On some curves of $AW(k)$-type

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Abstract

In this study, firstly, we consider Bertrand curves $\gamma : I \to \mathbb{E}^3$ with $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$. Then we show that there is no such a Bertrand curve of $AW(1)$-type and γ is of $AW(3)$-type if and only if it is a right circular helix. Furthermore, we study weak $AW(2)$-type and $AW(3)$-type conical geodesic curves in $\mathbb{E}^3$. We find the first and second curvatures of a conical geodesic curve to be of $AW(3)$-type.

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1 Introduction

In [5], K. Arslan and A. West defined the notion of $AW(k)$-type submanifolds. Since then, many works have been done related to $AW(k)$-type submanifolds, (see, for example, [1], [2], [3] and [4]). In [2], K. Arslan and the first author studied curves and surfaces of $AW(k)$-type. In this study, our aim is to carry out the results which were given in [2] to Bertrand curves and new special curves defined in [10] by S. Izumiya and N. Takeuchi.

2 Basic notions and properties

Let $\gamma : I \subseteq \mathbb{E} \to \mathbb{E}^n$ be a unit speed curve in $\mathbb{E}^n$. The curve γ is called a Frenet curve of osculating order d if its higher order derivatives $\gamma'(s), \gamma''(s), \gamma'''(s), ..., \gamma^{(d)}(s)$ are linearly dependent and $\gamma'(s), \gamma''(s), \gamma'''(s), ..., \gamma^{(d+1)}(s)$ are no longer linearly independent for all $s \in I$. To each Frenet curve of order d one can associate an orthonormal d-frame $\upsilon_1, \upsilon_2, ..., \upsilon_d$ along γ (such that $\gamma'(s) = \upsilon_1$) called the Frenet frame and $d-1$ functions $\kappa_1, \kappa_2, ..., \kappa_{d-1} : I \to \mathbb{R}$, called the Frenet curvatures, such that the Frenet formulas are defined in the usual way;

$$D_{\upsilon_1}\gamma'(s) = \kappa_1(s)\upsilon_2(s),$$

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$$D_{\upsilon_1}\upsilon_2(s) = -\kappa_1(s)\upsilon_1(s) + \kappa_2(s)\upsilon_3(s),$$

....

$$D_{\upsilon_1}\upsilon_i(s) = -\kappa_{i-1}(s)\upsilon_{i-1}(s) + \kappa_i(s)\upsilon_{i+1}(s),$$

$$D_{\upsilon_1}\upsilon_{i+1}(s) = -\kappa_i(s)\upsilon_i(s),$$

where D is the Levi-Civita connection of $\mathbb{E}^n$.

From now on we consider Frenet curves of osculating order 3 of $\mathbb{E}^n$. First we start with some well-known results;

Proposition 2.1. *Let γ be a Frenet curve of osculating order 3 then we have*

$$\gamma'(s) = \upsilon_1 \quad ; \quad \gamma''(s) = D_{\upsilon_1}\gamma'(s) = \kappa_1(s)\upsilon_2,$$

$$\gamma'''(s) = D_{\upsilon_1}D_{\upsilon_1}\gamma'(s) = -\kappa_1^2(s)\upsilon_1 + \kappa_1'(s)\upsilon_2 + \kappa_1(s)\kappa_2(s)\upsilon_3,$$

$$\begin{aligned}\gamma''''(s) = D_{\upsilon_1}D_{\upsilon_1}D_{\upsilon_1}\gamma'(s) = -3\kappa_1(s)\kappa_1'(s)\upsilon_1 + \{\kappa_1''(s) - \kappa_1^3(s) - \kappa_1(s)\kappa_2^2(s)\}\upsilon_2 \\ +\{2\kappa_1'(s)\kappa_2(s) + \kappa_2'(s)\kappa_1(s)\}\upsilon_3.\end{aligned}$$

Notation: Let us write

$$N_1(s) = \kappa_1(s)\upsilon_2, \tag{2.1}$$

$$N_2(s) = \kappa_1'(s)\upsilon_2 + \kappa_1(s)\kappa_2(s)\upsilon_3, \tag{2.2}$$

$$N_3(s) = \{\kappa_1''(s) - \kappa_1^3(s) - \kappa_1(s)\kappa_2^2(s)\}\upsilon_2 + \{2\kappa_1'(s)\kappa_2(s) + \kappa_2'(s)\kappa_1(s)\}\upsilon_3. \tag{2.3}$$

Corollary 2.2. *$\gamma'(s), \gamma''(s), \gamma'''(s)$ and $\gamma''''(s)$ are linearly dependent if and only if $N_1(s)$, $N_2(s)$ and $N_3(s)$ are linearly dependent.*

3 Curves of $AW(k)$-type

In the present section, we introduce Bertrand curves, slant helices, conical geodesic curves and give the main results.

