Normal transport surfaces in Euclidean 4-space $\mathbb{E}^4$

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Abstract. In the present paper we study normal transport surfaces in four-dimensional Euclidean space $\mathbb{E}^4$ which are the generalization of surface offsets in $\mathbb{E}^3$. We find some results of normal transport surfaces in $\mathbb{E}^4$ of evolute and parallel type. Further, we give some examples of these type of surfaces.

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1 Introduction

The geometric modeling of free-form curves and surfaces is of central importance for sophisticated CAD/CAM systems. Apart from the pure construction of these curves and surfaces, the analysis of their quality is equally important in the design and manufacturing process. It is for example very important to test the convexity of a surface, to pinpoint inflection points, to visualize flat points and to visualize technical smoothness of surface [8].

The 3D offsets or parallel surfaces are very widely used in many applications. These include tool path generation for 3N machining. However, 3D offsets are particularly important and useful as pre-process modifications to CAD geometry. Offset techniques for surfaces has been extensively studied by Mechawa ([12]) and Pham ([14]). Generally offsets of 3D models are achieved by first offsetting all surfaces of the model and then trimming and extending these offsets to reconstruct a closed 3D model ([4], [5]).

Focal surfaces are known in the field of line congruences. Line congruences have been introduced in the field of visualization by Hagen and Pottmann (see, [10]). Focal surfaces are also used as a surface interrogation tool to analyses the "quality" of the surface before further processing of the surface, for example in a NC-milling operation (see [8]). The generalized focal surfaces are related to hedgehog diagrams. Instead of drawing surface normals proportional to a surface value, only the point on the surface normal proportional to the function is drawing. The loci of all these points is the generalized focal surface. This method was introduced by Hagen and Hahmann ([8], [9]) and is based on the concept of focal surface which are known from line geometry. The focal surfaces are the loci of all focal points of special congruence, the normal

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congruence. Recently the present authors considered parallel and focal surfaces and their curvature properties (see [13]).

The normal transport surface $\widetilde{M}$ of M are generalization of offset surfaces to 4-dimensional Euclidean space $\mathbb{E}^4$ [6]. Observe that, evolute surfaces and parallel type surfaces in $\mathbb{E}^4$ are the special type normal transport surfaces [11], [3], [6]. Parallel type surface are widely used in geometry and mathematical physics.

The paper organized as follows. In section 2, we briefly considered basic concepts of surfaces in Euclidean spaces. In section 3, we consider some known results about the surfaces with flat normal bundle. In the final section, we consider normal transport surfaces in $\mathbb{E}^4$. Further we give some examples of evolute and parallel type surfaces in $\mathbb{E}^4$.

2 Preliminaries

In the present section we recall definitions and results of [6]. Let M be a local surface in $\mathbb{E}^{n+2}$ given with the regular patch $x(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space $T_p(M)$ to M at an arbitrary point $p = x(u,v)$ of M is spanned by $\{x_u, x_v\}$. Further, the coefficients of the first fundamental form of M are given by

$$g_{11} = \langle x_u, x_u \rangle, g_{12} = \langle x_u, x_v \rangle, g_{22} = \langle x_v, x_v \rangle,$$

where $\langle , \rangle$ is the Euclidean inner product. Let us denote the area element by

$$ds^2 = \sum_{i,j=1}^{2} g_{ij} du^i du^j,$$

and let us choose n linearly independent, orthogonal unit normal vectors N_α, $\alpha = 1, 2, ..., n$ spanning the normal space $T_p^{\perp}M$ at some point $p = x(u,v)$. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^{n+2} = T_pM \oplus T_p^{\perp}M$, where $T_p^{\perp}M$ is the orthogonal component of T_pM in $\mathbb{E}^{n+2}$. Let $\widetilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^{n+2}$ then the Gauss equation of the surface M is given by

$$x_{u^i u^j} = \widetilde{\nabla}_{x_{u^i}} x_{u^j} = \sum_{k=1}^{2} \Gamma_{ij}^k x_{u^k} + \sum_{\alpha=1}^{n} c_{ij}^{\alpha} N_\alpha,$$

where

$$c_{ij}^{\alpha} = \langle x_{u^i u^j}, N_\alpha \rangle ; \; c_{ij}^{\alpha} = c_{ji}^{\alpha},$$

are the coefficients of the second fundamental form and

$$\Gamma_{ij}^k = \sum_{l=1}^{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l} \right),$$

are the Christoffel symbols corresponding to $x(u,v)$.

