonality condition $\langle \mathrm{ad}(u_1)u_2, \mathrm{ad}(u_1)u_3\rangle = 0$. In fact, $\{e_1, e_2, e_3\}$ satisfies this orthogonality condition if and only if $\gamma = 0$.

Theorem 8.2 ([32])**.** *Let G be a 3-dimensional non-unimodular Lie group equipped with a left invariant contact Riemannian structure. Then the Lie algebra $\mathfrak{g}$ satisfies the commutation relations*

$$[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = -\gamma e_2,$$

with $e_3 = \xi$, $e_1 = -\varphi e_2 \in \mathfrak{u}^\perp$ and $\alpha \neq 0$. The Webster scalar curvature and the torsion invariant satisfy the relation:

$$4\sqrt{2}W < |\tau|.$$

The Lie algebra $\mathfrak{g} = \mathfrak{g}(\alpha, \gamma)$ is given explicitly by

$$\mathfrak{g}(\alpha,\gamma) = \left\{ \begin{pmatrix} (1+\alpha)x & \gamma x & y \\ 2x & x & z \\ 0 & 0 & x \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

with basis

$$e_1 = \begin{pmatrix} (1+\alpha) & \gamma & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The corresponding simply connected Lie group $G(\alpha, \gamma) = \exp \mathfrak{g}(\alpha, \gamma)$ is realized as $\mathbb{R}^3(x, y, z)$ with left invariant metric $\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \eta \otimes \eta$, where

$$\omega^1 = \frac{1}{2}dx, \quad \omega^2 = \frac{1}{2}(\alpha y + \gamma z)dx + dy, \quad \eta = dz + ydx.$$

The Levi-Civita connection of G is given by the following table:

Proposition 8.3. ([32, p. 251])

$$\begin{array}{ccc} \nabla_{e_1}e_1 = 0, & \nabla_{e_1}e_2 = -\frac{1}{2}(\gamma-2)e_3, & \nabla_{e_1}e_3 = \frac{1}{2}(\gamma-2)e_2 \\ \nabla_{e_2}e_1 = -\alpha e_2 - \frac{1}{2}(\gamma+2)e_3, & \nabla_{e_2}e_2 = \alpha e_1, & \nabla_{e_2}e_3 = \frac{1}{2}(\gamma+2)e_1 \\ \nabla_{e_3}e_1 = -\frac{1}{2}(\gamma+2)e_2, & \nabla_{e_3}e_2 = \frac{1}{2}(\gamma+2)e_1 & \nabla_{e_3}e_3 = 0. \end{array}$$

The endomorphism field h is given by

$$he_1 = -\frac{1}{2}\gamma e_1, \quad he_2 = \frac{1}{2}\gamma e_2.$$

The Riemannian curvature R is given by

$$\begin{aligned}
R(e_1,e_2)e_1 &= -\left\{\frac{1}{4}(\gamma^2-4\gamma-12)-\alpha^2\right\}e_2+\alpha\gamma e_3,\\
R(e_1,e_2)e_2 &= \left\{\frac{1}{4}(\gamma^2-4\gamma-12)-\alpha^2\right\}e_1,\\
R(e_1,e_3)e_1 &= \alpha\gamma e_2+\frac{1}{4}(3\gamma^2+4\gamma-4)e_3,\\
R(e_1,e_3)e_3 &= -\frac{1}{4}(3\gamma^2+4\gamma-4)e_1,\\
R(e_2,e_3)e_2 &= -\frac{1}{4}(\gamma+2)^2e_3,\\
R(e_2,e_3)e_3 &= \frac{1}{4}(\gamma+2)^2e_2,\\
R(e_1,e_2)e_3 &= -\alpha\gamma e_1.
\end{aligned}$$

$$H = K_{12} = \frac{1}{4}(\gamma^2-4\gamma-12)-\alpha^2, \quad K_{13} = -\frac{1}{4}(3\gamma^2+4\gamma-4), \quad K_{23} = \frac{1}{4}(\gamma+2)^2.$$

The Ricci curvatures are given by

$$\rho_{11} = -\alpha^2-2-2\gamma-\frac{\gamma^2}{2}, \quad \rho_{22} = -\alpha^2-2+\frac{\gamma^2}{2}, \quad \rho_{33} = 2-\frac{\gamma^2}{2}, \quad \rho_{23} = -\alpha\gamma.$$

The natural-reducibility obstruction U is given by

$$U(e_1,e_2) = -\frac{1}{2}(\alpha e_2+\gamma e_3), \quad U(e_1,e_3) = -e_2,$$

$$U(e_2,e_2) = \alpha e_1, \quad U(e_2,e_3) = \frac{1}{2}(\gamma+2)e_1.$$

The Lie algebra $\mathfrak{g}$ is classified by the Milnor's invariant $\mathcal{D} = -8\gamma/\alpha^2$.

