Immersions of constant mean curvature in hyperbolic space

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Abstract

This report contains a new approach to the topic of immersions of constant mean curvature $|H| > 1$ in $\mathbb{H}^3(-1)$, their *adjusted Gauss maps* (as harmonic maps in S^2) and their *adjusted frames* in $SU(2)$. We discuss correspondences between Euclidean CMC surfaces and their hyperbolic counterparts, while emphasizing the equivalence of the Weierstrass representations. We conclude with some remarks on foliations with cylinders of $\mathbb{H}^3$.

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1 Introduction

In this paper, we consider only Lawson cousins of the Euclidean CMC surfaces, that is, surfaces with constant mean curvature $|H| > 1$. To the case $|H| < 1$, consisting of solutions to the cosh-Gordon equation, the author has devoted a different study ([18]). Through another recent communication, the author has also given a unified approach of the two aforementioned cases through a generalized Weierstrass representation, which parallels (but does not identify with) the Aiyama-Akutagawa-Bryant representations existing for CMC surfaces of arbitrary positive mean curvature.

Several years ago, a method now referred to as DPW ([9]) was introduced for nonzero constant mean curvature surfaces (abbreviated as CMC) in $\mathbb{R}^3$, giving a characterization of these surfaces in terms of normalized (or meromorphic) potential, or alternatively, holomorphic potential as well as a method to construct all associate immersions based on loop group factorization. Our report does *not* contain a method in the spirit of DPW for CMC surfaces in hyperbolic space. Rather, it is a study of a Weierstrass-type representation using the spectral deformation method that [15] introduced, but from a different perspective and with a different scope.

Our study emphasizes the relationship of this representation with the one of CMC surfaces in $\mathbb{R}^3$. The essence of our investigation is that some Cartan theory can

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be performed for two different moving frames: the usual $SL(2,C)$ - valued frame F that corresponds to these surfaces in $\mathbb{H}^3$, as well as a $SU(2)$-valued frame $\widetilde{F}$ that corresponds to the usual orthonormal frame of their CMC cousins in $\mathbb{R}^3$. Accordingly, there exist two different Maurer-Cartan frames, $F^{-1}dF$ and $\widetilde{F}^{-1}d\widetilde{F}$. We found that the harmonic map into S^2 that has $\widetilde{F}$ as a lift is the only relevant harmonic map: starting from such a map, one constructs the frame $\widetilde{F}$ and correspondingly, a family of associated CMC surfaces in $\mathbb{R}^3$ on one hand, and on the other hand, a much larger set of associated families of CMC surfaces in $\mathbb{H}^3$ with mean curvatures H taking all the values in $[-\infty,-1)\bigcup(1,\infty]$. The Maurer-Cartan forms corresponding to these families form a loop of connections, indexed over a specific parameter that depends on the mean curvature. The null-curvature condition of this new loop of connections (see our Theorem 5) represents an integrability condition that generates CMC immersions of various mean curvatures $|H|>1$.

Our spectral deformations, potentials and DPW method are different from [13]. We study the normalized potential and show that it basically reduces to a 'Weierstrass pair': the Hopf differential, together with the holomorphic part of the metric conformal factor. We here analyze how the normalized potential is used in order to generate the *adjusted* SU(2) *frame*, the usual $\mathrm{SL}(2,\mathbb{C})$ frame, their two Maurer-Cartan forms, and the CMC immersion.

Among the advantages of this particular approach are the simple form of the normalized potential compared to other representations one may use, as well as the fact that one does not have to keep track of the monodromy representation. Also, although the factorizations are not explicit, the resulting frames and immersions are.

We would also like to point out the following:

Even for the spectral deformation from [13], the usual frame F is *not* r-unitary. Due to the off-diagonal entries of the Lax matrices, containing $H-1$ and $H+1$, the usual Maurer-Cartan form $F^{-1}\cdot dF$ is anything but su(2)-valued, even when the spectral parameter is in S^1.

The r-unitarization that is aimed in [13] takes place in our context for a different reason (see formula (4.9)).

2 Integrable Systems of Constant Mean Curvature Surfaces in Hyperbolic 3-Space $\mathbb{H}^3$

Let us consider the 4-dimensional Lorentzian space

$$\mathbb{R}^{3,1}=\{(x^0,x^1,x^2,x^3)|ds^2=-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2\}.$$

The hyperbolic 3-space is the spacelike 3-manifold

$$\mathbb{H}^3=\mathbb{H}^3(-1)=\{x\in\mathbb{R}^{3,1}|<x,x>=-1,x^0>0\}$$

of constant sectional curvature -1.

Note that the following correspondence

$$x=(x^0,x^1,x^2,x^3)\longmapsto x=\begin{pmatrix} x^0+x^3 & x^1+ix^2 \\ x^1-ix^2 & x^0-x^3\end{pmatrix}$$

provides an identification between $\mathbb{R}^{3,1}$ and the space of 2×2 Hermitian matrices. The complex Lie group $\mathrm{SL}(2,\mathbb{C})$ acts isometrically and transitively on $\mathbb{H}^3(-1)$ by

$$\begin{array}{rcl} \mathrm{SL}(2,\mathbb{C}) \times \mathbb{H}^3(-1) & \stackrel{\cdot}{\longrightarrow} & \mathbb{H}^3(-1) \\ (g,h) & \longmapsto & g \cdot h = ghg^*, \end{array}$$

where $g^* = \bar{g}^t$. Thus, $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$.