Further, the Weingarten equation of the surface M is given by

$$(N_\alpha)_{u^i} = \widetilde{\nabla}_{x_{u^i}} N_\alpha = -\sum_{k=1}^{2} c_\alpha^{ik} x_{u^k} + \sum_{\beta=1}^{n} T_i^{\alpha\beta} N_\beta, \tag{2.1}$$

where

$$c_\alpha^{ik} = \sum_{j=1}^{2} c_{ij}^\alpha g^{jk};\ c_\alpha^{ik} = c_\alpha^{ki},$$

are the Weingarten forms of M with respect to some unit normal vector N_α and

$$T_i^{\alpha\beta} = \langle (N_\alpha)_{u^i}, N_\beta \rangle\, ; T_i^{\alpha\beta} = -T_i^{\beta\alpha}, i = 1, 2,$$

are the torsion coefficients with $\alpha, \beta = 1, ..., n$ and

$$\left(g^{ij}\right)_{i,j=1,2} = \frac{1}{g}\begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{bmatrix},$$

where $g = g_{11}g_{22} - g_{12}^2$.

The Gaussian curvature of the surface M is defined by

$$K = \sum_{\alpha=1}^{n} K_\alpha, \quad K_\alpha = \frac{c_{11}^\alpha c_{22}^\alpha - (c_{12}^\alpha)^2}{g}.$$

where K_α is the α^{th} Gaussian curvature of the surface M. The Gaussian curvature vanishes identically for so-called flat surface. Observe that $K_\alpha = c_\alpha^{11} c_\alpha^{22} - (c_\alpha^{12})^2$.

The mean curvature vector field $\overrightarrow{H}$ of the surface M is defined by $\overrightarrow{H} = \sum_{\alpha=1}^{n} H_\alpha N_\alpha$, where

$$H_\alpha = \frac{1}{2}\sum_{i,j=1}^{2} g^{ij} c_{ij}^\alpha = \frac{g_{22}c_{11}^\alpha + g_{11}c_{22}^\alpha - 2g_{12}c_{12}^\alpha}{2g},$$

is the α^{th} mean curvature of the surface M with respect to the unit normal vector N_α. The mean curvature H of M is defined by $H = \left\|\overrightarrow{H}\right\|$. The mean curvature (vector) vanishes identically for the so-called minimal surface. Observe that

$$H_\alpha = \frac{c_\alpha^{11} + c_\alpha^{22}}{2}. \tag{2.2}$$

The curvature tensor of the normal bundle NM of the surface M is defined by

$$\begin{aligned} S_{ij}^{\alpha\beta} &= \left(T_i^{\alpha\beta}\right)_{u^j} - \left(T_j^{\alpha\beta}\right)_{u^i} + \sum_{\sigma=1}^{n}\left(T_i^{\alpha\sigma}T_j^{\sigma\beta} - T_j^{\alpha\sigma}T_i^{\sigma\beta}\right) \\ &= \sum_{m,n=1}^{2}\left(c_{1m}^\alpha c_{n2}^\beta - c_{2m}^\alpha c_{n1}^\beta\right) g^{mn}; 1 \leq \alpha, \beta \leq n. \end{aligned}$$