By using this table, the φ-Ricci tensor field is computed as

$$\rho^{\varphi}_{11} = \rho^{\varphi}_{22} = H = \frac{1}{4}(\gamma^2-4\gamma-12)-\alpha^2, \quad \rho^{\varphi}_{31} = 0, \quad \rho^{\varphi}_{32} = -\frac{1}{2}\alpha\gamma.$$

Hence G is $*$-Einstein. In particular, G is φ-Einstein if and only if $\gamma = 0$. As we saw in [32], G satisfies $\gamma = 0$ if and only if it is isometric to a Sasakian space form $\widetilde{\mathrm{SL}_2\mathbb{R}}$ of constant holomorphic sectional curvature $-3-\alpha^2 < -3$. Note that G with $\gamma = 0$ is *not* isomorphic to $\widetilde{\mathrm{SL}_2\mathbb{R}}$ as a Lie group.

$$\ell(e_1) = -\frac{1}{4}(\gamma+2)(3\gamma-2)e_1, \quad \ell(e_2) = \frac{1}{4}(\gamma+2)^2e_2. \tag{8.1}$$

Thus $\ell = 0$ if and only if $\gamma = -2$. In this case,

$$\lambda_1 = 1, \ \lambda_2 = -1.$$

The scalar curvature is $\mathrm{s} = -2\alpha^2 < 0$. $|Q\xi| = 2|\alpha| > 0$. This $G(\alpha, -2)$ has constant ξ-sectional curvature 0 and constant holomorphic sectional curvature $H = -\alpha^2$. This space is neither pseudo-symmetric and nor φ-Einstein.

Proposition 8.4. *Let $G = G(\alpha, \gamma)$ be a simply connected non-unimodular Lie group corresponding to $\mathfrak{g}(\alpha, \gamma)$ equipped with a left invariant contact Riemannian structure. Then $G(\alpha, \gamma)$ is an M_ℓ-manifold if and only if $\gamma = -2$.*

Remark 8.2. In our previous works [9], [10], [13], we studied pseudo-symmetry of contact 3-manifolds. In particular it is shown that a non-unimodular Lie group G is pseudo-symmetric if and only if $\gamma = 0$ [13]. In [20], 3-dimensional pseudo-symmetric Lie groups are investigated. In [8], 3-dimensional pseudo-symmetric real hypersurfaces in complex space forms are investigated.

The Lie group $G(\alpha, 0)$ is characterized as follows.

Proposition 8.5. *Let $G = G(\alpha, \gamma)$ be a simply connected non-unimodular Lie group corresponding to $\mathfrak{g}(\alpha, \gamma)$ equipped with a left invariant contact Riemannian structure. Then the following three conditions are mutually equivalent*:

- *G satisfies $\gamma = 0$.*
- *G is Sasakian. In this case, G is a Sasakian space form of constant holomorphic sectional curvature $-3 - \alpha^2 < -3$.*
- *G is pseudo-symmetric, that is, at least two of principal Ricci curvatures coincide.*
- *G is φ-Einstein.*

According to [5], a contact Riemannian 3-manifold is said to be

- *strongly locally φ-symmetric* if its characteristic reflections are isometric.
- *locally φ-symmetric* if
$$g((\nabla_X R)(Y, Z)V, W) = 0$$
for all X, Y, Z, V, $W \perp \xi$.

One can see that every strongly locally φ-symmetric contact Riemannian 3-manifold is locally φ-symmetric. Strong local φ-symmetry under local homogeneity is characterized as follows:

Proposition 8.6 ([5])**.** *A locally homogeneous contact Riemannian 3-manifold is strongly locally φ-symmetric if and only if its Reeb vector field is an eigenvector field of the Ricci operator.*

Corollary 8.7. *Let M be a locally homogeneous contact Riemannian 3-manifold. Then the following conditions are mutually erquivalent*:

- *ξ is an eigenvector field of Q.*
- *M is a generalized (κ, μ, ν)-space.*
- *M is strongly locally φ-symmetric.*

Every 3-dimensional unimodular Lie group equipped with left invariant contact Riemannian structure is strongly locally φ-symmetric. On the other hand, non-unimodular Lie groups, the following result was obtained.