Let M be a simply connected Riemann surface and $f : M \longrightarrow \mathbb{H}^3$ an immersion. Consider $(e^0 = f, e^1, e^2, e^3)$ the local orthonormal frame of the immersion f. Then we have

$$\begin{aligned} de^0 &= df = \omega_i e^i, \ i = 1,2, \\ de^j &= \omega_j e^0 + \omega_i^j e^i, \ i = 1,2,3, \end{aligned}$$

where $\omega_j^i = -\omega_i^j$ and $\omega_i^i = 0$.

For the adapted frame of the immersion f, Cartan's structure equations can be written on short as

$$\begin{aligned} d\omega_i &= \omega_i^j \wedge \omega_j \\ d\omega_j^i + \omega_k^i \wedge \omega_j^k + \omega_i \wedge \omega_j &= 0. \end{aligned}$$

Let $\sigma_i, \ i = 0,1,2,3$, be the following matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices are called Pauli spin matrices.

By the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{H}^3$, there exists a map F from an open set in M to $\mathrm{SL}(2,\mathbb{C})$ such that

$$F(\sigma_i) = F\sigma_i F^* = e^i, \ i = 0,1,2,3.$$

This map represents the local moving frame (or usual 'orthogonal' frame) associated to the immersion f. Let $\Omega := F^{-1}dF \in \mathfrak{sl}(2,\mathbb{C})$. The Gauss and Codazzi equations are equivalent to

$$d\Omega + \frac{1}{2}[\Omega \wedge \Omega] = 0,$$

which is the null curvature condition of the Maurer-Cartan (connection) form Ω.

It is well known [3, for example] that every surface with constant mean curvature (in $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$) admits conformal (isothermal) coordinates, $z = x + iy$, so that

$$I = ds^2 = df \otimes df = e^{2u} dz \otimes d\bar{z}.$$

So, we may rewrite $f : D \longrightarrow \mathbb{H}^3$ (by abuse of notation), with $D \in \mathbb{C}$ open, simply connected, and containing the origin.

Thus, $< f_z, f_z >=< f_{\bar{z}}, f_{\bar{z}} >= 0$, $< f_z, f_{\bar{z}} >= \frac{1}{2}e^{2u}$, where $f_z = \dfrac{1}{2}(f_x - if_y)$, $f_{\bar{z}} = \dfrac{1}{2}(f_x + if_y)$. We also have

$$< f_z, N >=< f_{\bar{z}}, N >= 0, \ < N, N >= 1.$$

The form $Qdz^2 :=< f_{zz}, N > dz^2$ is called Hopf differential.

It is also well known [3] that an immersion f has constant mean curvature if and only if the Hopf differential is holomorphic. The second fundamental form is defined as

$$II = - < df, dN >= ldx^2 + 2mdxdy + ndy^2.$$

Then

$$\begin{aligned} < f_{zz}, N > &= \frac{1}{4}(l - n - 2im) = Q, \\ < f_{z\bar{z}}, N > &= \frac{1}{4}(l + n) = \frac{1}{2}He^{2u}, \end{aligned}$$

where $N \equiv e_3$ represents the usual Gauss map (unit normal vector field on M). The Maurer-Cartan form Ω can be written as

$$\Omega = Adz + Bd\bar{z},$$

where

$$A = \begin{pmatrix} \frac{1}{2}u_z & \frac{1}{2}e^u(1+H) \\ -e^{-u}Q & -\frac{1}{2}u_z \end{pmatrix}, \ B = \begin{pmatrix} -\frac{1}{2}u_{\bar{z}} & e^{-u}\overline{Q} \\ \frac{1}{2}e^u(1-H) & \frac{1}{2}u_{\bar{z}} \end{pmatrix}.$$

The moving frame F satisfies the following Lax equations

$$\left\{ \begin{array}{lcl} F_z &=& FA \\ F_{\bar{z}} &=& FB \end{array} \right. . \tag{2.1}$$

The compatibility condition $F_{z\bar{z}} = F_{\bar{z}z}$ gives

$$A_{\bar{z}} - B_z - [A, B] = 0, \tag{2.2}$$

which can be written as

$$\left\{ \begin{array}{c} u_{z\bar{z}} - \dfrac{e^{2u}}{4}(1 - H^2) - e^{-2u}Q\overline{Q} = 0, \\ Q_{\bar{z}} = 0. \end{array} \right. \tag{2.3}$$

For the end of this paragraph, we would like to recall the following important result:

Theorem 1. *(Particular Case of Lawson Correspondence) There is a bijective correspondence between the space of isometric immersions of constant mean curvature $H > 0$ in $\mathbb{R}^3$ and the space of isometric immersions of constant mean curvature $\sqrt{H^2+1}$ in $\mathbb{H}^3(-1)$.*

3 Spectral Deformations

Let us consider an arbitrary immersion f, of metric conformal factor $e^{u(z,\bar{z})}$, constant mean curvature H and Hopf differential Qdz^2. We start with the desire to find isometric or quasi-isometric spectral deformations of this immersion. We are mainly interested in new surfaces characterized by the triple $(\widetilde{u}, \widetilde{H}, \widetilde{Q})$, such that they satisfy the same Gauss and Codazzi equations as the initial (u, H, Q). Here, it should be remarked that our approach is different from the one in [3] and [15]. More precisely, these references consider a spectral transformation given by the complex non-zero parameter λ such that

$$(1+H) \to \lambda(1+H),\ (1-H) \to \lambda^{-1}(1-H)$$

(see, for example, [15, formulas 23-25]); this accordingly changes the matrices A and B of the Lax system.