The equality $S_N^{\alpha\beta} = \frac{1}{\sqrt{g}} S_{12}^{\alpha\beta}$, is called the normal sectional curvature with respect to the plane $\Pi = span\{x_u, x_v\}$. For the case $n = 2$, the scalar curvature of its normal bundle is defined as

$$K_N = S_N^{12} = \frac{1}{\sqrt{g}} S_{12}^{12},$$

which is also called normal curvature of the surface M in $\mathbb{E}^4$. Observe that

$$K_N = \frac{1}{\sqrt{g}}\left(\left(T_2^{12}\right)_u - \left(T_1^{12}\right)_v\right). \tag{2.3}$$

3 Known results

Let M be a local surface in $\mathbb{E}^{n+2}$ given with the surface patch $x(u,v):(u,v)\in D\subset\mathbb{E}^2$. The mean curvature vector $\overrightarrow{H}$ is called parallel in the normal bundle if and only if

$$(H_\alpha)_u^\perp=0,(H_\alpha)_v^\perp=0,$$

or equivalently

$$(H_\alpha)_{u^i}=\sum_{\beta=1}^{n}H_\beta T_i^{\alpha\beta}. \tag{3.1}$$

for all $i=1,2$, $\alpha=1,...,n$ with respect to an arbitrary orthonormal frame $N_1,...N_n$ [6].

Proposition 3.1. *[6] The mean curvature vector $\overrightarrow{H}$ is called parallel in the normal bundle if and only if the squared mean curvature $\left\|\overrightarrow{H}\right\|^2$ of M is a constant function.*

Proof. Suppose $H_\alpha\neq 0$ and the mean curvature vector $\overrightarrow{H}$ is parallel in the normal bundle. Multiplying the first order differential equation (3.1) by H_α gives

$$H_\alpha\left(H_\alpha\right)_{u^i}=\sum_{\beta=1}^{n}H_\alpha H_\beta T_i^{\alpha\beta}=0,$$

for all $\alpha=1,...,n$. Summing over α shows

$$\frac{1}{2}\frac{\partial}{\partial u^i}\left\|\overrightarrow{H}\right\|^2=\sum_{\alpha=1}^{n}H_\alpha\left(H_\alpha\right)_{u^i}=\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}H_\alpha H_\beta T_i^{\alpha\beta}=0,$$

where the right hand side vanishes automatically due to the skew-symmetric of the torsion coefficients [6]. Thus, one get

$$\left\|\overrightarrow{H}\right\|^2=\sum_{\alpha=1}^{n}H_\alpha^2=const.$$

The converse statement of the proposition is trivial. □

Definition 3.1. A local surface of $\mathbb{E}^{n+2}$ is said to have flat normal bundle if and only if the orthonormal frame $N_1,...N_n$ of M is of torsion free.

Remark 3.2. The existence of flat normal bundle of M is equivalent to say that normal curvature K_N of M vanishes identically.

The following classification result due to Chen from [2].

Theorem 3.2. *Let M be an immersed surface in $\mathbb{E}^{n+2}$. If $\overrightarrow{H}\neq 0$ is parallel in the normal bundle then either M is a minimal surface of a hypersphere of $\mathbb{E}^{n+2}$, or it has flat normal bundle.*

4 Normal transport surfaces in $\mathbb{E}^4$

Let M and $\widetilde{M}$ be two smooth surfaces in Euclidean 4-space $\mathbb{E}^4$ and let $\varphi : M \to \widetilde{M}$ be a diffeomorphism. Then the surface $\widetilde{M}$ enveloping family of normal 2-planes to M is the normal transport of M in $\mathbb{E}^4$ [6]. Furthermore, let $\overrightarrow{x}$ be a position (radius) vector of $p \in M$, and $\widetilde{x}$ be the position (radius) vector of the point $\varphi(p) \in \widetilde{M}$. Then the mapping $\varphi : M \to \widetilde{M}$ has the form

$$\widetilde{x} = x + \overrightarrow{w}, \quad \overrightarrow{w} \in T_p^{\perp} M.$$