A curve $\gamma : I \to \mathbb{E}^3$ with $\kappa_1(s) \neq 0$ is called a *cylindrical helix* if the tangent lines of γ make a constant angle with a fixed direction. It has been well-known that the curve $\gamma(s)$ is a cylindrical helix if and only if $(\frac{\kappa_1}{\kappa_2})(s) =$ constant. If both $\kappa_1(s) \neq 0$ and $\kappa_2(s)$ are constant, it is, of course, a cylindrical helix. We call such a curve a *circular helix*. In [10], Izumiya and Takeuchi defined new special curves named slant helix and conical geodesic curve. A curve γ with $\kappa_1(s) \neq 0$ is called a *slant helix* if the principal normal lines of γ make a constant angle with a fixed direction. From this definition, Izumiya and Takeuchi obtained the following characterization:

Let $\gamma : I \to \mathbb{E}^3$ be a curve with $\kappa_1(s) \neq 0$. Then γ is a slant helix if and only if

$$\sigma(s) = \left(\frac{\kappa_1^2}{(\kappa_1^2 + \kappa_2^2)^{\frac{3}{2}}} \left(\frac{\kappa_2}{\kappa_1} \right)' \right)(s) \tag{3.4}$$

is a constant function.

Furthermore, a curve $\gamma : I \to \mathbb{E}^3$ with $\kappa_1(s) \neq 0$ is called a *conical geodesic curve* if $(\frac{\kappa_2}{\kappa_1})'$ is a constant function.

On the other hand, a curve $\gamma : I \to \mathbb{E}^3$ with $\kappa_1(s) \neq 0$ is called a Bertrand curve if there exist a curve $\overline{\gamma} : I \to \mathbb{E}^3$ such that the principal normal lines of γ and $\overline{\gamma}$ at $s \in I$ are equal. In this case $\overline{\gamma}$ is called a Bertrand mate of γ. Bertrand curves have the following properties (see, [7] and [9]):

Proposition 3.1. *Let $\gamma : I \to \mathbb{E}^3$ be a curve.*

i) Suppose that $\kappa_2(s) \neq 0$. Then γ is a Bertrand curve if and only if there exist nonzero real numbers A, B such that $A\kappa_1(s) + B\kappa_2(s) = 1$ for any $s \in I$. It follows from this fact that a circular helix is a Bertrand curve.

ii) Suppose that $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$. Then γ is a Bertrand curve if and only if there exist a nonzero real number A such that

$$A\left(\kappa_2'(s)\kappa_1(s) - \kappa_1'(s)\kappa_2(s)\right) - \kappa_2'(s) = 0. \tag{3.5}$$

In this case the Bertrand mate of γ is given by $\overline{\gamma}(s) = \gamma(s) + Av_2(s)$.

For more details related to special curves see also [8] and [10].

On the other hand, from (3.4) one can get easily the following proposition:

Proposition 3.2. *Let $\gamma : I \to \mathbb{E}^3$ be a curve with $\kappa_1(s) \neq 0$. Then γ is a slant helix if and only if there is a real number c such that*

$$\kappa_2'(s)\kappa_1(s) - \kappa_1'(s)\kappa_2(s) = c(\kappa_1^2(s) + \kappa_2^2(s))^{\frac{3}{2}}. \tag{3.6}$$

So using the above proposition we have:

Proposition 3.3. *Let $\gamma : I \to \mathbb{E}^3$ be a Bertrand curve with $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$. If γ is not a cylindrical helix and if there is a nonzero real number C such that*

$$\kappa_2'(s) = C(\kappa_1^2(s) + \kappa_2^2(s))^{\frac{3}{2}}, \tag{3.7}$$

then γ is a slant helix.