H and u are both real-valued, while Q is complex-valued. In our opinion, the following two spectral deformations have more geometric meaning:

A). A *positive real parameter* s is introduced in the second term of the Gauss-Codazzi equation (2.3) via $(1+H) \to s(1+H)$ and $(1-H) \to s^{-1}(1-H)$, so that the new $(\widetilde{u}, \widetilde{H}, \widetilde{Q})$ satisfy the same Gauss-Codazzi equation.

B). A *complex parameter of modulus one*, $\theta = e^{it}$, is introduced in the third term of the Gauss-Codazzi equation, via $Q \to \theta^{-2} Q$, so that the Gauss-Codazzi equation does not change.

A). **s-Spectral Deformations. Proper Deformations**

The name of spectral parameter comes from mathematical physics, where it was interpreted as a spectral parameter in a corresponding linear problem.

Definition 1. *We call s-spectral deformation of the constant mean curvature immersion f the effect (on the surface) of introducing the positive parameter s via $(1+H) \to s(1+H)$ and $(1-H) \to s^{-1}(1-H)$, respectively.*

This effect depends on the geometric interpretation we give this transformation, that is:

$$k(1+H^s) := s(1+H) \tag{3.4}$$

and

$$k(1-H^s) := s^{-1}(1-H), \tag{3.5}$$

where k is a nonzero real number.

As a direct consequence of equations above, we obtain :

$$k = \frac{s(1+H) + s^{-1}(1-H)}{2} \tag{3.6}$$

$$H^s = \frac{s(1+H) - s^{-1}(1-H)}{s(1+H) + s^{-1}(1-H)} \tag{3.7}$$

Theorem 2. *For any fixed positive parameter s, the s-spectral transformation*

$$(1+H) \to s(1+H)(= k(1+H^s)) \tag{3.8}$$

and

$$(1-H) \to s^{-1}(1-H)(= k(1-H^s)), \tag{3.9}$$

deforms an immersion f of metric $e^{2u}dzd\bar{z}$, Hopf differential Qdz^2 and mean curvature H, into a conformal immersion, f^s, of metric $e^{2u^s}dzd\bar{z} := k^2 \cdot e^{2u}dzd\bar{z}$, Hopf differential $Q^s dz^2 := k \cdot Qdz^2$ and mean curvature H^s, as defined by the formulas above.

Proof. Note that the Gauss-Codazzi equation satisfied by (u, H, Q) is equivalent to the following Gauss-Codazzi equation satisfied by (u^s, H^s, Q^s):

$$\begin{cases} u^s_{z\bar{z}} - \dfrac{e^{2u^s}}{4}[1-(H^s)^2] - e^{-2u^s}Q^s\overline{Q}^s = 0, \\ Q^s{}_{\bar{z}} = 0, \end{cases}$$

where $u^s := u + \ln|k|$, $H^s = \dfrac{s(1+H) - s^{-1}(1-H)}{s(1+H) + s^{-1}(1-H)}$, and $Q^s = kQ$. □

This spectral deformation of f to f^s may be interpreted as a substitute for similarity transformations, since similarity does not exist in hyperbolic 3-space $\mathbb{H}^3(-1)$.

Remark 1. Any such s-spectral deformation is interesting in itself; it *rescales* both the metric I and the 2-form $|Q|^2 dz \cdot d\bar{z}$ (by multiplication with the same positive constant) and so the new surface looks similar to the first one, although the mean curvature changes.

We say that the immersion f^s obtained from f via the spectral deformations (3.5) and (3.6) is *strongly conformal* to f.

Definition 2. *The s-spectral deformation is called proper if $s \neq 1$ and it leaves the metric unmodified, that is, $k = 1$ or $k = -1$ (see Theorem 2).*

Theorem 3. *The only proper s-deformation of the immersion f with $k = 1$ corresponds to the value $s = s_0 = \frac{1-H}{1+H}$, where H represents the mean curvature of the initial immersion f and satisfies $|H| < 1$. This deformation yields the immersion f^{s_0} with the same metric $u^{s_0} = u$, opposite mean curvature $H^{s_0} = -H$ and same Hopf differential $Q^{s_0} = Q$.*

The only proper s-deformation of the immersion f with $k = -1$ corresponds to the unique value $s_0 = \frac{H-1}{H+1}$, where H represents the mean curvature of the initial immersion f and satisfies $|H| > 1$. This deformation yields the immersion f^{s_0} with the same metric $u^{s_0} = u$, opposite mean curvature $H^{s_0} = -H$ and opposite Hopf differential $Q^{s_0} = -Q$.

Observation : The deformation mentioned in the second paragraph ($H^{s_0} = -H$, $Q^{s_0} = -Q$) represents a parallel surface to the given one. The distance between parallel surfaces is given by $q_0 := -\frac{1}{2}\ln s_0$.

Proof. Assume that $k = -1$. If $|H| = 1$, remark that no deformation occurs. We will assume assume $|H| \neq 1$. A simple computation provides $H = \dfrac{1+s}{1-s}$, so $s = \dfrac{H-1}{H+1}$, and the parameter is uniquely given by the value of the initial mean curvature H. That gives $H^s = \dfrac{s+1}{s-1} = -H$. It is easy to see that $s > 0$ implies $|H| = \left|\dfrac{1+s}{1-s}\right| > 1$. At the same time, $Q^s = kQ = -Q$.