where, $\overrightarrow{p\varphi(p)} = \overrightarrow{w}(p)$, $\overrightarrow{w}(p) \in T_p^{\perp} M$ is the normal vector to M. For the case

$$\overrightarrow{w}(p) = \sum_{i=1}^{2} f_i(u,v) N_i(u,v),$$

the normal transport surface $\widetilde{M}$ of M given by

$$\widetilde{M} : \widetilde{x}(u,v) = x(u,v) + \sum_{i=1}^{2} f_i(u,v) N_i(u,v), \tag{4.1}$$

where $f_i \ \ (i = 1, 2)$ are offset functions and $N_1, N_2 \in T_p^{\perp} M$ [6].

The tangent space to $\widetilde{M}$ at an arbitrary point $p = \widetilde{x}(u,v)$ of $\widetilde{M}$ is spanned by

$$\begin{aligned} \widetilde{x}_u &= x_u + f_1 \left(N_1\right)_u + f_2 \left(N_2\right)_u + (f_1)_u N_1 + (f_2)_u N_2, \\ \widetilde{x}_v &= x_v + f_1 \left(N_1\right)_v + f_2 \left(N_2\right)_v + (f_1)_v N_1 + (f_2)_v N_2. \end{aligned} \tag{4.2}$$

Further, using the Weingarten equation (2.1) we get

$$\begin{aligned} \left(N_1\right)_u &= -\left(c_1^{11} x_u + c_1^{12} x_v\right) + T_1^{12} N_2 \\ \left(N_2\right)_u &= -\left(c_2^{11} x_u + c_2^{12} x_v\right) - T_1^{12} N_1 \\ \left(N_1\right)_v &= -\left(c_1^{21} x_u + c_1^{22} x_v\right) + T_2^{12} N_2 \\ \left(N_2\right)_v &= -\left(c_2^{21} x_u + c_2^{22} x_v\right) - T_2^{12} N_2. \end{aligned} \tag{4.3}$$

So, substituting (4.3) into (4.2) we get

$$\begin{aligned} \widetilde{x}_u = &\left(1 - f_1 c_1^{11} - f_2 c_2^{11}\right) x_u - \left(f_1 c_1^{12} + f_2 c_2^{12}\right) x_v \\ &+ \left((f_1)_u - f_2 T_1^{12}\right) N_1 + \left((f_2)_u + f_1 T_1^{12}\right) N_2, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \widetilde{x}_v = &-\left(f_1 c_1^{21} + f_2 c_2^{21}\right) x_u + \left(1 - f_1 c_1^{22} - f_2 c_2^{22}\right) x_v \\ &+ \left((f_1)_v - f_2 T_2^{12}\right) N_1 + \left((f_2)_v + f_1 T_2^{12}\right) N_2. \end{aligned} \tag{4.5}$$

The normal transport surfaces in 3-dimensional Euclidean space $\mathbb{E}^3$ have the parametrization

$$\widetilde{M} : \widetilde{x}(u,v) = x(u,v) + F(u,v) \ N(u,v),$$

where $N(u,v) \in T_p^{\perp} M$ and F is a real valued function in the parameter (u,v). In fact, these surfaces are known as surface offsets in $\mathbb{E}^3$ and F is its offset function [7].

If the offset function depends on the principal curvatures k_1 and k_2 of M then one can choose the variable offset function as;

1. $F = k_1 k_2$, Gaussian curvature,
2. $F = \frac{1}{2}(k_1 + k_2)$, mean curvature,
3. $F = k_1^2 + k_2^2$, energy functional,
4. $F = |k_1| + |k_2|$, absolute functional,
5. $F = k_i, 1 \leq i \leq 2$, principal curvature,
6. $F = \frac{1}{k_i}$, focal points,
7. $F = const.$, parallel surface.

The different offset functions listed above can now be used the interrogate and visualize of the surfaces (see [13]). Using different offset functions, one can construct a one-parameter family of various normal transport surfaces from a given surface of 4-dimensional Euclidean space $\mathbb{E}^4$.

