Wave fronts, Integral Geometry and curvature measures

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Abstract

Let Y be an oriented non-euclidean space of constant curvature. Let $M \subset Y$ be a closed (compact, possibly with boundary), oriented, and immersed hypersurface. Given a point $y \in Y$, we can count the number (possibly with sign) of geodesics normal to M passing through y. We use this counting function to measure M geometrically in terms of curvature measures of M. As an example of the kind of result obtained, suppose that M is a closed, co-oriented, and immersed hypersurface in hyperbolic 3-space of constant curvature $-K$. If we average number of geodesics normal to M passing through points that lie in wavefronts to M of distance no greater than L from M, and then take the limit as $L \to \infty$, this average approaches

$$\frac{1}{2\pi}\int_{M^1} |K + \kappa_1(x)\kappa_2(x)|\, d\mathcal{A} + \frac{\sqrt{K}}{2\pi}\int_{M^2} |\kappa_1(x) + \kappa_2(x)|\, d\mathcal{A} + \frac{1}{2\pi}\int_{M^3} (K + \kappa_1(x)\kappa_2(x))\, d\mathcal{A}.$$

Here $\kappa_1(x)$ and $\kappa_2(x)$ are the principal curvature functions on M, and

$$\begin{aligned} M^1 &= \{x \in M \mid \kappa_1^2(x) > K, \kappa_2^2(x) > K\} \\ M^2 &= \{x \in M \mid \kappa_1^2(x) > K, \kappa_2^2(x) \le K\} \cup \{x \in M \mid \kappa_1^2(x) \le K, \kappa_2^2(x) > K\} \\ M^3 &= \{x \in M \mid \kappa_1^2(x) \le K, \kappa_2^2(x) \le K\} \end{aligned}$$

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1 Introduction

Let's start with two smooth n-dimensional Riemannian manifolds X^n and Y^n and a volume form Φ on Y. In this article Y will be an oriented non-euclidean space of

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constant curvature as follows.

Curvature $\lambda = 0$:	Euclidean Space:	$Y = \mathbb{E}^n$
Curvature $\lambda < 0$:	Hyperbolic Space:	$Y = \mathbb{H}^n(\lambda)$
Curvature $\lambda > 0$:	Spherical Space:	$Y = \mathbb{S}^n(\lambda)$

A fundamental relation from integral geometry, given a differentiable function $f : X^n \longrightarrow Y^n$, is

$$\boxed{\int_X |f^\star \Phi| = \int_{y \in Y} \# f^{-1}(y)\, \Phi.}$$

A classical example is where X is an immersed, closed, and oriented curve in $\mathbb{E}^2$, Y is the unit circle $\mathbb{S}^1$ with Φ as the standard arc-length form $d\theta$, and f is the Gauss map. In this case $\int_{\theta \in \mathbb{S}^1} \# f^{-1}(\theta)\, d\theta$ is the total absolute curvature of X, and $\# f^{-1}(\theta)$ is the number of points in X with oriented normals in the direction of θ.

As another example, suppose that we have a closed, oriented, and immersed curve M in $\mathbb{E}^2$ of length l, with unit speed parametrization $y(x) : 0 \leq x \leq l$, where $T(x) = y'(x)$ is directed in the positive direction along M. Choose unit normals $N(x)$ to M so that $T(x) \wedge N(x)$ induces the usual orientation on $\mathbb{E}^2$: we say that M is *co-oriented*. Then the (co-oriented) parallel curves or wave fronts M_t to M at a distance t from M can be parametrized by $y(x) + tN(x)$. If X denotes the normal bundle to M and $f : X \to \mathbb{E}^2$ denotes the exponential map, then we can orient X via f. If we consider all points X_L in X at a distance L or less from the zero-section, then $f(X_L)$ consists of those points that lie in a wave front M_t such that $-L \leq t \leq L$, giving a "tube" around M. If $\mathcal{A}$ denotes the area of $f(X_L)$, then

$$\frac{\int_{X_L} |f^\star \Phi|}{\mathcal{A}} = \frac{\int_{-L}^{L} \text{length}(M_t)\, dt}{\mathcal{A}} = \frac{\int_0^l \int_{-L}^{L} |1 - t\kappa(x)|\, dt\, dx}{\mathcal{A}} = \frac{\int_{y \in \mathbb{E}^2} \# f^{-1}(y)\, \Phi}{\mathcal{A}},$$

a value equal to the average number of normal segments (of length $\leq L$) through a point in $f(X_L)$. Note that

$$\frac{\int_{X_L} f^\star \Phi}{\mathcal{A}} = \frac{\int_{-L}^{L} \text{signed length}(M_t)\, dt}{\mathcal{A}} = \frac{\int_0^l 2L\, dx}{\mathcal{A}} = \frac{2lL}{\mathcal{A}}$$

and that the integrand $1 - t\kappa(x)$ becomes negative for a specific value of x when t is greater than the radius of curvature $\rho(x)$.

The Area Formula [4]. *Consider a Lipschitz function $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ for $m \leq n$. (1) If $X \subset \mathbb{R}^m$ is an $\mathcal{L}^m$ measurable set, then*

$$\int_X J_m f(x)\, d\mathcal{L}^m x = \int_{\mathbb{R}^n} \mathcal{N}(f \,|\, X, y)\, d\mathcal{H}^m y$$

where J denotes the Jacobian, $\mathcal{L}^m$ denotes m-dimensional Lebesgue measure, $\mathcal{H}^m$ denotes m-dimensional Hausdorff measure, and
$\mathcal{N}(f \,|\, X, y) \equiv card\{x \in X : f(x) = y\}$ denotes the counting function.

(2) More generally, if ϵ is an $\mathcal{L}^m$ integrable function, then

$$\int_{\mathbb{R}^m} \epsilon(x) J_m f(x)\, d\mathcal{L}^m x = \int_{\mathbb{R}^n} \sum_{f^{-1}\{y\}} \epsilon(x)\, d\mathcal{H}^m y.$$

Continuing our example above, we can write

$$\int_{X_L} f^\star \Phi = \int_{X_L} \epsilon(x,t) |f^\star \Phi| = \int_{\mathbb{E}^2} \sum_{f^{-1}(y)} \epsilon(x,t)\, \Phi$$

where $\epsilon(x,t) = \text{sign}(1 - t\kappa(x))$. Also, suppose that for a given x, $f(x,t)$ lies on a normal to M that meets M at $y(x)$. Then $\epsilon(x,t) = -1$ if the distance from $f(x,t)$ to $y(x)$ is a local maximum, $\epsilon(x,t) = +1$ if it is a local minimum, and $\epsilon(x,t) = 0$ otherwise. And so $2lL/A$ gives an average number of normal segments (of length $\leq L$) to M (counted with sign) through points in a tube of radius L about M.

An interesting and related application to the area formula is given by Langevin [11] who shows that

$$\int_{M^{n-1}} |\mathsf{K}|\, d\mathcal{A} = \int_{\mathbb{P}^{n-1}} |\mu|(M,L)\, d\Phi$$

where $M^{n-1} \subset \mathbb{E}^n$ is a closed, oriented, and immersed submanifold of dimension $n-1$ with induced volume form $d\mathcal{A}$, K is the Gaussian curvature function along M, and $|\mu|(M,L)$ is the number of critical points of the orthogonal projection of M onto $L \in \mathbb{P}^{n-1}$. Langevin's *General Exchange Theorem* generalizes this result to any finite dimension or codimension, and to non-euclidean spaces of constant curvature. By letting $\epsilon(x) = (-1)^{\text{critical}(x)}$ in the *Area formula*, where x is a critical point of the projection above, one may show that

$$\int_{M^{n-1}} \mathsf{K}\, d\mathcal{A} = \int_{\mathbb{P}^{n-1}} \mu(M,L)\, d\Phi$$

where

$$\mu(M,L) = \sum_{x \text{ critical}} (-1)^{\text{index}(x)}.$$

Now suppose that M^{n-1} is a closed, oriented, and immersed submanifold of dimension $n-1$ in an oriented non-euclidean space Y^n of constant curvature $\lambda = \epsilon K$ (where $\epsilon = \pm 1$ and K is a non-negative integer) of dimension n. We will say that M is *co-oriented* iff it and its normal bundle X are oriented in agreement with the orientation induced by that of Y^n. Give M, Y, and X respective local geodesic coordinates x, y, and (x,t). Note that the exponential map $f : X \longrightarrow Y$ is defined on all of X. We can pull back the Riemannian volume measure Φ of Y to X and attempt to write it in terms of the geometry of M, giving us a tube formula. This formula will then have an integral geometric interpretation via the normal counting function η_M since

$$\int_X |f^\star \Phi| = \int_{y \in Y} \# f^{-1}(y)\, \Phi = \int_Y \eta_M(y)\, \Phi,$$

where $\eta_M(y) \equiv \#f^{-1}(y)$.

Given a point $f(x,t) \in Y$ along a normal geodesic to M meeting M at the point $f(x,0)$, let $c_i, i = 1,2,3,\ldots,n-1$, denote the principal lines of curvature of M passing through the point $f(x,0)$. Now define $\epsilon_i(x,t)$ to be -1 if $f(x,t)$ is a local maximum of the distance function from $f(x,t)$ to c_i, and to be $+1$ if it is a local minimum. Define $\epsilon_i(x,t) = 0$ otherwise. Define $\epsilon(x,t) = \prod_{i=1}^{n-1} \epsilon_i(x,t)$. By part (2) of the area formula we may write

$$\int_X f^\star \Phi = \int_X \epsilon(x,t)\, |f^\star \Phi| = \int_Y \sum_{f^{-1}\{y\}} \epsilon(x,t)\, \Phi.$$

Henceforth the notation f, X, Y, η, ϵ, and Φ will refer to this model.