Similarly, for $k = 1$ one gets $s > 0$ implies $|H| = \left|\dfrac{1-s}{1+s}\right| < 1$; $H^s = \dfrac{s-1}{s+1} = -H$ and $Q^s = kQ = Q$.

□

Note that whenever s is not equal to 1, the deformation is proper iff $s = s_0 = \dfrac{|1-H|}{|1+H|}$, $H \neq -1, 1$.

In this work, we will use general s-deformations (strongly conformal deformations), and will specify those particular instances when deformations are proper (isometric).

B). **θ-Spectral Deformation:**

Definition 3. *We call θ-spectral deformation of the constant mean curvature immersion f the effect of introducing the $\mathbb{S}^1$-parameter $\theta = e^{it}$ such that the Hopf differential changes according to $Q \to \theta^{-2}Q$.*

The $\theta-$deformation does not change the metric or the mean curvature, only the Hopf differential. It gives the well-known family of *associated surfaces*.

4 The λ-Spectral Deformation

Let us consider a simply connected Riemann surface, immersed in $\mathbb{H}^3$. Let the immersion be f, of constant mean curvature H and Hopf differential Qdz^2.

Remark 2. It is easy to see that the two types of spectral deformations have different geometric effects on the surface. We will combine the two deformations, and introduce a parameter that covers both spectral deformations mentioned above.

Definition 4. *We define $\lambda = s \cdot \theta$, where $s > 0$, and $\theta = e^{it}$. We call λ generalized spectral parameter.*

Remark 3. For the case of an *isometric deformation* (that is a θ-deformation while $s = 1$, or a *theta*-deformation combined with a proper s-deformation), the mean curvature H and Hopf differential Q remain the same - up to an eventual change in sign. The corresponding λ parameter will then be of the following form:

$$\lambda = \begin{cases} e^{-2q+it}, & q = 0 \quad \textit{or} \quad q = \coth^{-1} H, \quad \text{if} \quad |H| > 1 \\ e^{-2q+it}, & q = 0 \quad \textit{or} \quad q = \tanh^{-1} H, \quad \text{if} \quad |H| < 1 \\ e^{it} & \qquad\qquad\qquad\qquad\qquad\quad\; \text{if} \quad |H| = 1 \end{cases} . \tag{4.10}$$

If $H \neq 0$, then the parameter λ is the same for the immersions of mean curvature H and $\dfrac{1}{H}$, respectively . Not only that these immersions correspond bijectively, but they have the same corresponding parameter s.

Also note the symmetry in terms of sign. Surfaces of mean curvature H and $-H$ respectively correspond to opposite values of q, and inverse values of s.

Definition 5. *By $\lambda(= s \cdot \theta)$-spectral deformation we mean the effect of performing both of the following deformations on the initial immersion f or mean curvature H:*
A). an s-deformation ($s > 0$),
B). a $\theta = e^{it}$-deformation.

Note that order does not matter: since s-deformations are independent from θ-deformations, they commute.

Case A). gives a *genuine* (and strongly conformal) surface deformation $f \longrightarrow f^s$ in general, as described in Theorem 3. The surface stays the same in just two cases: the trivial case $s = 1$ (identity) and the case of proper deformation ($s = s_0 = \frac{|1-H|}{|1+H|}$), both being isometries.

Case B). describes the associated family.

While performing a general λ-deformation, that is a θ-deformation and an s-deformation, keep in mind the changes described in Theorem 2. In terms of Lax matrices, we obtain:

$$A(s,\theta) = \begin{pmatrix} \frac{1}{2}u_z & \frac{s}{2}e^u(1+H) \\ -e^{-u}\theta^{-2}Q & -\frac{1}{2}u_z \end{pmatrix}, \tag{4.11}$$

$$B(s,\theta) = \begin{pmatrix} -\frac{1}{2}u_{\bar{z}} & e^{-u}\theta^2\overline{Q} \\ \frac{s^{-1}}{2}e^u(1-H) & \frac{1}{2}u_{\bar{z}} \end{pmatrix}. \tag{4.12}$$

Remark 4. For loop group reasons, we conjugate these matrices with the z-independent matrix

$$G = i\begin{pmatrix} 0 & \theta^{1/2} \\ \theta^{-1/2} & 0 \end{pmatrix}, \tag{4.13}$$

and obtain the matrices

$$A^\lambda = \begin{pmatrix} -\frac{1}{2}u_z & -\theta^{-1}\cdot e^{-u}Q \\ \theta^{-1}\cdot\frac{s}{2}e^u(1+H) & \frac{1}{2}u_z \end{pmatrix}, \tag{4.14}$$

$$B^\lambda = \begin{pmatrix} \frac{1}{2}u_{\bar{z}} & \theta\cdot\frac{s^{-1}}{2}e^u(1-H) \\ \theta\cdot e^{-u}\overline{Q} & -\frac{1}{2}u_{\bar{z}} \end{pmatrix}. \tag{4.15}$$

Note that these conjugated matrices will satisfy the Lax system and the compatibility condition associated to it.