In the following definition we construct some special normal transport surfaces in $\mathbb{E}^4$ which are the generalization of some generalized focal surfaces give before.

Definition 4.1. i) The normal transport surface $\widetilde{M}_H$ given with the parametrization

$$\widetilde{M}_H : \widetilde{x}(u,v) = x(u,v) + H_1(u,v)\ N_1(u,v) + H_2(u,v)\ N_2(u,v),$$

is called normal transport surface of H-type, where $f_\alpha(u,v) = H_\alpha$ $(\alpha = 1,2)$ are the offset functions.

ii) The normal transport surface $\widetilde{M}_K$ given with the parametrization

$$\widetilde{M}_K : \widetilde{x}(u,v) = x(u,v) + K_1(u,v)\ N_1(u,v) + K_2(u,v)\ N_2(u,v),$$

is called normal transport surface of K-type, where $f_\alpha(u,v) = K_\alpha$ $(\alpha = 1,2)$ are the offset functions.

4.1 Parallel surfaces in $\mathbb{E}^4$

Definition 4.2. The normal transport surface $\widetilde{M}$ of M is called parallel surface of M in $\mathbb{E}^4$ if the equality

$$\langle \widetilde{x}_{u_i}, N_\alpha \rangle = 0,\ 1 \leq i, \alpha \leq 2, \tag{4.6}$$

holds for all $N_\alpha \in T_p^\perp M$ [6].

If the functions f_1 and f_2 are constant then it is easy to see that $\widetilde{M}$ is a parallel surface of M and vice versa, at least if the surfaces are immersed in $\mathbb{E}^3$. The parallelity of $\widetilde{M}$ in $\mathbb{E}^4$ depends on the normal curvature K_N of M [6]. Parallel type surface are widely used in geometry and mathematical physics.

Let $\widetilde{M}$ be a parallel surface of M in $\mathbb{E}^4$. Then by use of (4.4) and (4.5) with (4.6) one can get

$$\begin{aligned} 0 &= \langle \widetilde{x}_u, N_1 \rangle = (f_1)_u - f_2 T_1^{12}, \\ 0 &= \langle \widetilde{x}_v, N_1 \rangle = (f_1)_v - f_2 T_2^{12}, \\ 0 &= \langle \widetilde{x}_u, N_2 \rangle = (f_2)_u + f_1 T_1^{12}, \\ 0 &= \langle \widetilde{x}_v, N_2 \rangle = (f_2)_v + f_1 T_2^{12}. \end{aligned} \tag{4.7}$$

Differentiating the first two equations and making use of the other equations shows us

$$(4.8)\qquad \begin{aligned} (f_1)_{uv} + f_1 T_2^{12} T_1^{12} - f_2\left(T_1^{12}\right)_v &= 0, \\ (f_1)_{vu} + f_1 T_1^{12} T_2^{12} - f_2\left(T_2^{12}\right)_u &= 0. \end{aligned}$$

Thus a computation of the left hand sides of (4.8) brings

$$-f_2\left\{\left(T_1^{12}\right)_v - \left(T_2^{12}\right)_u\right\} = 0.$$

So, by the use of (2.3) we can conclude that the normal curvature K_N of M vanishes identically [6]. Consequently, we obtain the following result of S. Fröhlich.

Theorem 4.1. *[6] The normal transport surface* $\widetilde{M}$ *of* M *is parallel if and only if* M *has flat normal bundle.*

We obtain the following result.

Corollary 4.2. *The normal transport surface* $\widetilde{M}$ *of* M *is parallel if and only if the squared sum of the offset functions is constant, i.e.,* $\sum_{i=1}^{2} f_i^2(u,v) = const.$

Proof. From the expressions in (4.7) we get

$$(4.9)\qquad \begin{aligned} (f_1)_u f_1 + (f_2)_u f_2 = 0, \\ (f_1)_v f_1 + (f_2)_v f_2 = 0. \end{aligned}$$

which completes the proof. □

We give the following example.