2 Volume Forms for Wave Fronts

Consider $\mathbb{R}^n_1 = \mathbb{R}^n \times \mathbb{R}^1$ with the Lorentzian $(n,1)$ space-time metric $ds^2 = -dy_1^2 - dy_2^2 - \cdots - dy_n^2 + d\tau^2$ where we write $(y,\tau) = (y_1, y_2, \ldots, y_n, \tau)$ for a point $(y,\tau) \in \mathbb{R}^n_1$. If we consider the quadratic form $q = -(\sum_{i=1}^n y_i^2) + \tau^2$ with projective group $PO(n)$, then $PO(n)$ acts on the Grassmannian $G_{n+1,1}$ of lines through the origin with three orbits: *(i)* the isotropic cone of lines $q^{-1}(0)$, *(ii)* the set of lines where q is positive definite, and *(iii)* the set of lines where q is negative definite. The geometry of case *(ii)* concerns us here: Let $Y = \mathbb{H}^n(\lambda) = q^{-1}(1/K)$. The geodesics of $\mathbb{H}^n(\lambda)$ are precisely the intersections of 2-planes in $\mathbb{R}^n_1$ through the origin with $\mathbb{H}^n(\lambda)$. If p is a point of $\mathbb{H}^n(\lambda)$, then the vector p in $\mathbb{R}^n_1$ is perpendicular to the tangent space of $\mathbb{H}^n(\lambda)$ at p. Now if M^{n-1} is a co-oriented submanifold of $\mathbb{H}^n(\lambda)$, where p belongs to M and ν_p is the positively oriented unit normal to $M \subset \mathbb{H}^n(\lambda)$ at p, so that $q(\nu_p) = -1$, then the positively oriented geodesic in $\mathbb{H}^n(\lambda)$ normal to M at p has a unit speed parameterization given by $(\cosh(t\sqrt{K}))p + (1/\sqrt{K})(\sinh(t\sqrt{K}))\nu_p, t \in \mathbb{R}$. Given $t \in \mathbb{R}$, let

$$M_t = \{y \in Y \mid y = (\cosh(t\sqrt{K}))p + \frac{1}{\sqrt{K}}(\sinh(t\sqrt{K}))\nu_p,\ \ p \in M\}.$$

And so M_t is the wave front or parallel subvariety to M at distance t. We will also let the notation M_t denote $f^{-1}(M_t) \subset X$ as appropriate.

The interested reader can look at the last two sections of this paper for a review of the hyperbolic trigonometric functions as well as some identities that will be used implicitly throughout this paper. Henceforth we will use the following notation: Given λ, let $\mathfrak{c}(t) = \cos(t\sqrt{\lambda})$, $\mathfrak{s}(t) = \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}$, and $\mathfrak{t}(t) = \mathfrak{s}(t)/\mathfrak{c}(t)$ (so if $\lambda = 0$, then $\mathfrak{c}(t) = 1$ and $\mathfrak{s}(t) = t$). We may then express a wave front to M as

$$M_t = \{y \in Y \mid y = \mathfrak{c}(t)p + \mathfrak{s}(t)\nu_p,\ \ p \in M\}.$$

Define the symmetric polynomials on the principal curvature functions

$\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ of $M \subset Y$ as follows:

$$\begin{aligned}
\Pi_0 &= 1 \\
\Pi_1 &= \sum_{1 \le i \le n-1} \kappa_i \\
\Pi_2 &= \sum_{1 \le i < j \le n-1} \kappa_i \kappa_j \\
\Pi_3 &= \sum_{1 \le i < j < k \le n-1} \kappa_i \kappa_j \kappa_k \\
\Pi_4 &= \sum_{1 \le i < j < k < l \le n-1} \kappa_i \kappa_j \kappa_k \kappa_l \\
&\vdots \\
\Pi_{n-1} &= \prod_{1 \le i \le n-1} \kappa_i = \mathsf{K}.
\end{aligned}$$

We then define the i^{th} *curvature measure* of M by

$$\boxed{\mathcal{M}_i = \int_M \Pi_i \, d\mathcal{A}}$$

and the i^{th} *absolute curvature measure* of M by

$$\boxed{|\mathcal{M}_i| = \int_M |\Pi_i| \, d\mathcal{A}.}$$

So $\mathcal{M}_0 = |\mathcal{M}_0| = \text{Volume}(M) \equiv \mathcal{V}(M)$. Please see [15] for some interesting results concerning Π_i and $\mathcal{M}_i$.

The notation below will follow that of [12]. Suppose that $Y = \mathbb{H}^n(\lambda)$, and let $\rho_1, \rho_2, \ldots, \rho_{n-1}$ be the principal radii of curvature and C_i, $i = 1, \ldots, n-1$, the lines of curvature of $M \subset \mathbb{H}^n(\lambda)$, and let dx_i be the element of arc length of C_i.

Definition 1. *A closed wave front to an immersed hypersurface in $\mathbb{H}^n(\lambda)$ is reversible iff the functions ρ_i have finite values at every point of the wave front, for $i = i, \ldots, n-1$.*

So for any Y we have that $d\mathcal{A} = dx_1 \wedge \cdots \wedge dx_{n-1}$, and then the signed volume of the wave front M_t may be computed as $\int_M d\mathcal{A}_t$ where

$$\boxed{d\mathcal{A}_t = \bigwedge_{i=1}^{n-1} (\mathfrak{c}(t) - \kappa_i \mathfrak{s}(t)) \, dx_i}$$

$= \sum_{j=0}^{n-1} (-1)^j \mathfrak{c}^{n-1-j}(t) \mathfrak{s}^j(t) \Pi_j \, d\mathcal{A}$ so that $\int_M d\mathcal{A}_t = \sum_{j=0}^{n-1} (-1)^j \mathfrak{c}^{n-1-j}(t) \mathfrak{s}^j(t) \mathcal{M}_j$. Note that $d\mathcal{A}_t(x)$ is defined for generic values of t and x.

Suppose now that $M \subset \mathbb{H}^n(\lambda)$ is also reversible. Given $x \in M$, let $R_i(x)$, $i = 1, \ldots, n-1$, denote the distance to the contact point of the normal to M at x with the envelope of the normals to M along C_i, so that [14]

$$\rho_i(x) = \mathfrak{t}(R_i(x)).$$

Let $d\alpha_i$ denote the angle between two infinitesimally adjacent normals to M along C_i at x. Then for $M \subset \mathbb{H}^n(\lambda)$ we have

$$dx_i = \sqrt{K}\mathfrak{s}(R_i)\, d\alpha_i.$$

In [2] Anisov shows that for a reversible front in $\mathbb{H}^n$, the oriented volume of the front is a polynomial in $\cosh t$ and $\sinh t$ of degree $n-1$, and that the oriented volume of a closed front in $\mathbb{S}^n$ (or $\mathbb{E}^n$) to an immersed hypersurface is a polynomial in $\sin t$ and $\cos t$ (or t) of degree $n-1$. So we have seen how to extend these results:

Lemma 1. *Let M be a closed, co-oriented, and immersed submanifold of $\mathbb{H}^n(\lambda)$, $\mathbb{S}^n(\lambda)$, or $\mathbb{E}^n$. Then the oriented volume of any wave front M_t is a polynomial in $\mathfrak{c}(t)$ and $\mathfrak{s}(t)$ of degree $n-1$. In particular*

$$\boxed{\mathcal{V}(M_t) \equiv Vol(M_t) = \int_M d\mathcal{A}_t = \sum_{j=0}^{n-1} (-1)^j \mathfrak{c}^{n-1-j}(t)\mathfrak{s}^j(t)\mathcal{M}_j.}$$

Note that in the euclidean case this says that

$$\boxed{\mathcal{V}(M_t) = \int_M d\mathcal{A}_t = \sum_{j=0}^{n-1} (-1)^j t^j \mathcal{M}_j.}$$

3 The Number of Normals: A Select History

Given a point $y \in Y$, recall that $\eta_M(y)$ is the number of normals to M passing through y. Others have looked at problems from integral geometry and geometric probability about η in Euclidean space. For example, in [9] Hann shows that

$$2 \le N(B) \le 12$$

if B is a convex planar body where

$$N(B) = \frac{\mathcal{J}(B)}{\mathcal{V}(B)} \qquad \text{and} \qquad \mathcal{J}(B) = \int_B \eta_{\partial B}(b)\, d\mathcal{A}$$

(so $N(B)$ is the average number of normals to ∂B passing through a point in B). For example, if we consider B to be a square in $\mathbb{E}^2$, then

$$N(B) = \frac{\mathcal{J}(B)}{\text{Area}(B)} = \frac{\int_B \eta_M(b)\, dx\, dy}{\mathcal{A}(B)} = 8$$

since every point in the interior of B has exactly 8 normals to ∂B passing through it, including the normals passing through the vertices of the square. To see that every point in the interior of B has exactly 8 normals to ∂B passing through it, view B as the limit of squares B_r whose sharp corners have been replaced with quarter circles of radius r, letting $r \to 0^+$, noting that set of points in the interior of B_r with exactly 8 normals to ∂B_r passing through them approaches the set B.