Proof. Since γ is a Bertrand curve with $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$ the condition (3.5) holds. Furthermore, if γ is not a cylindrical helix then $\kappa_2'(s)\kappa_1(s) - \kappa_1'(s)\kappa_2(s) \neq 0$. So from (3.5), there is a nonzero real number A such that

$$\kappa_2'(s) = A\left(\kappa_2'(s)\kappa_1(s) - \kappa_1'(s)\kappa_2(s)\right). \tag{3.8}$$

If the condition (3.7) is satisfied then from (3.6) and (3.8) it is easy to see that γ is a slant helix. □

Definition 3.4. *Frenet curves (of osculating order 3) are*
i) of type weak $AW(2)$ if they satisfy

$$N_3(s) = \langle N_3(s), N_2^*(s)\rangle N_2^*(s), \tag{3.9}$$

ii) of type weak $AW(3)$ if they satisfy

$$N_3(s) = \langle N_3(s), N_1^*(s)\rangle N_1^*(s), \tag{3.10}$$

where

$$N_1^*(s) = \frac{N_1(s)}{\|N_1(s)\|}, \tag{3.11}$$

and

$$N_2^*(s) = \frac{N_2(s) - \langle N_2(s), N_1^*(s)\rangle N_1^*(s)}{\|N_2(s) - \langle N_2(s), N_1^*(s)\rangle N_1^*(s)\|}, \tag{3.12}$$

(see, [2]).

Definition 3.5. *Frenet curves are*
i) of type $AW(1)$ if they satisfy $N_3(s) = 0$,
ii) of type $AW(2)$ if they satisfy

$$\|N_2(s)\|^2 N_3(s) = \langle N_3(s), N_2(s)\rangle N_2(s), \tag{3.13}$$

iii) of type $AW(3)$ if they satisfy

$$\|N_1(s)\|^2 N_3(s) = \langle N_3(s), N_1(s)\rangle N_1(s), \tag{3.14}$$

(see [5], for the general case).

Using the above definitions, we have the following propositions:

Proposition 3.6. *[2]. Let γ be a Frenet curve of order 3. Then γ is of type $AW(1)$ if and only if*

$$\kappa_1''(s) - \kappa_1^3(s) - \kappa_2^2(s)\kappa_1(s) = 0 \quad ; \quad \kappa_2(s) = \tfrac{c}{\kappa_1^2},\ c = \text{constant}. \tag{3.15}$$

Proposition 3.7. *[2]. Let γ be a Frenet curve of order 3. Then γ is of type $AW(2)$ if and only if*

$$2(\kappa_1'(s))^2\kappa_2(s) + \kappa_1(s)\kappa_1'(s)\kappa_2'(s) = \kappa_1''(s)\kappa_1(s)\kappa_2(s) - \kappa_1^4(s)\kappa_2(s) - \kappa_1^2(s)\kappa_2^3(s). \tag{3.16}$$

Proposition 3.8. *[2]. Let γ be a Frenet curve of order 3. Then γ is of type $AW(3)$ if and only if*

$$2\kappa_2(s)\kappa_1'(s) + \kappa_2'(s)\kappa_1(s) = 0, \tag{3.17}$$

and the solution of this differential equation is $\kappa_2(s) = \frac{c}{\kappa_1^2}$, c =constant.

Proposition 3.9. *[2]. Let γ be a Frenet curve of order 3. Then γ is of weak $AW(2)$-type if and only if*

$$\kappa_1''(s) - \kappa_1^3(s) - \kappa_1(s)\kappa_2^2(s) = 0. \tag{3.18}$$

Theorem 3.10. *There is no any Bertrand curve $\gamma : I \to \mathbb{E}^3$ with $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$ of $AW(1)$-type.*

Proof. Now suppose that $\gamma : I \to \mathbb{E}^3$ is a Bertrand curve of type $AW(1)$ with $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$. Then the equations (3.5) and (3.15) hold on γ. Differentiating the equation

$$\kappa_2(s) = \frac{c}{\kappa_1^2} \tag{3.19}$$

we get

$$\kappa_2'(s) = \frac{-2c\kappa_1'}{\kappa_1^3}. \tag{3.20}$$

So substituting (3.19) and (3.20) into (3.5) we obtain

$$\kappa_1(s) = \frac{2}{3A}. \tag{3.21}$$

Hence substituting (3.21) into the first equation of (3.15) we have

$$\frac{8}{27A^3} + \frac{27A^3c^2}{8} = 0,$$

which gives us $A^6 < 0$. Since A is a real number, this is impossible. So γ cannot be a $AW(1)$-type Bertrand curve. This proves the theorem. □