Example 4.3. The normal transport surface $\widetilde{M}$ of M is given with the patch

$$\widetilde{X}(u,v) = X(u,v) + r\cos u\ N_1(u,v) + r\sin u\ N_2(u,v),$$

is a parallel surface of M in $\mathbb{E}^4$.

Let M be a non-minimal local surface in $\mathbb{E}^4$ and $\widetilde{M}_H$ its normal transport surface. If $\widetilde{M}_H$ is a parallel surface of M in $\mathbb{E}^4$ then by Theorem 4.1, M has vanishing normal curvature. Furthermore, by the use of (4.9) we get

$$\begin{aligned} (H_1)_u H_1 + (H_2)_u H_2 &= 0, \\ (H_1)_v H_1 + (H_2)_v H_2 &= 0. \end{aligned}$$

Thus, $\left\|\overrightarrow{H}\right\|^2 = \sum_{\alpha=1}^{2} H_\alpha^2$ is a constant function. So, we conclude that the mean curvature vector $\overrightarrow{H}$ of M is parallel in the normal bundle. Thus, we have proved the following result.

Theorem 4.3. *Let* M *be a non-minimal local surface in* $\mathbb{E}^4$*. Then the normal transport surface* $\widetilde{M}_H$ *of* M *in* $\mathbb{E}^4$ *is parallel if and only if the mean curvature vector* $\overrightarrow{H}$ *of* M *is parallel in the normal bundle.*

Let M be a non-flat local surface in $\mathbb{E}^4$ and $\widetilde{M}_K$ its normal transport surface. If $\widetilde{M}_K$ is a parallel surface of M in $\mathbb{E}^4$ then by Theorem 4.1, $\widetilde{M}_K$ has vanishing normal curvature. Furthermore, by the use of (4.9) we get

$$\begin{aligned} (K_1)_u K_1 + (K_2)_u K_2 &= 0, \\ (K_1)_v K_1 + (K_2)_v K_2 &= 0. \end{aligned}$$

Thus, we conclude that $K = \sum_{\alpha=1}^{2} K_\alpha^2$ is a constant function, i.e., M has constant Gauss curvature. Thus, we have proved the following result.

Theorem 4.4. *Let M be a non-flat local surface in $\mathbb{E}^4$. Then the normal transport surface $\widetilde{M}_K$ of M in $\mathbb{E}^4$ is parallel if and only if the Gaussian curvature of M is a non-zero constant.*

4.2 Evolute surfaces in $\mathbb{E}^4$

Definition 4.4. The normal transport surface $\widetilde{M}$ of M is called evolute surface of M in $\mathbb{E}^4$ if the equality

$$\left\langle \widetilde{x}_{u_i}, x_{u_j} \right\rangle = 0,\ 1 \le i, j \le 2, \tag{4.10}$$

holds for all $x_{u_j} \in T_p M$.

Observe that, The tangent 2-planes at a point $p \in M$ and at the corresponding point $\varphi(p) \in \widetilde{M}$ are mutually orthogonal, and the vector $\overrightarrow{p\varphi(p)} = \overrightarrow{w}(p)$, $\overrightarrow{w}(p) \in T_p^{\perp} M$ is the normal vector to M [3].

Let $\widetilde{M}$ be a evolute surface of M in $\mathbb{E}^4$. Then by use of (4.4) with (4.10) we can get