Suppose that $p \in \mathbb{E}^n$ and $I = \{(p, X) \in p \times X \mid p = f(x,t)\}$. Now define the map $e : p \times X \setminus I \longrightarrow \mathbb{S}^{n-1}$ by

$$e(p,x,t) = \frac{p - f(x,t)}{||p - f(x,t)||}$$

and let $d\mathcal{O}_{n-1}$ be the pull-back under $e^\star$ of the volume element of $\mathbb{S}^{n-1}$. If we orient $p \times X$ via the orientation on X, then White [18] has shown that

$$\boxed{\frac{1}{\mathcal{O}_{n-1}} \int_{p \times \partial X_{[a,b]}} d\mathcal{O}_{n-1} = L(p, M_b) - L(p, M_a) = (-1)^{n-1} \mathcal{I}(f,p)}$$

where $L(p, M_t)$ is the linking number of p with the wave front M_t, $\mathcal{O}_{n-1}$ is the volume of $\mathbb{S}^{n-1}$, and $\mathcal{I}(f,p)$ is the algebraic number of intersections of p with $f(X_{[a,b]})$, where $X_{[a,b]}$ consists of those points in X whose signed distance to M along some normal is in $[a,b]$.

In particular, if we compactify each fiber of $X_{[0,L]}$ by letting $L \longrightarrow \infty$ and adding a point at infinity (see [17] for further details), obtaining the manifold $X_{[0,\infty]}$, then White shows that

$$\frac{1}{\mathcal{O}_{n-1}} \int_{p \times \partial X_{[0,\infty]}} d\mathcal{O}_{n-1} = \frac{1}{\mathcal{O}_{n-1}} \int_M \mathsf{K}\, d\mathcal{A} = \frac{\mathcal{M}_{n-1}}{\mathcal{O}_{n-1}} = (-1)^{n-1} \mathcal{I}(f,p).$$

This is due partially to the fact that $d\mathcal{O}_{n-1}$ involves the Jacobian of the Gauss map. If we consider instead the one point compactification $X_{[-\infty,\infty]}$ of X, then

$$\frac{1}{\mathcal{O}_{n-1}} \int_{p \times \partial X_{[-\infty,\infty]}} d\mathcal{O}_{n-1} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ \mathcal{I}(f,p), & \text{if } n \text{ is odd.} \end{cases}$$

In this paper we will examine averages of the number of normals to M in the spirit of Hann, but we will consider the image of the entire normal bundle in the spirit of White, largely restricting ourselves to the euclidean and hyperbolic cases. What we we will get is something more than just an average number of normals.

Let B_L denote a solid ball of radius L in Y, $X_L = X_{[-L,L]}$, and $X_L^+ = X_{[0,L]}$. Keep in mind that as L increases, the volume of $f(X_L)$ and $f(X_L^+)$ will grow proportionally, in the limit, with that of B_L. We then define the following measures.

$\mathcal{N}(M) = \lim_{L\to\infty} \frac{1}{\mathcal{V}(B_L)} \int_{X_L} \lvert f^\star \Phi\rvert$	$\mathcal{N}^+(M) = \lim_{L\to\infty} \frac{1}{\mathcal{V}(B_L)} \int_{X_L^+} \lvert f^\star \Phi\rvert$
$\mathcal{I}(M) = \lim_{L\to\infty} \frac{1}{\mathcal{V}(B_L)} \int_{X_L} f^\star \Phi$	$\mathcal{I}^+(M) = \lim_{L\to\infty} \frac{1}{\mathcal{V}(B_L)} \int_{X_L^+} f^\star \Phi$
$\mathcal{S}(M) = \lim_{L\to\infty} \frac{\int_{M_L} d\mathcal{A}_L}{\mathcal{A}(\partial B_L)}$	$\lvert\mathcal{S}\rvert(M) = \lim_{L\to\infty} \frac{\int_{M_L} \lvert d\mathcal{A}_L\rvert}{\mathcal{A}(\partial B_L)}$

In defining $\mathcal{S}$ and $|\mathcal{S}|$ for hyperbolic and euclidean spaces, our aim is to measure M "from infinity" using the normal counting function η. We divide by $\mathcal{A}$ since the volume of the wave front M_L and the volume of B_L will grow proportionately, in the limit, as $L \to \infty$.

$\mathcal{N}$ and $\mathcal{I}$ also measure M using η, but now the entire ambient space Y is used, as all possible observation points are taken into account. For $\mathcal{N}^+$ and $\mathcal{I}^+$ we count the number of normals taking sign into account.

The formulas for $\mathcal{N}$, $\mathcal{N}^+$, $\mathcal{I}$, $\mathcal{I}^+$, $\mathcal{S}$, and $|\mathcal{S}|$ make no sense if $Y = \mathbb{S}^n(\lambda)$. In this case, let us redefine $\mathcal{N}$ and $\mathcal{I}$, taking into account the fact that each geodesic circle in $\mathbb{S}^n(\lambda)$ has length $2\pi/\sqrt{\lambda}$.

$\mathcal{N}(M) = \frac{\int_{X_{\pi/\sqrt{\lambda}}} \lvert f^\star\Phi\rvert}{\mathcal{V}(\mathbb{S}^n(\lambda))}$	$\mathcal{I}(M) = \frac{\int_{X_{\pi/\sqrt{\lambda}}} f^\star\Phi}{\mathcal{V}(\mathbb{S}^n(\lambda))}$

Note that all of these measures vary continuously as the image of M in Y is smoothly deformed. The fact that not all of these measures remain constant under smooth deformations will be seen below.

It is not difficult to compute $\mathcal{S}$ and $|\mathcal{S}|$, which we do below for the sake of completeness. Computations of $\mathcal{N}$, $\mathcal{N}^+$, $\mathcal{I}$, and $\mathcal{I}^+$ are left to the remaining sections.

Theorem 1. *If M is a closed, co-oriented, and immersed hypersurface in $\mathbb{H}^n(\lambda)$ or $\mathbb{E}^n$, then*

$$|\mathcal{S}|(M) = \begin{cases} \frac{1}{\mathcal{O}_{n-1}} \int_M |\Pi_{n-1}|\,\Phi = \frac{|\mathcal{M}_{n-1}|}{\mathcal{O}_{n-1}}, & \text{if } Y = \mathbb{E}^n; \\ \frac{\sqrt{K^{n-1}}}{\mathcal{O}_{n-1}} \int_M \left|\Pi_0 - \frac{\Pi_1}{\sqrt{K}} + \frac{\Pi_2}{K} - \frac{\Pi_3}{\sqrt{K^3}} + \cdots + (-1)^{n-1}\frac{\Pi_{n-1}}{\sqrt{K^{n-1}}}\right| \Phi, & \text{if } Y = \mathbb{H}^n(\lambda). \end{cases}$$

Proof.

$$\begin{aligned} |\mathcal{S}|(M) &= \lim_{L\to\infty} \frac{\int_{M_L} |d\mathcal{A}_L|}{\mathcal{A}(\partial B_L)} \\ &= \lim_{L\to\infty} \frac{\int_{M_L} |d\mathcal{A}_L|}{\mathfrak{s}^{n-1}(L)\mathcal{O}_{n-1}} \\ &= \lim_{L\to\infty} \frac{\int_M \left|\sum_{i=1}^n (-1)^{i-1}\mathfrak{c}^{n-i}(L)\mathfrak{s}^{i-1}(L)\Pi_{i-1}\right|}{\mathfrak{s}^{n-1}(L)\mathcal{O}_{n-1}} \\ &= \lim_{L\to\infty} \frac{1}{\mathcal{O}_{n-1}} \int_M \left|\sum_{i=1}^n (-1)^{i-1}\mathfrak{t}^{i-n}(L)\Pi_{i-1}\right|. \end{aligned}$$

Since $\lim_{L\to\infty} \mathfrak{t}^j(L) = \frac{1}{\sqrt{K^j}}$ if $Y = \mathbb{H}^n(\lambda)$, and

$$\lim_{L\to\infty} \mathfrak{t}^j(L) = \begin{cases} 0, & \text{if } j < 0; \\ 1, & \text{if } j = 0. \end{cases}$$

if $Y = \mathbb{E}^n$, the result follows. □

Corollary 1. *If M is a closed, co-oriented, and immersed hypersurface in $\mathbb{H}^n(\lambda)$ or $\mathbb{E}^n$, then*

$$\mathcal{S}(M) = \begin{cases} (-1)^{n-1}\frac{\mathcal{M}_{n-1}}{\mathcal{O}_{n-1}}, & \text{if } Y = \mathbb{E}^n; \\ \frac{\sqrt{K^{n-1}}}{\mathcal{O}_{n-1}}\left[\mathcal{M}_0 - \frac{\mathcal{M}_1}{\sqrt{K}} + \frac{\mathcal{M}_2}{K} - \frac{\mathcal{M}_3}{\sqrt{K^3}} + \cdots + (-1)^{n-1}\frac{\mathcal{M}_{n-1}}{\sqrt{K^{n-1}}}\right], & \text{if } Y = \mathbb{H}^n(\lambda). \end{cases}$$

For example, suppose that M is a sphere of radius R in $\mathbb{H}^3$, so that outward pointing normals to M determine the co-orientation on M. Then M has area $4\pi\sinh^2 R$ with constant principal curvatures $\kappa_1 = \kappa_2 = -\coth R$, so that

$$\mathcal{S}(M) = \frac{1}{4\pi}\left[4\pi\sinh^2 R + 4\pi\sinh^2 R \cdot 2\coth R + 4\pi\sinh^2 R \cdot \coth^2 R\right] = 1 + \sinh 2R + 2\sinh^2 R,$$

and so clearly $\mathcal{S}(M)$ is not constant under deformations of M. This contrasts with the case where M is smoothly isotopic to a sphere in $\mathbb{E}^3$, for then $\mathcal{S}(M) = \chi(M) = 2$ by corollary 1 and the Gauss-Bonnet theorem (see below). Note that for the hyperbolic plane $\mathbb{H}^2$, corollary 1 says that $S(M) = \frac{1}{2\pi}$(length - total curvature).