Remark 5. While looking for the right type of spectral transformation, eventually an isometric one, instead of our deformation, one may have been tempted to perform the traditional one (see also [13]) : $Q \longrightarrow \lambda^{-1}Q$, with $\lambda \in C^*$, hoping to obtain a frame F with the property $F \cdot \overline{F(\bar{\lambda}^{-1})}^t = I$, which in particular would be unitary for λ in S^1. Note that this type a deformation does not lead to such a frame.

Actually, if we made such a choice, the off-diagonal terms that contain $H+1$ and $H-1$ would destroy the hope for a su(2)-valued Maurer-Cartan form!

The λ deformation we just introduced is convenient, in the sense that the Maurer-Cartan form becomes a su(2)-valued form for a specific real value c of the parameter s, and all values of θ in S^1.

Via our λ deformation, the frame F changes to F^λ (which can be considered fixed at a point $p \in M$). The Lax system

$$\begin{cases} F^\lambda_z &= F^\lambda A^\lambda \\ F^\lambda_{\bar{z}} &= F^\lambda B^\lambda \end{cases}$$

can be also written as

$$(F^\lambda)^{-1} dF^\lambda = \Omega^\lambda, \tag{4.16}$$

so that the Maurer-Cartan form Ω^λ writes

$$\Omega^\lambda = A^\lambda dz + B^\lambda d\bar{z} \tag{4.17}$$

A solution F^λ of the equation (4.7) above, together with the initial condition $F^\lambda(0,0,\lambda) = I$, in a simply connected domain D, $F^\lambda : D \longrightarrow \Lambda^s\mathrm{SL}(2,\mathbb{C})$, is called extended frame corresponding to the spectral deformations $f \longmapsto f^s$, and $Q \longmapsto \theta^{-2}Q$.

Here, $\Lambda^s\mathrm{SL}(2,\mathbb{C})$ represents the "twisted" loop group over $\mathrm{SL}(2,\mathbb{C})$ given by the automorphism

$$\sigma : g \longmapsto (\mathrm{Ad}\sigma_3)(g),$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\Lambda^s\mathrm{SL}(2,\mathbb{C}) := \{g : C_s \longrightarrow \mathrm{SL}(2,\mathbb{C}) | g(-\lambda) = \sigma(g(\lambda))\}$, where s is the absolute value of the parameter λ and C_s is the circle of center O and radius s in the complex plane.

Note these maps could be also written as $g_s(\theta) : S_1 \longrightarrow \mathrm{SL}(2,C)$, with s fixed, real and positive, and the property $g_s(-\theta) = \sigma(g_s(\theta))$, and that there is no significant difference between these loop groups and the usual loop group considered in [6].

We will denote $\Lambda\mathrm{SL}(2,\mathbb{C}) := \{g : S^1 \longrightarrow \mathrm{SL}(2,\mathbb{C}) | g(-\theta) = \sigma(g(\theta))\}$.

It is customary to denote by $\Lambda^*_-\mathrm{SL}(2,\mathbb{C})$ the set of all maps of $\Lambda\mathrm{SL}(2,\mathbb{C})$ that can be holomorphically extended outside the disk enclosed by the circle, and equal to identity at infinity. Also, $\Lambda_+\mathrm{SL}(2,\mathbb{C})$ stands for those maps that can be holomorphically extended inside the same disk.

Similar notations are used for $\Lambda\mathrm{SU}(2)$.

In order to make such a loop group into a complete Banach Lie group, we consider the same H^p-norm for $p > \frac{1}{2}$ as used in [6]. Elements of this loop group are matrices with off-diagonal entries that are odd in θ and diagonal entries that are even in θ. We view the elements as *formal series in* θ.

Whenever we use loop group factorizations, we will always split inside the loop group $\Lambda\mathrm{SL}(2,\mathbb{C})$. The reason why we use loop group factorizations is related to the methods of constructing surfaces starting from the generalized Weierstrass representation formula. Such a method was first presented in [6].

Theorem 4. *For any associated family of CMC surfaces with given frame $F = F(\theta)$, $\theta \in S^1$, and mean curvature $|H| > 1$, there exists a certain s-deformation, for a specific $s = c$ (that depends on H), which generates a unitary frame $\widetilde{F} = \widetilde{F}(\theta) \in \Lambda\mathrm{SU}(2)$. The unitary frame $\widetilde{F}$ represents the lift of a harmonic map $\widetilde{N}$ in S^2.*

Proof. It is easy to see that choosing $s = c := \sqrt{|\frac{H-1}{H+1}|}$ gives the only deformation that makes (changes) the Maurer-Cartan Ω into an su(2)-valued form $\widetilde{\Omega}$.

Remark that as λ we approaches $\lambda_0 = c \cdot \theta$, the mean curvature H^c will go to infinity, and this particular deformation degenerates. From the Gauss-Codazzi equations, it follows that there exists a map $\widetilde{F}$ from D to $SU(2)$ such that $\widetilde{F}^{-1}d\widetilde{F} = \widetilde{\Omega}$. The harmonic map $\widetilde{N}$ represents the natural projection of the frame $\widetilde{F}$ to S^2. □

Definition 6. *We call $\widetilde{F}$ the adjusted frame of F and the form $\widetilde{F}^{-1}d\widetilde{F}$ the adjusted Maurer-Cartan form.*

Hence, the explicit form of the adjusted Maurer-Cartan is

(4.18)
$$\widetilde{\Omega} = \begin{pmatrix} -\frac{1}{2}u_z & -\theta^{-1}\cdot e^{-u}Q \\ \theta^{-1}\cdot \frac{1}{2}e^{u}\sqrt{H^2-1} & \frac{1}{2}u_z \end{pmatrix} dz + \begin{pmatrix} \frac{1}{2}u_{\bar{z}} & -\theta\cdot\frac{1}{2}e^{u}\sqrt{H^2-1} \\ \theta\cdot e^{-u}\overline{Q} & -\frac{1}{2}u_{\bar{z}} \end{pmatrix} d\bar{z}.$$

5 Weierstrass Type Representation Formula for Surfaces of Constant Mean Curvature in $\mathbb{H}^3$

Let M be any simply connected Riemann surface immersed in $\mathbb{H}^3$, via immersion f, corresponding to the moving frame F.

