Infinitesimal holomorphically projective transformations on tangent bundles with complete lift connection

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Dedicated to Professor Shigeyoshi Fujimura on his sixtieth birthday

Abstract

Let $(M,\ g)$ be a Riemannian manifold and TM its tangent bundle with complete lift connection and adapted almost complex structure. We determine the infinitesimal holomorphically projective transformation on TM. Furthermore, if TM admits a non-affine infinitesimal holomorphically projective transformation, then M and TM are locally flat.

M.S.C. 2000: 53C20, 53C15.
Key words: infinitesimal holomorphically projective transformation, complete lift connection, adapted almost complex structure.

§1. Introduction

Let M be an n-dimensional manifold and TM its tangent bundle. We denote by $\mathfrak{T}_s^r(M)$ the set of all tensor fields of type $(r,\ s)$ on M. Similarly, we denote by $\mathfrak{T}_s^r(TM)$ the corresponding set on TM.

Let ∇ be an affine connection on M. A vector field V on M is called an *infinitesimal projective transformation* if there exists a 1-form Ω on M such that

$$(L_V\nabla)(X,\ Y) = \Omega(X)Y + \Omega(Y)X$$

for any $X,\ Y \in \mathfrak{T}_0^1(M)$, where L_V is the Lie derivation with respect to V. In this case Ω is called the *associated 1-form* of V. Especially, if $\Omega = 0$, then V is called an *infinitesimal affine transformation.*

Next let $(M,\ J)$ be an almost complex manifold with affine connection ∇. A vector field V on M is called an *infinitesimal holomorphically projective transformation* if there exists a 1-form Ω on M such that

$$(L_V\nabla)(X,\ Y) = \Omega(X)Y + \Omega(Y)X - \Omega(JX)JY - \Omega(JY)JX$$

for any $X,\ Y \in \mathfrak{T}_0^1(M)$. In this case Ω is also called the *associated 1-form* of V. Especially, if $\Omega = 0$, then V is the infinitesimal affine transformation. S. Ishihara [3] has

Differential Geometry - Dynamical Systems, Vol.7, 2005, pp. 42-48.

introduced the notion of infinitesimal holomorphically projective transformation, and S. Tachibana and S. Ishihara [6] investigated infinitesimal holomorphically projective transformations on Kaehlerian manifolds. In [1] we have proved that (1) an infinitesimal holomorphically projective transformation is infinitesimal isometry on a compact Kaehlerian manifold with non-positive constant scalar curvature and (2) a compact Kaehlerian manifold M with constant scalar curvature is holomorphically isometric to a complex projective space with Fubini-Study metric if M admits a non-isometric infinitesimal holomorphically projective transformation.

It is well-known that there are several lift connections of ∇ on TM ([7, 8]). In our previous paper [2], we study the infinitesimal holomorphically projective transformation on TM with horizontal lift connection and proved the following:

Theorem A. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with horizontal lift connection and adapted almost complex structure. A vector field $\widetilde{V}$ is an infinitesimal holomorphically projective transformation with the associated 1-form $\widetilde{\Omega}$ on TM if and only if there exist $\varphi,\ \psi \in \mathfrak{T}_0^0(M),\ B = (B^h),\ D = (D^h) \in \mathfrak{T}_0^1(M),\ A = (A_i{}^h),\ C = (C_i{}^h) \in \mathfrak{T}_1^1(M)$ satisfying*

$$(1)\quad (\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) = (B^h + y^a A_a{}^h + 2\varphi y^h - y^h y^a \Psi_a,\ D^h + y^a C_a{}^h + 2\psi y^h + y^h y^a \Phi_a),$$

$$(2)\quad (\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar{i}}) = (\partial_i \psi,\ \partial_i \varphi) = (\Psi_i,\ \Phi_i),$$

$$(3)\quad \nabla_j \Phi_i = 0, \qquad \nabla_j \Psi_i = 0,$$

$$(4)\quad \nabla_j A_i{}^h = \Phi_i \delta_j^h - \Phi_j \delta_i^h,$$

$$(5)\quad \nabla_j C_i{}^h = \Psi_i \delta_j^h - \Psi_j \delta_i^h - K_{aji}{}^h B^a,$$

$$(6)\quad L_B \Gamma_{ji}{}^h = \nabla_j \nabla_i B^h + K_{aji}{}^h B^a = \Psi_j \delta_i^h + \Psi_i \delta_j^h,$$

$$(7)\quad \nabla_j \nabla_i D^h = -\Phi_j \delta_i^h - \Phi_i \delta_j^h,$$

$$(8)\quad A_k{}^a K_{aji}{}^h + 2\varphi K_{kji}{}^h = 0,$$

$$(9)\quad \Psi_l K_{kji}{}^h = 0,$$

where $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) := \widetilde{V}^a E_a + \widetilde{V}^{\bar{a}} E_{\bar{a}} = \widetilde{V},\ (\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar{i}}) := \widetilde{\Omega}_a dx^a + \widetilde{\Omega}_{\bar{a}} \delta y^a = \widetilde{\Omega},\ \nabla$ denotes the Riemannian connection on M, $\Gamma_{ji}{}^h$ the coefficients of ∇ and $K = (K_{kji}{}^h)$ the Riemannian curvature tensor of (M, g) defined by $K_{kji}{}^h := \partial_k \Gamma_{ji}{}^h - \partial_j \Gamma_{ki}{}^h + \Gamma_{ji}{}^a \Gamma_{ka}{}^h - \Gamma_{ki}{}^a \Gamma_{ja}{}^h$.