Note also that for a point sphere P in hyperbolic space, $S(M) = 1$, as might be expected since, on average, through every point on the sphere at infinity there passes exactly one normal through the (point) sphere P. However, the same can be said for any sphere, so $S(M)$ does not just count the average number of normals through a point belonging to the sphere at infinity.

Recall that $\mathcal{M}_{n-1} = \int_M \mathsf{K}\, d\mathcal{A}$ and $|\mathcal{M}_{n-1}| = \int_M |\mathsf{K}|\, d\mathcal{A}$ give the total and absolute total Gaussian curvature of M respectively.

4 Gauss-Bonnet and Curvature Measures

Theorem 2. Hopf's Theorem *[6] [7] [10]. Let M^{2k} be an even-dimensional, compact, and immersed submanifold of $\mathbb{E}^{2k+1}$ with no boundary. Then $\mathcal{M}_{2k} = \int_M \mathsf{K}\, d\mathcal{A} = \mathcal{O}_{2k}\chi(M)$. (Recall that if M^{2k-1} is odd-dimensional, then $\chi(M^{2k-1}) = 0$.)*

Theorem 3. Gauss-Bonnet Theorem *[7]. Let $M = \partial N$ where M is a compact and embedded submanifold and N a region of $\mathbb{E}^{2k+1}$. Then $\mathcal{M}_{2k} = \int_M \mathsf{K}\, d\mathcal{A} = \frac{1}{2}\mathcal{O}_{2k}\chi(N)$.*

Corollary 2. *If M is a closed, co-oriented, and immersed hypersurface of $\mathbb{E}^n$, then*

$$\mathcal{S}(M) = \mathcal{I}^+(M) = \begin{cases} \chi(M) = \mathcal{M}_{n-1}/\mathcal{O}_{n-1}, & \textit{if } n \textit{ is odd;} \\ -\mathcal{M}_{n-1}/\mathcal{O}_{n-1}, & \textit{if } n \textit{ is even.} \end{cases}$$

$$\mathcal{I}(M) = \begin{cases} 2\chi(M), & \textit{if } n \textit{ is odd;} \\ 0, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. For example,

$$\begin{aligned} \mathcal{I}^+(M) &= \lim_{L\to\infty} \frac{1}{\mathcal{V}(B_L)} \int_{X_L^+} f^\star \Phi \\ &= \lim_{L\to\infty} \frac{1}{\mathcal{V}(B_L)} \int_0^L \int_M \sum_{i=0}^{n-1} (-1)^i t^i \Pi_i \, d\mathcal{A}\, dt \\ &= \lim_{L\to\infty} \frac{n}{L^n \mathcal{O}_{n-1}} \int_0^L \int_M \sum_{i=0}^{n-1} (-1)^i t^i \Pi_i \, d\mathcal{A}\, dt \\ &= \lim_{L\to\infty} \frac{n}{L^n \mathcal{O}_{n-1}} \int_0^L \sum_{i=0}^{n-1} (-1)^i t^i \mathcal{M}_i \, dt \\ &= \lim_{L\to\infty} (-1)^{n-1} \frac{nL^n \mathcal{M}_{n-1}}{nL^n \mathcal{O}_{n-1}} \\ &= \frac{(-1)^{n-1}\mathcal{M}_{n-1}}{\mathcal{O}_{n-1}}. \end{aligned}$$

□

The reader may wish to compare these results with the equalities

$$\frac{1}{\mathcal{O}_{n-1}} \int_{p\times\partial X_{[0,\infty]}} d\mathcal{O}_{n-1} = \frac{1}{\mathcal{O}_{n-1}} \int_M \mathsf{K}\, d\mathcal{A} = \frac{\mathcal{M}_{n-1}}{\mathcal{O}_{n-1}} = (-1)^{n-1}\mathcal{I}(f,p).$$

We now wish to see how the above results extend to the hyperbolic plane and 3-space, and to do so we wish to note the following theorem.

Theorem 4. Gauss-Bonnet Theorem *[14]. Let $M^{n-1} = \partial N^n$ where M^{n-1} is a compact and embedded submanifold and N^n is a region of a non-euclidean space Y^n of constant curvature λ and dimension n, where $Y^n = \mathbb{P}^n(\lambda)$ if $\lambda > 0$ and $Y^n = \mathbb{H}^n(\lambda)$ if $\lambda < 0$. Then*

$$c_{n-1}\mathcal{M}_{n-1} + c_{n-3}\mathcal{M}_{n-3} + \cdots + c_3\mathcal{M}_3 + c_1\mathcal{M}_1 + \sqrt{K^n}\mathcal{V}(N) = \frac{1}{2}\mathcal{O}_n\chi(N)$$

for n even and

$$c_{n-1}\mathcal{M}_{n-1} + c_{n-3}\mathcal{M}_{n-3} + \cdots + c_2\mathcal{M}_2 + c_0\mathcal{M}_0 = \frac{1}{2}\mathcal{O}_n\chi(N)$$

for n odd, where

$$c_i = \frac{\mathcal{O}_n}{\mathcal{O}_i \mathcal{O}_{n-i-1}} \sqrt{K^{n-i-1}}.$$

It is timely now to mention the following interesting result [3] about curvature measures for a compact region N in $\mathbb{E}^n$ with smooth boundary:

$$\boxed{\mathcal{M}_i(\partial N) = \begin{cases} 2\mathcal{M}_i(N), & \text{if } n-i \text{ is odd;} \\ 0, & \text{if } n-i \text{ is even.} \end{cases}}$$

In particular, this says that $\chi(\partial N) = 2\chi(N)$ when n is odd.

5 Calculations for $\mathcal{N}(M)$ when $n = 2$

5.1 The euclidean case.

Theorem 5. *Let M be a closed, co-oriented, and immersed planar curve. Then*

$$\mathcal{N}(M) = \frac{1}{\pi}|\mathcal{M}_1|.$$

Proof. For large values of L,

$$\begin{aligned} \mathcal{N}(M) &= \lim_{L\to\infty} \frac{\int_M \int_{-L}^{L} |1 - t\kappa(x)|\, dt\, dx}{\pi L^2} \\ &= \pm \lim_{L\to\infty} \frac{\int_M \left[\int_{-L}^{1/\kappa(x)} (1 - t\kappa(x))\, dt - \int_{1/\kappa(x)}^{L} (1 - t\kappa(x))\, dt\right] dx}{\pi L^2}, \end{aligned}$$

where we take $+$ if $\kappa(x) > 0$ and $-$ if $\kappa(x) < 0$. Since we are taking the limit as $L \to \infty$, only the limits of integration L and $-L$ will make a non-zero contribution to the limit. Those parts of the curve where $\kappa(x) = 0$ contribute only a value of zero to $\mathcal{N}(M)$. So

$$\mathcal{N}(M) = \lim_{L\to\infty} \frac{\int_M \left|L^2\kappa(x)\right|\, dx}{\pi L^2} = \frac{1}{\pi}|\mathcal{M}_1|.$$

□

Corollary 3. *Let M be a closed, co-oriented, and immersed planar curve. Then*

$$\mathcal{N}^+(M) = \frac{1}{2\pi}|\mathcal{M}_1|.$$

Proof. The proof is similar to that for Theorem 5. □

For example, if M is locally convex, then for all points y far enough away from M, we must have $\eta_M(y) = |2c|$, where the constant c is the winding number of M. For large values of L we will have that $\lim_{L\to\infty} \frac{\text{Area}(f(X_L))}{\pi L^2} = 1$. So then

$$\mathcal{N}(M) = \lim_{L\to\infty} \frac{\int_{X_L} |\ f^\star\Phi|}{\pi L^2} = \lim_{L\to\infty} \frac{\int_{\mathbb{E}^2} \eta_M(y)\, \Phi}{\pi L^2} = \lim_{L\to\infty} \frac{|2c|\, \pi L^2}{\pi L^2} = 2|c|,$$

so that the total absolute curvature $|\mathcal{M}_1|$ equals $2\pi|c|$, a well known formulation.

The following generalizes a theorem of Alvarez [1].

5.2 The spherical case.

Theorem 6. *Let M be a closed, co-oriented, and immersed curve on $\mathbb{S}^2(\lambda)$. Then*

$$\mathcal{N}(M) = \int_{\mathbb{S}^2(\lambda)} \eta_M(y)\, dx = 4\int_M \sqrt{\frac{\kappa^2(x)}{\lambda} + 1}\, dx.$$

Proof. We can write

$$\begin{aligned}\int_{\mathbb{S}^2(\lambda)} \eta_M(y)\, dx &= \int_M \int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}} \left|\left(\cos(t\sqrt{\lambda}) - \frac{\kappa_x(0)}{\sqrt{\lambda}}\sin(t\sqrt{\lambda})\right)\right| dt\, dx \\ &= \int_M \int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}} |\mathfrak{c}(t) - \kappa_x(0)\mathfrak{s}(t)|\; dt\, dx \\ &= 4\int_M \sqrt{\frac{\kappa^2(x)}{\lambda} + 1}\, dx.\end{aligned}$$

□

For example, if M is a geodesic circle in $\mathbb{S}^2$, then $\kappa(x) = 0$ along M and $\eta_M(y) = 2$ for all points $y \in \mathbb{S}^2$. So $\int_{\mathbb{S}^2} \eta_M(y)\, \Phi = 2 \cdot 4\pi$ while $4\int_M \sqrt{\kappa^2(x)+1}\, dx = 4 \cdot 2\pi$.