$$\begin{aligned} 0 &= \langle \widetilde{x}_u, x_u \rangle = \left(1 - f_1 c_1^{11} - f_2 c_2^{11}\right) g_{11} - \left(f_1 c_1^{12} + f_2 c_2^{12}\right) g_{21}, \\ 0 &= \langle \widetilde{x}_u, x_v \rangle = \left(1 - f_1 c_1^{11} - f_2 c_2^{11}\right) g_{12} - \left(f_1 c_1^{12} + f_2 c_2^{12}\right) g_{22}, \\ 0 &= \langle \widetilde{x}_v, x_u \rangle = -\left(f_1 c_1^{12} + f_2 c_2^{12}\right) g_{11} + \left(1 - f_1 c_1^{22} - f_2 c_2^{22}\right) g_{21}, \\ 0 &= \langle \widetilde{x}_v, x_v \rangle = -\left(f_1 c_1^{12} + f_2 c_2^{12}\right) g_{12} + \left(1 - f_1 c_1^{22} - f_2 c_2^{22}\right) g_{22}. \end{aligned} \tag{4.11}$$

From now on we assume that the surface patch $x(u, v)$ satisfies the metric condition $g_{12} = 0$. So the equations in (4.11) turn into

$$\begin{aligned} f_1 c_1^{11} + f_2 c_2^{11} &= 1, \\ f_1 c_1^{22} + f_2 c_2^{22} &= 1, \\ f_1 c_1^{12} + f_2 c_2^{12} &= 0. \end{aligned} \tag{4.12}$$

Consequently by the use of (4.12) with (2.2) we get

$$f_1 H_1 + f_2 H_2 = 1. \tag{4.13}$$

So, we obtain the following result.

Theorem 4.5. *Let M be local surface in $\mathbb{E}^4$ with $g_{12} = 0$. Then the normal transport surface $\widetilde{M}$ in $\mathbb{E}^4$ is evolute surface of M if and only if the first and second mean curvatures H_1, H_2 satisfies the condition (4.13).*

Corollary 4.6. *Let M be local surface in $\mathbb{E}^4$ with $g_{12} = 0$. Then the normal transport surface $\widetilde{M}_H$ in $\mathbb{E}^4$ is evolute surface of M if and only if the mean curvature of M is equal to one.*

In [3] M. A. Cheshkova gave the following results.

Theorem 4.7. *[3] Let M be local surface in $\mathbb{E}^4$. If the normal transport surface $\widetilde{M}$ in $\mathbb{E}^4$ is evolute surface of M then M has flat normal bundle.*

Theorem 4.8. *[3] The minimal surfaces have no evolutes.*

Example 4.5. Let M is a translation surface $x(u, v) = \alpha(u) + \beta(v)$ in $\mathbb{E}^4$, then the translation curves $\alpha(u) = (\alpha_1(u), \alpha_2(u), 0, 0)$ and $\beta(v) = (0, 0, \beta_1(v), \beta_2(v))$ are plane curves of mutually orthogonal 2-planes. The surface $\widetilde{M}$ is a translation surface, and its translation curves $\widetilde{\alpha}(u)$, $\widetilde{\beta}(u)$ are the evolutes of the curves $\alpha(u)$, $\beta(u)$. If $u, v, \kappa_\alpha, \kappa_\beta$ and $\{t_\alpha, n_\alpha\}, \{t_\beta, n_\beta\}$ are the arc length, the curvature, and the Frenet frame of the curves $\alpha(u)$ and $\beta(v)$, correspondingly, then

$$\begin{aligned}
\widetilde{x}(u, v) &= \alpha(u) + \frac{1}{\kappa_\alpha} n_\alpha(u) + \beta(v) + \frac{1}{\kappa_\beta} n_\beta(v) \\
&= \alpha(u) + \beta(v) + \frac{1}{\kappa_\alpha} n_\alpha(u) + \frac{1}{\kappa_\beta} n_\beta(v) \\
&= x(u, v) + \frac{1}{\kappa_\alpha} n_\alpha(u) + \frac{1}{\kappa_\beta} n_\beta(v).
\end{aligned}$$

The tangent space to $\widetilde{M}$ at an arbitrary point $p = \widetilde{x}(u, v)$ of $\widetilde{M}$ is spanned by

$$\widetilde{x}_u = \left(\frac{1}{\kappa_\alpha}\right)' n_\alpha(u), \quad \widetilde{x}_v = \left(\frac{1}{\kappa_\beta}\right)' n_\beta(v).$$

Hence the normal transport surface $\widetilde{M}$ of M satisfies the equality $\langle \widetilde{x}_{u_i}, x_{u_j} \rangle = 0$, and $\widetilde{M}$ is the evolute of M [3].