Theorem 8.8 ([5])**.** *Let $G(\alpha, \gamma)$ be a 3-dimensional non-unimodular Lie group equipped with a left invariant contact Riemannian structure as before, then*

- *if $\gamma < 0$, none of $G(\alpha, \gamma)$ is locally φ-symmetric,*
- *if $\gamma = 0$, then $G(\alpha, \gamma)$ is a Sasakian locally φ-symmetric,*
- *if $\gamma > 0$, then none of $G(\alpha, \gamma)$ is strongly φ-symmetric, but there exists one which is locally φ-symmetric. The Lie group $G(\alpha, \gamma)$ is locally φ-symmtric only when $\gamma = 2$.*

8.3

Ghosh and Sharma [15] introduced the notion of *Jacobi (κ, μ)-contact space* as a generalization of a contact (κ, μ)-space and K-contact manifold.

Definition 8.3. A contact Riemannian manifold $(M, \varphi, \xi, \eta, g)$ is said to be a *Jacobi (κ, μ)-contact space* if M satisfies

$$\ell = -\kappa\varphi^2 + \mu\, h \tag{8.2}$$

for real constants κ and μ.

One can see that every contact (κ, μ)-space is a Jacobi (κ, μ)-contact space. Indeed in the (κ, μ, ν)-contact condition (4.1) with constant κ, μ and $\nu = 0$, substitution $Y = \xi$ implies (8.2).

Here we give *new proper examples* of Jacobi (κ, μ)-contact 3-spaces.

Proposition 8.9. *Every 3-dimensional non-unimodular Lie group with left invariant contact Riemannian structure is a Jacobi (κ, μ)-contact space. Except the Sasakian case $G(\alpha, 0)$, $G(\alpha, \gamma)$ is not a contact (κ, μ)-space.*

Proof. Let $G = G(\alpha, \gamma)$ be a non-unimodular Lie group equipped with left invariant contact Riemannian structure as before. Then the characteristic Jacobi operator ℓ is given by (8.1). On the other hand, for any constants κ and μ, the operator $-\kappa\varphi^2 + \mu\, h$ is computed as

$$(-\kappa\varphi^2 + \mu\, h)e_1 = (\kappa - \gamma\mu/2)e_1, \quad (-\kappa\varphi^2 + \mu\, h)e_2 = (\kappa + \gamma\mu/2)e_2. \tag{8.3}$$

First we assume that $\gamma \neq 0$. In this case, Comparing (8.3) with (8.1), we obtain that $G(\alpha, \gamma)$ is Jacobi (κ, μ)-contact space with

$$\kappa = -\frac{1}{4}(\gamma^2 - 4), \quad \mu = \gamma + 2.$$

Next suppose that $\gamma = 0$. In this case $G(\alpha, 0)$ is Sasakian and hence it is Jacobi (κ, μ)-contact space. □

In addition here we give a non-homogeneous example of Jacobi (κ, μ)-contact space.

Example 8.4. In [33], Perrone gave the following example of 3-dimensional weakly φ-symmetric space which is neither homogeneous nor strongly φ-symmetric. Let M be the open submanifold $\{(x,y,z) \in \mathbb{R}^3(x,y,z) \mid x \neq 0\}$ of Cartesian 3-space $\mathbb{R}^3$ together with a contact form $\eta = xydx + dz$. The Reeb vector field of this contact 3-manifold is $\xi = \partial/\partial z$. Take a global frame field

$$e_1 = -\frac{2}{x}\frac{\partial}{\partial y},\ e_2 = \frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z},\ e_3 = \xi$$

and define a Riemannian metric g by the condition $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. Moreover, define an endomorphism field φ by $\varphi e_1 = e_2,\ \varphi e_2 = -e_1$ and $\varphi\xi = 0$. Then (φ, ξ, η, g) is the associated almost contact metric structure of (M, η). The endomorphism field h satisfies $he_1 = e_1,\ he_2 = -e_2$. Hence M is non-Sasakian. Perrone showed that this contact Riemannian 3-manifold is non-homogeneous. We obtain

$$\ell = 4h.$$

Namely, $(M, \varphi, \xi, \eta, g)$ is a Jacobi (κ, μ)-contact manifold with $\kappa = 0$ and $\mu = -4$.

Proposition 8.10. *The example due to Perrone satisfies*

- *Jacobi (κ, μ)-contact manifold, but non contact (κ, μ)-space,*
- *neither pseudo-symmetric, nor H-contact,*
- *weakly φ-symmetric, but not strongly φ-symmetric,*
- *non-homogeneous.*

Remark 8.5. In a separate publication [12], contact Riemannian 3-manifolds whose characteristic Jacobi operator is invariant under the Reeb flows are investigated.