It is well-known that for every local framing F and connection form $\Omega := F^{-1}dF$, we have the identity (Maurer-Cartan equation):

$$d\Omega + \frac{1}{2}[\Omega \wedge \Omega] = 0.$$

An arbitrary $\lambda \in C^*$ deformation transforms Ω into $\Omega^\lambda := (F^\lambda)^{-1}dF^\lambda = A^\lambda dz + B^\lambda d\bar{z}$, which can be also written as

$$\Omega^\lambda = \Omega_1^{'}dz + \Omega_0 + \Omega_1^{''}d\bar{z},$$

where $\Omega_0 = \Omega_0^{'}dz + \Omega_0^{''}d\bar{z}$. Here Ω_0 is, as usual, a one form with values on the diagonal elements of $\mathfrak{sl}(2,\mathbb{C})$, and the rest of the terms are off-diagonal. Hence,

$$\Omega_0^{'} = \begin{pmatrix} -\frac{1}{2}u_z & 0 \\ 0 & \frac{1}{2}u_z \end{pmatrix},$$
$$\Omega_0^{''} = \begin{pmatrix} \frac{1}{2}u_{\bar{z}} & 0 \\ 0 & -\frac{1}{2}u_{\bar{z}} \end{pmatrix},$$
$$\Omega_1^{'} = \begin{pmatrix} 0 & -\theta^{-1}\cdot e^{-u}Q \\ \theta^{-1}\cdot\frac{s}{2}e^u(1+H) & 0 \end{pmatrix},$$
$$\Omega_1^{''} = \begin{pmatrix} 0 & \theta\cdot\frac{s^{-1}}{2}e^u(1-H) \\ \theta\cdot e^{-u}\overline{Q} & 0 \end{pmatrix}.$$

Here $\lambda = s\cdot\theta$ and $\theta = e^{it}$, as usual.

Let us consider the associate form $\widetilde{\Omega}$ which is a $\mathfrak{su}(2)$-valued 1-form and hence decomposed via Cartan decomposition, as $\mathfrak{su}(2) = \mathfrak{k}\oplus\mathfrak{p}$, where $\mathfrak{k}$ is the diagonal part, and $\mathfrak{p}$ is the off-diagonal one. Thus, the form $\widetilde{\Omega}$ writes $\widetilde{\Omega} = \widetilde{\Omega}_0 + \widetilde{\Omega}_1$. Let $\widetilde{\Omega}_1 = \widetilde{\Omega}_1^{'} + \widetilde{\Omega}_1^{''}$ be the splitting into $(1,0)$ and, respectively, $(0,1)$-forms. We compare matrices $\widetilde{\Omega}_1^{'}$ and $\widetilde{\Omega}_1^{''}$ to their correspondents from Ω^λ, namely $\Omega_1^{'}$ and $\Omega_1^{''}$. A straight-forward computation leads us to the following result:

Theorem 5. *Let $\widetilde{N} : D \longrightarrow S^2$ be a non-conformal harmonic map with lift $\widetilde{F} = \widetilde{F}(\theta) : D \longrightarrow \Lambda\mathrm{SU}(2)$, where D is a simply connected domain as before.*

Let $\widetilde{\Omega}(\theta) = \widetilde{F}^{-1}d\widetilde{F} = \widetilde{\Omega}_1^{'} + \widetilde{\Omega}_0 + \widetilde{\Omega}_1^{''}$. Let $a > 0$ be an arbitrary real constant, and let

$$\beta_1^{'}(a) = \frac{1}{4}\cdot(a-1)\cdot(\sigma_0-\sigma_3)\widetilde{\Omega}_1^{'}(\sigma_0+\sigma_3)dz,$$

respectively

$$\beta_1^{''}(a) = \frac{1}{4}\cdot(a^{-1}-1)\cdot(\sigma_0+\sigma_3)\widetilde{\Omega}_1^{''}(\sigma_0-\sigma_3)d\bar{z}$$

Let
$$\Omega = \Omega(a,\theta) := \widetilde{\Omega}(\theta) + \beta_1^{'}(a) + \beta_1^{''}(a).$$

Then we have the following:
i).
$$d\Omega(a,\theta) + \frac{1}{2}[\Omega(a,\theta)\wedge\Omega(a,\theta)] = 0.$$

ii).If F is a $\mathrm{SL}(2,\mathbb{C})$-valued solution of $\Omega = F^{-1}dF$, then $f = F\cdot F^*$ is a conformal immersion with isolated singularities and constant mean curvature $H = \frac{a^2+1}{a^2-1}$.