5 An application

Rotation surfaces were studied in [15] by Vranceanu as surfaces in $\mathbb{E}^4$ which are defined by the following parametrization

$$M : x(u, v) = (r(v) \cos v \cos u, r(v) \cos v \sin u, r(v) \sin v \cos u, r(v) \sin v \sin u) \tag{5.1}$$

where $r(v)$ is a real valued non-zero function.

We have the following result.

Theorem 5.1. *Let $\widetilde{M}$ be a normal transport surface of the Vranceanu surface M given with the parametrization (4.1). If $\widetilde{M}$ is an evolute surface of M in $\mathbb{E}^4$ then*

$$\widetilde{M} : \widetilde{x}(u,v) = \lambda\mu e^{\mu v}(-\sin v\cos u, -\sin v\sin u, \cos v\cos u, \cos v\sin u),$$

where λ and μ are non zero constants.

Proof. Let M be a Vranceanu surfaces given with the parametrization (5.1). We choose a moving frame $\{X_u, X_v, N_1, N_2\}$ such that X_u, X_v are tangent to M and N_1, N_2 normal to M as given the following (see, [16]):

$$\begin{aligned} x_u &= r(-\cos v\sin u, \cos v\cos u, -\sin v\sin u, \sin v\cos u), \\ x_v &= (B(v)\cos u, B(v)\sin u, C(v)\cos u, C(v)\sin u), \\ N_1 &= \frac{1}{A}(-C(v)\cos u, -C(v)\sin u, B(v)\cos u, B(v)\sin u), \\ N_2 &= (-\sin v\sin u, \sin v\cos u, \cos v\sin u, -\cos v\cos u), \end{aligned}$$

where

$$\begin{aligned} A(v) &= \sqrt{r^2(v) + (r'(v))^2}, \ \ B(v) = r'(v)\cos v - r(v)\sin v, \\ C(v) &= r'(v)\sin v + r(v)\cos v. \end{aligned}$$

Suppose that $\widetilde{M}$ is the normal transport surface of the Vranceanu surface M in $\mathbb{E}^4$ then we have

$$\begin{aligned} \langle \widetilde{x}_u, x_u \rangle &= r^2(v) - f_1\left(\frac{r^2(v)}{\sqrt{r^2(v) + (r'(v))^2}}\right), \\ \langle \widetilde{x}_u, x_v \rangle &= f_2 r(v), \\ \langle \widetilde{x}_v, x_u \rangle &= f_2 r(v), \\ \langle \widetilde{x}_v, x_v \rangle &= r^2(v) + (r'(v))^2 + f_1\left(\frac{r(v)r''(v) - 2(r'(v))^2 - r^2(v)}{\sqrt{r^2(v) + (r'(v))^2}}\right). \end{aligned} \tag{5.2}$$

Furthermore, if $\widetilde{M}$ is an evolute surface of the Vranceanu surface M in $\mathbb{E}^4$ then using (4.10) with (5.2) we obtain

$$\begin{aligned} f_2 &= 0, \\ f_1 &= \sqrt{r^2(v) + (r'(v))^2}. \end{aligned} \tag{5.3}$$

Moreover, from the first and fourth equations of (5.2) one can get

$$r(v)r''(v) - (r'(v))^2 = 0$$

which has a non-trivial solution

$$r(v) = \lambda e^{\mu v} \tag{5.4}$$

As a consequence of (5.3) with (5.4) we get the desired result. □

Remark 5.1. It is known that a Vranceanu surface given with $r(v) = \lambda e^{\mu v}$ is the flat surface which has vanishing normal curvature [1].

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