Proposition 3.11. *Let $\gamma : I \to \mathbb{E}^3$ be a Bertrand curve with $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$. Then γ is of $AW(2)$-type if and only if there is a nonzero real number A such that*

$$\begin{aligned} &(\kappa_1'(s))^2\kappa_2(s)(2\text{-}A\kappa_1(s)) + A\kappa_1^2(s)\kappa_1'(s)\kappa_2'(s) \\ =\ &\kappa_1''(s)\kappa_1(s)\kappa_2(s)\text{-}\kappa_1^4(s)\kappa_2(s)\text{-}\kappa_1^2(s)\kappa_2^3(s). \end{aligned} \tag{3.22}$$

Proof. Assume that γ is of $AW(2)$-type Bertrand curve with $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$. Then (3.5) and (3.16) hold on γ. So by the common solution of these equations, one can easily get (3.22). □

Theorem 3.12. *Let $\gamma : I \to \mathbb{E}^3$ be a Bertrand curve with $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$. Then γ is of $AW(3)$-type if and only if γ is a right circular helix.*

Proof. Now suppose that $\gamma : I \to \mathbb{E}^3$ is a Bertrand curve of $AW(3)$-type with $\kappa_1(s) \neq 0$ and $\kappa_2(s) \neq 0$. Then the equations (3.5) and (3.17) hold on γ. Since the equation (3.19) is also a solution for $AW(3)$-type curves, we substitute the equations (3.19) and (3.20) into (3.5) we find again (3.21). So substituting (3.21) into (3.19) we get

$$\kappa_2(s) = \frac{9A^2c}{4}.$$

Since $\kappa_1(s)$ and $\kappa_2(s)$ are nonzero constants, γ is a right circular helix. The converse statement is trivial. Hence our theorem is proved. □

Proposition 3.13. *Let $\gamma : I \to \mathbb{E}^3$ be a conical geodesic curve with $\kappa_2(s) \neq 0$. Then γ is of weak $AW(2)$-type if and only if there exists a real number c such that*

$$k_2''(s) - k_1^2(s)k_2(s) - k_2^3 = ck_1'(s). \tag{3.23}$$

Proof. Since γ is a conical geodesic curve, by definition there is a real number c_1 such that

$$k_2'(s)k_1(s) - k_1'(s)k_2(s) = c_1k_1^2(s). \tag{3.24}$$

Differentiating (3.24) we have

$$k_2''(s)k_1(s) - k_1''(s)k_2(s) = 2c_1k_1(s)k_1'(s). \tag{3.25}$$

If we put $2c_1 = c$ then combining (3.25) and (3.18) we get (3.23). □

Corollary 3.14. *Let $\gamma : I \to \mathbb{E}^3$ be a weak $AW(2)$-type conical geodesic curve. If $k_2(s)$ is nonzero constant then*

$$k_1(s) = \tan\left(\frac{c_1(s+c)}{c_2}\right)c_1, \tag{3.26}$$

where c, c_1 and c_2 are real constants.

Proof. If $k_2(s)$ is nonzero constant, say $k_2(s) = c_1$ then from (3.23), we have

$$-c_1(k_1^2(s) + c_1^2) = ck_1'(s).$$

Putting $c_2 = -\frac{c}{c_1}$, the last equation can be written as

$$(k_1^2(s) + c_1^2) = c_2k_1'(s).$$

Hence the solution of the last differential equation gives us (3.26). □

Theorem 3.15. *Let $\gamma : I \to \mathbb{E}^3$ be a conical geodesic curve. Then γ is of $AW(3)$-type if and only if the curvatures of γ are of the form*

$$k_1(s) = (c_1s + c_2)^{\frac{-1}{3}} \tag{3.27}$$

and

$$k_2(s) = c(c_1s + c_2)^{\frac{2}{3}}, \tag{3.28}$$

where c, c_1 and c_2 are real constants.

Proof. Since γ is of type $AW(3)$, the condition (3.17) holds on γ. On the other hand, since γ is a conical geodesic curve, the condition (3.24) also holds. So combining (3.17) and (3.24) we obtain there exist a real number C such that

$$3k_1'(s) = Ck_1^4(s).$$

So the solution of the last differential equation gives us (3.27). Hence, from Proposition 3.8 we obtain (3.28). This proves the theorem. □

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