5.3 The hyperbolic case.

Theorem 7. *Let M be a closed, co-oriented, reversible, and immersed curve M in $\mathbb{H}^2(\lambda)$. Then*

$$\int_M \sqrt{\frac{\kappa^2}{K} - 1}\, dx = \int_{M_t} \sqrt{\frac{\kappa_t^2}{K} - 1}\, dx = |\mathcal{M}_1|$$

is a geometric invariant for the family of wave fronts to M.

Proof. Since M is reversible, $\kappa^2 > K$ along M. Now $\int_M \sqrt{\frac{\kappa^2}{K} - 1}\, dx$ is a geometric invariant for the wave fronts to M since

$$\int_M \sqrt{\frac{\kappa^2}{K} - 1}\, dx = \int_M \sqrt{\frac{1}{K\mathfrak{t}^2(R)} - 1}\, dx = \int_M d\alpha = \int_{M_t} d\alpha_t = \int_{M_t} \sqrt{\frac{\kappa_t^2}{K} - 1}\, dx.$$

Note that $|\mathcal{M}_1| = \int_M d\alpha = \int_{M_t} d\alpha_t$ as any normal to M is a normal to M_t and visa-versa. □

The calculations for $\mathcal{N}(M)$ when $\lambda < 0$ are a bit more complex than when $\lambda = 0$. Recall that $\lambda = -K$ if $\lambda < 0$. In $\mathbb{H}^2(\lambda)$ the volume of a hyperbolic ball B_L of radius L is given by

$$\mathcal{V}(B_L) = 2\pi \int_0^L \mathfrak{s}(t)\, dt = \frac{2\pi}{K}(\mathfrak{c}(L) - 1).$$

Now suppose that $\kappa^2(x) \leq K$. Then

$$\begin{aligned}
\mathcal{N}(M) &= \lim_{L\to\infty} \frac{1}{\mathcal{V}(B_L)} \int_{X_L} |f^\star \Phi| \\
&= \lim_{L\to\infty} \frac{K \int_M \int_{-L}^{L} (\mathfrak{c}(t) - \kappa(x)\mathfrak{s}(t))\, dt\, dx}{2\pi(\mathfrak{c}(L)-1)} \\
&= \lim_{L\to\infty} \int_{-L}^{L} \left(\frac{K\mathcal{M}_0 \mathfrak{c}(t)}{2\pi\mathfrak{c}(L)} - \frac{K\mathcal{M}_1 \mathfrak{s}(t)}{2\pi\mathfrak{c}(L)} \right) dt \\
&= \lim_{L\to\infty} \left(\frac{2\sqrt{K}\mathcal{M}_0 \mathfrak{s}(L)}{2\pi\mathfrak{c}(L)} \right) \\
&= \frac{\sqrt{K}\mathcal{M}_0}{\pi} = \frac{\sqrt{K} l}{\pi}.
\end{aligned}$$

One may similarly show that

$$\mathcal{N}^+(M) = \frac{1}{2\pi} \left[\sqrt{K}\mathcal{M}_0 - \mathcal{M}_1 \right].$$

Now suppose that $\kappa^2(x) > K$. Then

$$\mathcal{N}(M) = \lim_{L\to\infty} \frac{K \int_M \int_{-L}^{L} |\mathfrak{c}(t) - \kappa(x)\mathfrak{s}(t)|\, dt\, dx}{2\pi(\mathfrak{c}(L)-1)}$$

and so (for L large enough) $\int_{-L}^{L} |\mathfrak{c}(t) - \kappa(x)\mathfrak{s}(t)|\, dt =$

$$\pm \left[\int_{-L}^{\mathfrak{t}^{-1}(1/\kappa(x))} [\mathfrak{c}(t) - \kappa(x)\mathfrak{s}(t)]\ dt - \int_{\mathfrak{t}^{-1}(1/\kappa(x))}^{L} [\mathfrak{c}(t) - \kappa(x)\mathfrak{s}(t)]\ dt \right]$$

where we take $+$ if $\kappa(x) > \sqrt{K}$ and $-$ if $\kappa(x) < -\sqrt{K}$. So

$$\int_{-L}^{L} |\mathfrak{c}(t) - \kappa(x)\mathfrak{s}(t)|\, dx = \pm 2 \left[-\frac{\kappa^2(x)}{K\sqrt{\kappa^2(x) - K}} + \frac{1}{\sqrt{\kappa^2(x) - K}} + \frac{\kappa(x)}{K}\mathfrak{c}(L) \right].$$

Thus

$$\mathcal{N}(M) = \lim_{L\to\infty} \frac{K \int_M 2|\kappa(x)|\mathfrak{c}(L)\, dx}{2K\pi(\mathfrak{c}(L)-1)} = \frac{1}{\pi}\int_M |\kappa(x)|\, dx = \frac{|\mathcal{M}_1|}{\pi}.$$

Similarly one may show that

$$\mathcal{N}^+(M) = \begin{cases} \frac{1}{2\pi}\left[\mathcal{M}_1 - \sqrt{K}\mathcal{M}_0\right], & \text{if } \kappa(x) > \sqrt{K}; \\ \frac{1}{2\pi}\left[\sqrt{K}\mathcal{M}_0 - \mathcal{M}_1\right], & \text{if } \kappa(x) < -\sqrt{K}. \end{cases}$$

Now let

$$M = M^1 \cup M^2 \cup M^3$$

where

$$M^1 = \{x \in M \mid \kappa^2(x) \leq K\}$$

$$M^2 = \{x \in M \mid \kappa(x) > \sqrt{K}\}$$

$$M^3 = \{x \in M \mid \kappa(x) < -\sqrt{K}\}$$

are disjoint, co-oriented, and immersed curves, and so our calculations above apply to each such manifold (note that M^2 and M^3 are both reversible). Let $\mathcal{M}_i^j$ denote $\int_{M^j} \Pi_i \, d\mathcal{A}$ and $|\mathcal{M}_i^j|$ denote $\int_{M^j} |\Pi_i| \, d\mathcal{A}$. We have then proved the following theorem.

Theorem 8. *Let M be a closed, co-oriented, and immersed curve in the hyperbolic plane of curvature $-K$. Then*

$$\mathcal{N}(M) = \frac{1}{\pi}\left(|\mathcal{M}_1^2| + |\mathcal{M}_1^3|\right) + \frac{\sqrt{K}}{\pi}\mathcal{M}_0^1$$

and

$$\mathcal{N}^+(M) = \frac{\sqrt{K}}{2\pi}\left(\mathcal{M}_0^1 - \mathcal{M}_0^2 + \mathcal{M}_0^3\right) + \frac{1}{2\pi}\left(-\mathcal{M}_1^1 + \mathcal{M}_1^2 - \mathcal{M}_1^3\right).$$

The Gauss-Bonnet formula for a convex polygonal region Q in $\mathbb{H}^2$ says that $\int_{\partial Q} \kappa \, dx - \mathcal{F} = 2\pi\chi(Q)$ where $\mathcal{F}$ is the area of Q. So Theorem 8 implies that $\mathcal{N}(\partial Q) = 2 + \frac{l+\mathcal{F}}{\pi}$ where l is the length of ∂Q.

6 Calculations for $\mathcal{N}(M)$ when $n = 3$

In $\mathbb{H}^3(\lambda)$ the volume of a hyperbolic ball B_L of radius L is given by

$$\begin{aligned}
\mathcal{V}(B_L) &= \mathcal{O}_2 \int_0^L \mathfrak{s}^2(t)\, dt \\
&= 4\pi \left[\frac{1}{2K}\mathfrak{c}(t)\mathfrak{s}(t) - \frac{t}{2K}\right]_0^L \\
&= \frac{2\pi}{K}\mathfrak{c}(L)\mathfrak{s}(L) - \frac{2\pi L}{K}.
\end{aligned}$$

Now $\mathcal{N}(M)$ is the limit as $L \to \infty$ of

$$\frac{1}{\mathcal{V}(B_L)} \int_M \int_{-L}^{L} |\mathfrak{c}(t) - \kappa_1(x)\mathfrak{s}(t)||\mathfrak{c}(t) - \kappa_2(x)\mathfrak{s}(t)|\, dt\, d\mathcal{A}.$$

The antiderivative of $f_x(t) = [\mathfrak{c}(t) - \kappa_1(x)\mathfrak{s}(t)]\,[\mathfrak{c}(t) - \kappa_2(x)\mathfrak{s}(t)]$ is

$$\frac{t}{2} + \frac{1}{2}\mathfrak{s}(t)\mathfrak{c}(t) - \frac{\kappa_1(x)+\kappa_2(x)}{4K}(\mathfrak{c}^2(t) + K\mathfrak{s}^2(t)) + \frac{\kappa_1(x)\kappa_2(x)}{2K}(\mathfrak{c}(t)\mathfrak{s}(t) - t)$$

and so in the limit, as $L \to \infty$, the only limits of integration that will contribute a non-zero value to $\mathcal{N}(M)$ are $\pm L$. There are, however, several cases to consider.