Proof. One may also see Theorem 4.4, [15], for the construction of a form that is similar to Ω.

Let us now assume that $\widetilde{\Omega}(\theta) = \widetilde{F}^{-1}d\widetilde{F}$ is of the form given by equation (4.9). We are looking for a parameter $\lambda \in \mathbb{C}^*$ such that Ω^λ coincides with $\Omega(a,\theta)$. By direct computation, we obtain that $s = |\lambda|$ must satisfy the relation $s = a\cdot\sqrt{|\frac{H-1}{H+1}|}$. Therefore, $\lambda = s\cdot\theta \in C_s$, where s is uniquely determined by a, from the above formula. The frame F^λ is a solution to $\Omega = F^{-1}dF$, and its corresponding immersion is $f^\lambda = F^\lambda \cdot {F^\lambda}^*$, with mean curvature H^s. We substitute s in the simplified formula (3.4) and obtain $H^s = \frac{a^2+1}{a^2-1}$. This proves ii). □

Remark 6. Assuming that the initial Maurer-Cartan form $\widetilde{\Omega}$ corresponds to the adjusted frame $\widetilde{F}$ of a certain family of CMC surfaces with initial mean curvature H, let us compute the matrices explicitely. We obtain

$$\beta_1^{'} = (a-1)\begin{pmatrix} 0 & 0 \\ \theta^{-1}\cdot \frac{1}{2}e^u\sqrt{H^2-1} & 0 \end{pmatrix} dz,$$

respectively

$$\beta_1^{''} = (a^{-1}-1)\begin{pmatrix} 0 & -\theta\cdot\frac{1}{2}e^u\sqrt{H^2-1} \\ 0 & 0 \end{pmatrix} d\bar{z}$$

The sum of these matrices is $\Lambda\mathrm{sl}(2,\mathbb{C})$-valued. Note that the first defines a $(1,0)$ form in θ^{-1}, while the other one is a $(0,1)$ form in θ. These expressions will be of use in the next section.

Note that while Ω is not su(2)-valued, $\beta_1^{'}+\beta_1^{''}$ measures its 'defect' from su(2). In a sense, this measures the 'defect' of the Gauss map N from being a harmonic map in the symmetric space S^2.

Let us now denote

$$G(\theta) = F^\lambda \cdot \widetilde{F}^{-1}(\theta) \tag{5.19}$$

A very important remark is that $G\cdot G^* = F\cdot F^* = f^\lambda$. We view the matrix G exclusively as a function of θ.

6 Normalized Weierstrass Representation

The notion of normalized potential was introduced in the most general case - for harmonic maps in symmetric spaces, and their extended frames - [9]. Next, [6] and [21] gave the expression and computation of this potential in particular for the case of constant mean curvature surfaces in Euclidean space. We recall the following adaptation of (see [21]):

Theorem 6. *Let $\widetilde{N} : D \to S^2$ be a harmonic map based at identity, and $\widetilde{F}(\theta) : D \to \Lambda\mathrm{SU}(2,\mathbb{C})$ an extended frame corresponding to it. Then there exists a discrete subset S of $D-0$ such that for any $z \in D-S$ we have $\widetilde{F}(z,\theta) = \widetilde{F}_-(z,\theta)\cdot\widetilde{F}_+(z,\theta)$, with $\widetilde{F}_-(\theta) \in \Lambda_-^*\mathrm{SL}(2,\mathbb{C})$ and $\widetilde{F}_+(\theta) \in \Lambda_+\mathrm{SL}(2,\mathbb{C})$. Remark that here minus and plus refer to the power series in θ. The form $P(z) = \widetilde{F}_-^{-1}d\widetilde{F}_-\theta$ is a meromorphic (1,0) - form on D, with poles in S. This form is called meromorphic potential or normalized potential.*

Conversely, any such harmonic map $\widetilde{N}$ can be constructed from a meromorphic potential by integration, obtaining first $\widetilde{F}_- : D-S \in \Lambda_-^*\mathrm{SL}(2,\mathbb{C})$ where the discrete *subset S consists of poles of P and then obtaining an extended frame $\widetilde{F}$ of f via the Iwasawa factorization* $\Lambda\mathrm{SL}(2,\mathbb{C}) = \Lambda\mathrm{SU}(2) \cdot \Lambda_+^B\mathrm{SL}(2,\mathbb{C})$, $\widetilde{F}_- = \widetilde{F} \cdot \widetilde{F}_+^{-1}$.

For details on the Iwasawa factorization, one may consult [14] and [6]. Note that this type of decomposition may be done in minus-plus form or in plus-minus form (with different, unique factors).

The above stated theorem has the following important consequence:

Theorem 7. *The normalized potential corresponding to constant mean surfaces ($|H| > 1$) in the hyperbolic space is identical to the one corresponding to their Euclidean correspondents.*

Proof. The Lawson correspondence is performed via the same harmonic maps. More precisely, the harmonic maps that represent Gauss maps for the Euclidean CMC surfaces correspond to the adjusted Gauss maps of their hyperbolic counterparts.

□

In view of the above theorem and using formula (3.24) of [21], we deduce the normalized potential corresponding to a CMC surface with $|H| > 1$ in hyperbolic space, of Hopf differential Qdz^2 and metric factor $u(z,\bar{z})$, as

$$P = \begin{pmatrix} 0 & -e^{-2h(z)+h(0)}Q \\ \frac{1}{2}e^{2h(z)-h(0)}\sqrt{H^2-1} & 0 \end{pmatrix} dz$$

where $h(z) := u(z,0)$ is the *holomorphic part* of $u(z,\bar{z})$.