8.4

Here we introduce the following notion which is a generalization of Jacobi (κ, μ)-contact condition:

Definition 8.6. A contact Riemannian manifold M is said to be a *generalized Jacobi (κ, μ)-contact space* if

$$\ell = -\kappa\varphi^2 + \mu\, h$$

for some functions κ and μ.

We would like to point out the existence of generalizedJacobi (κ, μ)-contact spaces. To this end we recall critical metric conditions for contact 3-manifolds.

For a contact 3-manifold (M, η), we denote by $\mathcal{M}(\eta)$ the space of all Riemannian metrics associated to η. Since the volume element dv_g of an associated metric is $d\eta \wedge \eta/2$ (see *eg.* [4]), all the metrics in $\mathcal{M}(\eta)$ has same volume element.

Theorem 8.11 ([30]). *Let $(M, \varphi, \xi, \eta, g)$ be a compact contact Riemannian 3-manifold. Then the metric g is a critical point of the total scalar curvature functional on the space of all associated metric to η if and only if $\nabla_\xi h = 0$.*

On the other hand, Chern and Hamilton studied the following functional on $\mathcal{M}(\eta)$.

$$E_{\mathrm{CH}}(g) = \int_M \frac{1}{2}|\tau|^2\, dv_g.$$

The functional E_{CH} is called the *Chern-Hamilton energy.*

Theorem 8.12 ([7]). *An associated metric g of a compact contact 3-manifold (M, η) is a critical point of the Chern-Hamiltion energy if and only if $\nabla_\xi h = 2h\varphi$.*

It is known that every contact Riemannian manifold satisfies (*cf.* [4, p. 112]):

$$\ell = -h^2 - \varphi^2 + \varphi(\nabla_\xi h).$$

Now assume that M is non-Sasakian contact Riemannian 3-manifold and denote by λ the positive eigenvalue of h. Then one can see that

$$\ell = (\lambda^2 - 1)\varphi^2 + \varphi(\nabla_\xi h).$$

From this formula we obtain the following fact.

Proposition 8.13. *Let M be a contact Riemannian 3-manifold. If M satisfis $\nabla_\xi h = 0$, then the characteristic Jacobi operator satisfies*

$$\ell = -\kappa\, \varphi^2,$$

where κ is a smooth function. In case $\kappa = 1$, M is Sasakian.

On contact Riemannian 3-manifolds satisfying $Q\varphi = \varphi Q$, the equation $\nabla_\xi h = 0$ holds. The following example is due to Blair [3].

Example 8.7. Let $f(x, y)$ be a smooth non-constant function on $\mathbb{R}^3(x, y, z)$ bounded below by a positive constant c. Take a contact form $\eta = (dz - y dx)/2$ and consider a compatible metric

$$g = \frac{1}{4}\begin{pmatrix} (e^{zf} + (1+f^2)e^{-zf} - 2)f^2 + y^2 & (e^{zf}-1)/f & -y \\ (e^{zf}-1)/f & e^{zf} & 0 \\ -y & 0 & 1 \end{pmatrix}.$$

We equip an almost contact structure associated to (η, g) (for more detail, see [3] or [4, p. 226]). Then the resulting contact Riemannian 3-manifold satisfies $\nabla_\xi h = 0$ and $Q\varphi \neq \varphi Q$. The characteristic Jacobi operator satisfies $\ell = -\kappa\varphi^2$ with $\kappa = f(x, y)^2$.

Now let M be a contact Riemannian 3-manifold satisfying $\nabla_\xi h = 2h\varphi$. Then we have

$$\varphi\nabla_\xi h = 2h.$$

Hence the characteristic Jacobi operator satisfies

$$\ell = (\lambda^2 - 1)\varphi^2 + 2h.$$

Proposition 8.14. *Let M be a contact Riemannian 3-manifold satisfying $\nabla_\xi h = 2h\varphi$. Then M is a generalizedJacobi $(\kappa, 2)$-contact space.*

Let $\overline{M}^2(c)$ be a Riemannian 2-manifold of constant curvature $c = \pm 1$, then its unit tangent sphere bundle $M := T_1\overline{M}^2(c)$ equipped with standard contact Riemannian structure satisfies $\nabla_\xi h = 2h\varphi$ and constant eigenvalue $\lambda = 0$ for $c = 1$ and $\lambda = 2$ for $c = -1$ (see [4, p. 209]). In case $c = 1$, as is well known, M is Sasakian. In case $c = -1$, M is non-Sasakian and Jacobi (κ, μ)-contact space with $\kappa = -3$ and $\mu = 2$. More strongly, M is a contact $(-3, 2)$-space.

We conclude this paper with the following problem.

Problem 8.8. Classify all 3-dimensional Jacobi (κ, μ)-contact spaces.

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