Suppose first that $\kappa_1^2(x) > K$ and $\kappa_2^2(x) > K$, so that $\mathfrak{c}(t) - \kappa_1(x)\mathfrak{s}(t)$ and $\mathfrak{c}(t) - \kappa_2(x)\mathfrak{s}(t)$ are each sometimes negative somewhere in the interval of integration $(-L, L)$, for large values of L. So we may write in place of $\int_{-L}^{L} |f_x(t)|\, dt$ the following:

$$\pm\left[L + \mathfrak{s}(L)\mathfrak{c}(L) + \frac{\kappa_1(x)\kappa_2(x)}{K}(\mathfrak{c}(L)\mathfrak{s}(L) - L)\right],$$

where we use $+$ if both $\kappa_1(x)$ and $\kappa_2(x)$ have the same sign, and we use $-$ otherwise. Again, removing those terms which contribute zero to the limit, we obtain

$$\mathcal{N}(M) = \pm \lim_{L\to\infty} \int_M \frac{K\mathfrak{c}(L)\mathfrak{s}(L)\left(1 + \frac{\kappa_1(x)\kappa_2(x)}{K}\right)}{2\pi\mathfrak{s}(L)\mathfrak{c}(L)}\, d\mathcal{A} =$$
$$= \pm \int_M \left(\frac{K}{2\pi} + \frac{\kappa_1(x)\kappa_2(x)}{2\pi}\right) d\mathcal{A} = \frac{1}{2\pi}\int_M |K + \kappa_1(x)\kappa_2(x)|\, d\mathcal{A}.$$

Now what if exactly one of the principal curvatures satisfies the equation $\kappa_i^2(x) \leq K$? WLOG let us suppose that $\kappa_1^2(x) \leq K, \kappa_2^2(x) > K$. So we may write in place of $\int_{-L}^{L} |f_x(t)|\, dt$ the following:

$$\pm\frac{\kappa_1(x) + \kappa_2(x)}{2K}(\mathfrak{c}^2(L) + K\mathfrak{s}^2(L)),$$

where we use $+$ if $\kappa_2(x)$ is positive and $-$ otherwise. Again, removing those terms which contribute zero to the limit, we obtain

$$\mathcal{N}(M) = \pm \lim_{L\to\infty} \int_M \frac{\left(\frac{\kappa_1(x)+\kappa_2(x)}{2K}\right)(\mathfrak{c}^2(L) + K\mathfrak{s}^2(L))}{\frac{2\pi}{K}\mathfrak{s}(L)\mathfrak{c}(L)}\, d\mathcal{A} =$$
$$= \pm\sqrt{K}\int_M \frac{\kappa_1(x) + \kappa_2(x)}{2\pi}\, d\mathcal{A} = \frac{\sqrt{K}}{2\pi}\int_M |\kappa_1(x) + \kappa_2(x)|\, d\mathcal{A} = \frac{\sqrt{K}}{2\pi}|\mathcal{M}_1|.$$

Finally, if both $\kappa_1^2(x) \leq K$ and $\kappa_2^2(x) \leq K$, then $\mathcal{N}(M)$ is the limit of

$$\frac{\int_M \int_{-L}^{L} (\mathfrak{c}(t) - \kappa_1(x)\mathfrak{s}(t))\,(\mathfrak{c}(t) - \kappa_2(x)\mathfrak{s}(t))\, dt\, d\mathcal{A}}{\mathcal{V}(B_L)}$$

as $L \to \infty$, which gives us (much as in the first case)

$$\mathcal{N}(M) = \lim_{L\to\infty} \int_M \frac{K\mathfrak{c}(L)\mathfrak{s}(L)\left(1 + \frac{\kappa_1(x)\kappa_2(x)}{K}\right)}{2\pi\mathfrak{s}(L)\mathfrak{c}(L)}\, d\mathcal{A} =$$
$$= \int_M \frac{1}{2\pi}\left(K + \kappa_1(x)\kappa_2(x)\right) d\mathcal{A} = \frac{K\mathcal{M}_0}{2\pi} + \frac{\mathcal{M}_2}{2\pi}.$$

In conclusion, we have that

$$M = M^1 \cup M^2 \cup M^3$$

where

$$\begin{aligned} M^1 &= \{x \in M \mid \kappa_1^2(x) > K, \kappa_2^2(x) > K\} \\ M^2 &= \{x \in M \mid \kappa_1^2(x) > K, \kappa_2^2(x) \leq K\} \cup \{x \in M \mid \kappa_1^2(x) \leq K, \kappa_2^2(x) > K\} \\ M^3 &= \{x \in M \mid \kappa_1^2(x) \leq K, \kappa_2^2(x) \leq K\} \end{aligned}$$

are disjoint, co-oriented, and immersed manifolds and so our calculations above apply to each such manifold. We then have that $\mathcal{N}(M) = \mathcal{N}(M^1) + \mathcal{N}(M^2) + \mathcal{N}(M^3)$, and so have proved the following theorem.

Theorem 9. *Let M be a closed, co-oriented, and immersed hypersurface in hyperbolic space of constant curvature $-K$. Then*

$$\mathcal{N}(M) = \frac{1}{2\pi}\int_{M^1} |K + \kappa_1(x)\kappa_2(x)|\, d\mathcal{A} + \frac{\sqrt{K}}{2\pi}\int_{M^2} |\kappa_1(x) + \kappa_2(x)|\, d\mathcal{A} + \frac{1}{2\pi}\int_{M^3} (K + \kappa_1(x)\kappa_2(x))\, d\mathcal{A}.$$

The Gauss-Bonnet formula for the boundary $M = \partial Q$ of a convex polygonal region Q in $\mathbb{H}^3$ says that $\mathcal{M}_2 - \mathcal{M}_0 = 4\pi\chi(Q)$. Now $\int_{M^1} |K + \kappa_1(x)\kappa_2(x)|\, d\mathcal{A} = |\mathcal{M}_2| = \mathcal{M}_2$ as the absolute total curvature is concentrated at the vertices, $\int_{M^2} |\kappa_1(x) + \kappa_2(x)|\, d\mathcal{A} = |\mathcal{M}_1| = -\mathcal{M}_1$ as the mean curvatures is concentrated along the edges, and $\int_{M^3} (K + \kappa_1(x)\kappa_2(x))\, d\mathcal{A} = \mathcal{M}_0$ as the area is concentrated along the faces. So $\mathcal{N}(\partial Q) = 2 + \frac{2\mathcal{M}_0 - \mathcal{M}_1}{2\pi}$.

The proofs of the following corollaries are quite similar to the proof given for Theorem 9.

Corollary 4. *Let M be a closed, co-oriented, and immersed hypersurface in hyperbolic space of constant curvature $-K$. Then*

$$\mathcal{N}^+(M) = \frac{K}{4\pi}\int_M \left|\Pi_0 - \frac{\Pi_1}{\sqrt{K}} + \frac{\Pi_2}{K}\right| d\mathcal{A}.$$

Corollary 5. *Let M be a closed, co-oriented, and immersed hypersurface in $\mathbb{E}^n$. Then*

$$\mathcal{N}(M) = 2\frac{|\mathcal{M}_{n-1}|}{\mathcal{O}_{n-1}}.$$

Corollary 6. *Let M be a closed, co-oriented, and immersed hypersurface in $\mathbb{E}^n$. Then*

$$\mathcal{N}^+(M) = \frac{|\mathcal{M}_{n-1}|}{\mathcal{O}_{n-1}}.$$

7 Calculations for $\mathcal{I}(M)$

Recall that we have already found formulations for $\mathcal{I}(M)$ for euclidean spaces. In [13] Santalo calculated $I(M)$ for projective space $\mathbb{P}^n$ where $M = \partial Q$, Q a connected domain with class C^3 boundary. Below we generalize to the case of closed and immersed hypersurfaces of $\mathbb{S}^n(\lambda)$.

Lemma 2. *For $k \in \{2, 3, 4, \dots\}$ and $l \in \{1, 2, 3, \dots, k-1\}$,*

$$\frac{\mathcal{O}_{2k+1}}{\mathcal{O}_{2l}\mathcal{O}_{2k-2l}} = \frac{\pi}{2^{l+1}} \frac{(2l-1)!!(2k-2l-1)!!}{l!(2k)(2k-2)(2k-4)\cdots(2l+2)}.$$

Proof.

$$\frac{\mathcal{O}_{2k+1}}{\mathcal{O}_{2l}\mathcal{O}_{2k-2l}} = \frac{1}{2}\frac{\Gamma(l+\frac{1}{2})\Gamma(k-l+\frac{1}{2})}{\Gamma(k+1)} = \frac{\pi}{2^{k+1}}\frac{(2l-1)!!(2k-2l-1)!!}{k!}.$$

Now since $2^k k! = 2^l l!(2k)(2k-2)\cdots(2l+2)$ (proof by double induction) the lemma is proved. □

For the spherical case when $\lambda > 0$,

$$\mathcal{I}(M) = \frac{\sqrt{\lambda^n}}{\mathcal{O}_n}\int_{X_{\pi/\sqrt{\lambda}}} f^\star\Phi = \frac{\sqrt{\lambda^n}}{\mathcal{O}_n}\int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}}\int_M d\mathcal{A}_t\,dt =$$
$$= \sqrt{\lambda^n}\sum_{j=0}^{n-1}(-1)^j\frac{\mathcal{M}_j}{\mathcal{O}_n}\int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}}\mathfrak{c}^{n-1-j}(t)\mathfrak{s}^j(t)\,dt.$$

The terms $\int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}}\mathfrak{c}^{n-1-j}(t)\mathfrak{s}^j(t)\,dt$ will be zero whenever j is odd, so we can assume that j is even. So if $j = 2l$ is even and $n = 2k+1$ is odd ($l = 0, 1, 2, \dots, k$), then $n-1-j = 2k-2l$ is even and formulas (3) and (5) imply that

$$\begin{aligned}
\int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}}\mathfrak{c}^{n-1-j}(t)\mathfrak{s}^j(t)\,dt &= \int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}}\mathfrak{c}^{2k-2l}(t)\mathfrak{s}^{2l}(t)\,dt \\
&= \frac{(2k-2l-1)!!}{(2k)(2k-2)\cdots(2l+2)\sqrt{\lambda^{2l+1}}}\int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}}\mathfrak{s}^{2l}(t)\,dt \\
&= \frac{(2k-2l-1)!!}{(2k)(2k-2)\cdots(2l+2)}\frac{(2l-1)!!}{2^l l!\lambda^{l+1}}2\pi.
\end{aligned}$$

This can be simplified to

$$\frac{4}{\lambda^{l+1}}\frac{\mathcal{O}_n}{\mathcal{O}_{2l}\mathcal{O}_{2k-2l}}$$

by the lemma above.