Remark that $\theta^{-1}P$ can be deduced directly from the form $\widetilde{\Omega}_1'$, as we had expected.

Note that we did not use Lawson's correspondence in order to obtain this result. In some other words, we did not 'cheat', by replacing some Euclidean mean curvature k with its hyperbolic correspondent $\sqrt{k^2+1}$. Nevertheless, the expression $\sqrt{H^2-1}$ in the normalized potential P is the same with the Lawson-correspondent Euclidean mean curvature. But this simply occurred as a byproduct of our loop group techniques!

In the spirit of Wu, the holomorphic part $e^{2h(z)}$ of the conformal factor $e^{2u(z,\bar{z})}$ in the induced metric $e^{2u}dzd\bar{z}$ on a CMC immersion f in the hyperbolic space is meromorphic on the entire domain D. This meromorphic function and the Hopf differential uniquely determine the induced metric and the surface, up to spectral deformations.

7 Foliations of the Hyperbolic Space with Constant Mean Curvature Surfaces

Due to Lawson's correspondence, examples of CMC surfaces $|H| > 1$ abound. One of the best sources of examples and surface construction using loop group factorization techniques is [13]. Based on these methods, N. Schmitt wrote programs that produce hyperbolic analogues of some CMC surfaces in $\mathbb{E}^3$, such as CMC bubbletons, CMC cylinders, Smyth surfaces, and N-noids. The pictures of these CMC surfaces in $\mathbb{H}^3$ can be viewed at the GANG's gallery of CMC surfaces (http://www.gang.umass.edu/). Among these, Delauney surfaces present a particular interest in the area, due to at least two facts: a). A CMC ($|H| > 1$) surface of finite topology and exactly two ends is a Delauney surface; b). properly embedded annular ends off CMC surfaces with $|H| > 1$ converge to Delauney ends. These examples are particularly interesting topics in themselves, and they go beyond the scope of this article. Instead, we will turn our attention to the notion of parallel CMC surfaces.

Chopp and Velling found numerical evidences that $\mathbb{H}^3$ can be foliated by CMC discs (with all the values $|H| < 1$ as mean curvatures) that share a common Jordan curve boundary in S^2_∞. They conjectured this result in [5]. The author's proof to their conjecture is included in her report [18] on CMC $|H| < 1$ surfaces in $\mathbb{H}^3$.

Towards a $|H| > 1$ analogue of this foliation, the author investigated parallel surfaces to a given surface in $\mathbb{H}^3$ (in Poincaré spherical model). It is known (see e.g. [15]) that given the immersion $f : D \to \mathbb{H}^3$, with normal N, in the linear model of $\mathbb{H}^3$, the surface $f^q = \cosh q \cdot f + \sinh q \cdot N$ is the surface parallel to f at distance q and its normal is $N^q = \sinh q \cdot f + \cosh q \cdot N$. If F is the usual frame of $f = FF^*$, then the immersion is given by $f^q = F \begin{pmatrix} e^q & 0 \\ 0 & e^{-q} \end{pmatrix} F^*$.

This parallel surface can be obtained by a gauge transformation $F \mapsto F \cdot U$, where $U = \begin{pmatrix} e^{q/2} & 0 \\ 0 & e^{-q/2} \end{pmatrix}$. Moreover, every gauge transformation that gives a new strongly conformal immersion is of this form (see [15]).

In the particular case when $q = \frac{1}{2} \ln |\frac{H+1}{H-1}|$ (when the spectral deformation is proper, via parameter $s = e^{-2q} = s_0 = |\frac{H-1}{H+1}|$), then the new immersion has the same mean curvature and Hopf differential, with only a change in sign.

Making q run through all possible values, we obtain all the parallel surfaces to a given immersed surface.

Let us now consider one of the simplest examples of constant mean curvature surfaces, the round cylinder. E.g., the hyperbolic cousin of the round cylinder of mean curvature $H = 2$ in $\mathbb{R}^3$ is a cylinder of mean curvature $H = \sqrt{5}$ in $\mathbb{H}^3$ (Fig. 1). In terms of Weierstrass representation, both are given by the same matrix P, namely $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dz$

With these considerations, Theorem 5 has the following particular consequence:

Theorem 8. *There exists a foliation with round cylinders of the hyperbolic 3-space* $\mathbb{H}^3$*. In Poincaré spherical model, these cylinders have two antipodes in common on the ideal boundary* S^2_∞*, and mean curvatures ranging between* 1 *and infinity. The limiting cases* $H = 1$ *and* $H = \infty$ *correspond to the ideal boundary and the diameter, respectively.*

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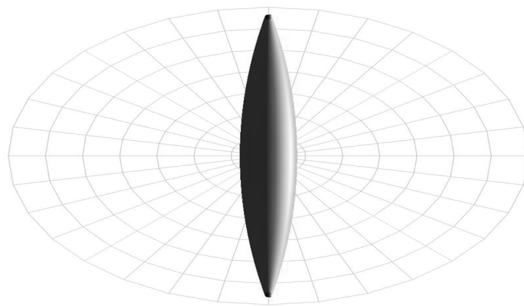

Figure 1: Round cylinder in the Poincaré spherical model of $\mathbb{H}^3$.

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