Now if $j = 2l$ is even and $n = 2k+2$ is even ($l = 0, 1, 2, \ldots, k$), then $n - 1 - j = 2k + 1 - 2l$ is odd and formula (4) implies that

$$\int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}} \mathfrak{c}^{n-1-j}(t)\mathfrak{s}^j(t)\,dt = \int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}} \mathfrak{c}^{2k+1-2l}(t)\mathfrak{s}^{2l}(t)\,dt$$

vanishes.

So in the elliptic case when $n = 2k+1$ is odd, we have shown that

$$\begin{aligned}\mathcal{I}(M) &= \frac{\sqrt{\lambda^n}}{\mathcal{O}_n}\int_X f^\star\Phi = \frac{\sqrt{\lambda^n}}{\mathcal{O}_n}\int_{\frac{\pi}{\sqrt{\lambda}}}^{\frac{\pi}{\sqrt{\lambda}}}\int_M d\mathcal{A}_t\,dt = \sqrt{\lambda^n}\sum_{l=0}^{k}\frac{\mathcal{M}_{2l}}{\mathcal{O}_n}\frac{4}{\lambda^{l+1}}\frac{\mathcal{O}_n}{\mathcal{O}_{2l}\mathcal{O}_{n-1-2l}} \\ &= 4\sqrt{\lambda^n}\sum_{l=0}^{k}\frac{\mathcal{M}_{2l}}{\lambda^{l+1}\mathcal{O}_{2l}\mathcal{O}_{n-1-2l}} = \frac{4}{\sqrt{\lambda}}\sum_{l=0}^{k}\lambda^{(n-1-2l)/2}\frac{\mathcal{M}_{2l}}{\mathcal{O}_{2l}\mathcal{O}_{n-1-2l}} = \\ &= \frac{4}{\sqrt{\lambda}\mathcal{O}_n}(c_{n-1}\mathcal{M}_{n-1} + c_{n-3}\mathcal{M}_{n-3} + \cdots + c_2\mathcal{M}_2 + c_0\mathcal{M}_0).\end{aligned}$$

We have thus proved the following theorem.

Theorem 10. *Let M be a closed, co-oriented, and immersed hypersurface in $\mathbb{S}^n(\lambda)$. If n is even, then $\mathcal{I}(M) = 0$. Otherwise, if n is odd, then*

$$\mathcal{I}(M) = \frac{4}{\sqrt{\lambda}\mathcal{O}_n}(c_{n-1}\mathcal{M}_{n-1} + c_{n-3}\mathcal{M}_{n-3} + \cdots + c_2\mathcal{M}_2 + c_0\mathcal{M}_0)$$

where the constants c_i are those from the Gauss-Bonnet Theorem. If $M^{2n} = \partial N$, then $\mathcal{I}(M) = 2\chi(N)/\sqrt{\lambda}$.

7.1 The hyperbolic case.

If $\lambda < 0$, then let B_L denote a hyperball of radius L containing M, and recall that X_L denotes the fiber subspace of X where each fiber is isomorphic to $[-L, L]$. We want to compute

$$\mathcal{I}(M) = \lim_{L\to\infty}\frac{\int_{X_L} f^\star\Phi}{\mathcal{V}(B_L)} = \lim_{L\to\infty}\frac{\int_{-L}^{L}\int_M d\mathcal{A}_t\,dt}{\mathcal{V}(B_L)}.$$

Now

$$\int_M d\mathcal{A}_t = \sum_{j=0}^{n-1}(-1)^j\mathcal{M}_j\mathfrak{c}^{n-1-j}(t)\mathfrak{s}^j(t)$$

and the volume of a hyperbolic n-dimensional ball B_L of radius L is given by

$$\mathcal{V}(B_L) = \mathcal{O}_{n-1}\int_0^L \mathfrak{s}^{n-1}(t)\,dt.$$

However, the only term of $\mathcal{V}(B_L)$ that will be of significance in the limit is

$$\mathcal{O}_{n-1}\int_0^L\left(\frac{e^{t\sqrt{K}}}{2\sqrt{K}}\right)^{n-1}dt$$

which can be replaced by

$$\frac{\mathcal{O}_{n-1}e^{(n-1)L\sqrt{K}}}{(n-1)\sqrt{K^n}2^{n-1}}.$$

Now for any n we can write

$$\mathcal{I}(M) = \lim_{L\to\infty} \frac{\int_{X_L} f^\star\Phi}{\mathcal{V}(B_L)} = \lim_{L\to\infty} \frac{(n-1)\sqrt{K^n}2^{n-1}\int_{-L}^{L}\int_M d\mathcal{A}_t\,dt}{\mathcal{O}_{n-1}e^{(n-1)L\sqrt{K}}}$$

which is the limit of the following integral as $L \to \infty$:

$$\sum_{j=0}^{n-1} \frac{(-1)^j(n-1)\sqrt{K^n}2^{n-1}\mathcal{M}_j}{\mathcal{O}_{n-1}e^{(n-1)L\sqrt{K}}} \int_{-L}^{L} \mathfrak{c}^{n-1-j}(t)\mathfrak{s}^j(t)\,dt.$$

The terms

$$\int_{-L}^{L} \mathfrak{c}^{n-1-j}(t)\mathfrak{s}^j(t)\,dt$$

will be zero whenever j is odd. Assuming that j is even, in the limit we can replace the integral immediately above with

$$\int_{-L}^{L} \left[\frac{e^{(n-1)t\sqrt{K}}}{2^{n-1}\sqrt{K^j}} + \frac{e^{(1-n)t\sqrt{K}}}{2^{n-1}\sqrt{K^j}}\right] dt.$$

We can simplify, in the limit, as follows:

$$\lim_{L\to\infty} \frac{\int_{-L}^{L} \mathfrak{c}^{n-1-j}(t)\mathfrak{s}^j(t)\,dt}{\mathcal{V}(B_L)} = \lim_{L\to\infty} \frac{2e^{(n-1)L\sqrt{K}}}{(n-1)2^{n-1}\sqrt{K^{j+1}}} \cdot \frac{(n-1)2^{n-1}\sqrt{K^n}}{\mathcal{O}_{n-1}e^{(n-1)L\sqrt{K}}} = $$
$$= 2\frac{\sqrt{K^{n-1-j}}}{\mathcal{O}_{n-1}}$$

So we have proved the following theorem.

Theorem 11. *Let M be a closed, co-oriented, and immersed hypersurface in hyperbolic space of constant curvature $-K$. Then*

$$\mathcal{I}(M) = 2\frac{\sqrt{K^{n-1}}}{\mathcal{O}_{n-1}}\left(\frac{\mathcal{M}_{n-1}}{\sqrt{K^{n-1}}} + \frac{\mathcal{M}_{n-3}}{\sqrt{K^{n-3}}} + \cdots + \frac{\mathcal{M}_2}{K} + \mathcal{M}_0\right)$$

if n is odd, and

$$\mathcal{I}(M) = 2\frac{\sqrt{K^{n-1}}}{\mathcal{O}_{n-1}}\left(\frac{\mathcal{M}_{n-1}}{\sqrt{K^{n-1}}} + \frac{\mathcal{M}_{n-3}}{\sqrt{K^{n-3}}} + \cdots + \frac{\mathcal{M}_3}{\sqrt{K^3}} + \frac{\mathcal{M}_1}{\sqrt{K}}\right)$$

if n is even. (Compare with Corollary 2.)

Corollary 7. *Let M be a closed, co-oriented, and immersed hypersurface in hyperbolic space of constant curvature $-K$. Then*

$$\mathcal{I}^+(M) = \frac{\sqrt{K^{n-1}}}{\mathcal{O}_{n-1}} \left(\frac{\mathcal{M}_{n-1}}{\sqrt{K^{n-1}}} + \frac{\mathcal{M}_{n-3}}{\sqrt{K^{n-3}}} + \cdots + \frac{\mathcal{M}_2}{K} + \mathcal{M}_0 \right)$$

if n is odd, and

$$\mathcal{I}^+(M) = \frac{\sqrt{K^{n-1}}}{\mathcal{O}_{n-1}} \left(\frac{\mathcal{M}_{n-1}}{\sqrt{K^{n-1}}} + \frac{\mathcal{M}_{n-3}}{\sqrt{K^{n-3}}} + \cdots + \frac{\mathcal{M}_3}{\sqrt{K^3}} + \frac{\mathcal{M}_1}{\sqrt{K}} \right)$$

if n is even.

*Proof.*For any n we can write

$$\mathcal{I}^+(M) = \lim_{L\to\infty} \frac{\int_{X_L^+} f^\star \Phi}{\mathcal{V}(B_L)} = \lim_{L\to\infty} \frac{(n-1)\sqrt{K^n} 2^{n-1} \int_0^L \int_M d\mathcal{A}_t \, dt}{\mathcal{O}_{n-1} e^{(n-1)L\sqrt{K}}}$$

which is the limit of the following integral as $L \to \infty$:

$$\sum_{j=0}^{n-1} \frac{(-1)^j (n-1)\sqrt{K^n} 2^{n-1} \mathcal{M}_j}{\mathcal{O}_{n-1} e^{(n-1)L\sqrt{K}}} \int_0^L \mathfrak{c}^{n-1-j}(t) \mathfrak{s}^j(t) \, dt.$$

The terms

$$\int_0^L \mathfrak{c}^{n-1-j}(t) \mathfrak{s}^j(t) \, dt$$

can be replaced in the limit with

$$\int_0^L \frac{e^{(n-1)t\sqrt{K}}}{2^{n-1}\sqrt{K^j}} \, dt.$$

We can simplify, in the limit, as follows:

$$\lim_{L\to\infty} \frac{\int_0^L \mathfrak{c}^{n-1-j}(t)\mathfrak{s}^j(t)\, dt}{\mathcal{V}(B_L)} = \lim_{L\to\infty} \frac{e^{(n-1)L\sqrt{K}}}{(n-1)2^{n-1}\sqrt{K^{j+1}}} \cdot \frac{(n-1)2^{n-1}\sqrt{K^n}}{\mathcal{O}_{n-1} e^{(n-1)L\sqrt{K}}} = \\ = \frac{\sqrt{K^{n-1-j}}}{\mathcal{O}_{n-1}}$$

□

8 Construction of the Hyperbolic Functions

Following the notation of Fenchel [5], we define the *extended complex cylinder*

$$\mathbb{A}_\infty = \mathbb{A} \cup \{+\infty, -\infty\}$$

as the two-point compactification of the additive group $\mathbb{A} = \mathbb{C}/(2\pi i\mathbb{Z}) = \mathbb{R}\oplus i\mathbb{R}/(2\pi\mathbb{Z})$. We can then perform arithmetic on the two points at infinity in the obvious way, such as $\theta + \infty = +\infty$, where $\theta \in \mathbb{A}_\infty \setminus \{-\infty\}$. Let $\mathbb{C}_\infty$ be the Riemann sphere, or one point compactification $\mathbb{C} \cup \infty$ of the additive group $\mathbb{C}$, and give $\mathbb{A}_\infty$ and $\mathbb{C}_\infty$ their usual topologies. We can lift the exponential map $\exp : \mathbb{A} \to \mathbb{C}$ to a homeomorphism $\exp : \mathbb{A}_\infty \to \mathbb{C}_\infty$ with inverse $\log : \mathbb{C}_\infty \to \mathbb{A}_\infty$ by defining $\exp(-\infty) = 0$ and $\exp(+\infty) = +\infty$.

We can then define the following hyperbolic trigonometric functions from $\mathbb{A}_\infty$ to $\mathbb{C}_\infty$, each determining a two-sheeted covering of $\mathbb{C}_\infty$ with two branch points each ($\pm i$ for $\sinh\theta$ and ± 1 for the other three):

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2} \qquad \cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$$
$$\tanh\theta = \frac{e^{2\theta} - 1}{e^{2\theta} + 1} \qquad \cosh\theta = \frac{e^{2\theta} + 1}{e^{2\theta} - 1}$$

Their inverses will be two-valued, but we will define them (as well as the hyperbolic trigonometric functions themselves) on extended real values with principal values as determined below:

$$\begin{array}{lcl}
\theta = \sinh^{-1}\phi \text{ for } \phi \in \mathbb{R}\cup\{\pm\infty\} & \iff & \theta \in \mathbb{R}\cup\{\pm\infty\} \text{ and } \sinh\theta = \phi \\
\theta = \cosh^{-1}\phi \text{ for } \phi \in [1, +\infty] & \iff & 0 \le \theta \le +\infty \text{ and } \cosh\theta = \phi \\
\theta = \tanh^{-1}\phi \text{ for } -1 \le \phi \le 1 & \iff & -\infty \le \theta \le +\infty \text{ and } \tanh\theta = \phi \\
\theta = \coth^{-1}\phi \text{ for } \phi \le -1, \phi \ge 1 & \iff & -\infty \le \theta < 0, 0 < \theta \le +\infty \text{ and } \coth\theta = \phi
\end{array}$$

We may then write, for θ real:

$$\begin{array}{ll}
\sinh\theta = -i\sin(i\theta) & \tanh\theta = -i\tan(i\theta) \\
\cosh\theta = \cos(i\theta) & \coth\theta = i\cot(i\theta) \\
\tanh^{-1}\theta = \frac{\tan^{-1}(i\theta)}{i} & \sinh^{-1}\theta = -i\sin^{-1}(i\theta) \\
\coth^{-1}\theta = \frac{\cot^{-1}(-i\theta)}{i} & \cosh^{-1}\theta = i\cos^{-1}\theta
\end{array}$$

If $\lambda < 0$, then the operators $\mathfrak{s}$, $\mathfrak{c}$, and $\mathfrak{t}$ on $\mathbb{R}\cup\{+\infty, -\infty\}$ satisfy the following formulas.

$$\begin{array}{ll}
\mathfrak{s}(t) = \frac{\sinh(t\sqrt{K})}{\sqrt{K}} = \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} & \text{and} \quad \mathfrak{s}^{-1}(t) = \frac{1}{\sqrt{\lambda}}\sin^{-1}(\sqrt{\lambda}t) \\
\mathfrak{c}(t) = \cosh(t\sqrt{K}) = \cos(t\sqrt{\lambda}) & \\
\mathfrak{t}(t) = \frac{\tanh(t\sqrt{K})}{\sqrt{K}} = \frac{\tan(t\sqrt{\lambda})}{\sqrt{\lambda}} & \text{and} \quad \mathfrak{t}^{-1}(t) = \frac{1}{\sqrt{\lambda}}\tan^{-1}(\sqrt{\lambda}t)
\end{array}$$

A few other useful identities follow.

$$\tanh^{-1}\theta = \sinh^{-1}\frac{\theta}{\sqrt{1-\theta^2}} = \cosh^{-1}\frac{1}{\sqrt{1-\theta^2}}$$

$$\frac{d\mathfrak{s}}{d\theta} = \mathfrak{c}(\theta) \qquad \frac{d\mathfrak{c}}{d\theta} = \lambda\mathfrak{s}(\theta)$$

$$\mathfrak{c}^2 + K\mathfrak{s}^2 = 1$$

So one may show that

$$\mathfrak{s}\left(\mathfrak{t}^{-1}(\theta)\right) = \frac{\theta}{\sqrt{1-K\theta^2}} \quad \text{and} \quad \mathfrak{c}\left(\mathfrak{t}^{-1}(\theta)\right) = \frac{1}{\sqrt{1-K\theta^2}}.$$

9 More Useful Identities

More useful identities are as follows, where n is a natural number.

$$\mathcal{O}_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \tag{9.1}$$

where $\mathcal{O}_n$ is the surface area of the n-dimensional unit sphere $\mathbb{S}^n$ and Γ is the gamma function:

$$\Gamma(n) = (n-1)! \quad \text{and} \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}. \tag{9.2}$$

For $p \neq -2, -4, -6, \dots, -2n$, we have

$$\int \mathfrak{s}^p(t)\mathfrak{c}^{2n}(t)\,dt = \tag{9.3}$$

$$\frac{\mathfrak{s}^{p+1}(t)}{2n+p}\left[\mathfrak{c}^{2n-1}(t) + \sum_{i=1}^{n-1} \frac{(2n-1)(2n-3)\cdots(2n-2i+1)}{(2n+p-2)(2n+p-4)\cdots(2n+p-2i)}\mathfrak{c}^{2n-2i-1}(t)\right] + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\cdots(p+2)\sqrt{\lambda}}\int \mathfrak{s}^p(t)\,dt$$

For $p \neq -1, -3, -5, \dots, -2n+1$, we have

$$\int \mathfrak{s}^p(t)\mathfrak{c}^{2n+1}(t)\,dt = \tag{9.4}$$

$$\frac{\mathfrak{s}^{p+1}(t)}{2n+p+1}\left[\mathfrak{c}^{2n}(t) + \sum_{i=1}^{n} \frac{2^i n(n-1)(n-2)\cdots(n-i+1)}{(2n+p-1)(2n+p-3)\cdots(2n+p-2i+1)}\mathfrak{c}^{2n-2i}(t)\right]$$

For l a natural number we have

$$\int \mathfrak{s}^{2l}(t)\,dt = \tag{9.5}$$

$$-\frac{\mathfrak{c}(t)}{2l\lambda}\left[\mathfrak{s}^{2l-1}(t) + \sum_{i=1}^{l-1} \frac{(2l-1)(2l-3)\cdots(2l-2i+1)}{(2\lambda)^i(l-1)(l-2)\cdots(l-i)}\mathfrak{s}^{2l-2i-1}(t)\right] + \frac{(2l-1)!!}{2^l l!\lambda^l}t$$

If $B_r \subset \mathbb{E}^n$ is a solid ball of radius r, then

$$\mathrm{Vol}(B_r) = \frac{\pi^{n/2}r^n}{(n/2)!} = \frac{r^n}{n}\mathcal{O}_{n-1}. \tag{9.6}$$

If B_r is a solid ball of radius r in a oriented non-euclidean space Y^n of constant curvature $\lambda < 0$, then

$$\text{(9.7)} \qquad \mathrm{Vol}(B_r) = \mathcal{O}_{n-1} \int_0^r \mathfrak{s}^{n-1}(t)\, dt.$$

Formulas (1) and (2) were adapted from [16], formulas (3), (4) and (5) were adapted from [8], and formulas (6) and (7) were taken from [7].

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