Theorem B. *Let (M, g) be a complete Riemannian manifold and TM its tangent bundle with horizontal lift connection and adapted almost complex structure. If TM admits a non-affine infinitesimal holomorphically projective transformation, then M and TM are locally flat.*

Let (M, g) be a Riemannian manifold and ∇ its Riemannian connection. The horizontal lift connection of ∇ does not coincide with the Riemannian connection of horizontal lift metric of g to TM with respect to ∇. Two connections coincide if and only if M is locally flat. On the other hand the complete lift connection of

∇ is the Riemannian connection of complete lift metric of g. Here, since g satisfies $\nabla g = 0$, the complete lift metric of g coincides with the horizontal lift metric of g (see [7, 8]). Moreover, in the case of horizontal lift connection it is necessary to assume that M is complete to prove Theorem B, but not necessary in the case of complete lift connection (see Theorem 2 stated below). Therefore, in this paper we investigate the case of complete lift connection and prove the following:

Theorem 1. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with complete lift connection and adapted almost complex structure. A vector field $\widetilde{V}$ is an infinitesimal holomorphically projective transformation with associated 1-form $\widetilde{\Omega}$ on TM if and only if there exist $\varphi,\ \psi \in \mathfrak{T}^0_0(M),\ B = (B^h),\ D = (D^h) \in \mathfrak{T}^1_0(M),\ A = (A_i{}^h),\ C = (C_i{}^h) \in \mathfrak{T}^1_1(M)$ satisfying*

$$(1)\quad (\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) = (B^h + y^a A_a{}^h + 2\varphi y^h - y^h y^a \Psi_a,\ D^h + y^a C_a{}^h + 2\psi y^h + y^h y^a \Phi_a),$$

$$(2)\quad (\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar{i}}) = (\partial_i \psi,\ \partial_i \varphi) = (\Psi_i,\ \Phi_i),$$

$$(3)\quad \nabla_j \Phi_i = 0, \qquad \nabla_j \Psi_i = 0,$$

$$(4)\quad \nabla_j A_i{}^h = \Phi_i \delta^h_j - \Phi_j \delta^h_i,$$

$$(5)\quad \nabla_j C_i{}^h = \Psi_i \delta^h_j - \Psi_j \delta^h_i - K_{aji}{}^h B^a,$$

$$(6)\quad L_B \Gamma_{ji}{}^h = \nabla_j \nabla_i B^h + K_{aji}{}^h B^a = \Psi_j \delta^h_i + \Psi_i \delta^h_j,$$

$$(7)\quad L_D \Gamma_{ji}{}^h = \nabla_j \nabla_i D^h + K_{aji}{}^h D^a = -\Phi_j \delta^h_i - \Phi_i \delta^h_j,$$

$$(8)\quad A_k{}^a K_{aji}{}^h + 2\varphi K_{kji}{}^h = 0,$$

$$(9)\quad \Phi_l K_{kji}{}^h = 0, \qquad \Psi_l K_{kji}{}^h = 0,$$

$$(10)\quad B^a \nabla_a K_{kji}{}^h = K_{kji}{}^a C_a{}^h - K_{aji}{}^h C_k{}^a - K_{kai}{}^h \nabla_j B^a - K_{kja}{}^h \nabla_i B^a,$$

where $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) := \widetilde{V}^a E_a + \widetilde{V}^{\bar{a}} E_{\bar{a}} = \widetilde{V}$ and $(\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar{i}}) := \widetilde{\Omega}_a dx^a + \widetilde{\Omega}_{\bar{a}} \delta y^a = \widetilde{\Omega}$.

Theorem 2. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with complete lift connection and adapted almost complex structure. If TM admits a non-affine infinitesimal holomorphically projective transformation, then M and TM are locally flat.*

In the present paper everything will be always discussed in the C^∞-category, and manifolds will be assumed to be connected and dimension $n > 1$.

The authors would like to express their hearty thanks to the referee for the valuable suggestions.

§2. Preliminaries

In this section we shall give some definitions and formulae on TM for later use (for details, see [7, 8]). Let (M, g) be a Riemannian manifold, ∇ the Riemannian connection of g and $\Gamma_{ji}{}^h$ the coefficients of ∇, i.e., $\Gamma_{ji}{}^a \partial_a := \nabla_{\partial_j} \partial_i$, where $\partial_h = \frac{\partial}{\partial x^h}$ and (x^h) is the local coordinates of M.

Adapted frame of TM.

We define a local frame $\{E_i,\ E_{\bar{i}}\}$ of TM as follows:

$$E_i := \partial_i - y^b \Gamma_{ib}{}^a \partial_{\bar{a}} \quad \text{and} \quad E_{\bar{i}} := \partial_{\bar{i}}, \tag{2.1}$$

where $(x^h,\ y^h)$ is the induced coordinates of TM derived from the local coordinates (x^h) of M and $\partial_{\bar{i}} := \frac{\partial}{\partial y^i}$. This frame $\{E_i,\ E_{\bar{i}}\}$ is called the *adapted frame* of TM. Then $\{dx^h,\ \delta y^h\}$ is the dual frame of $\{E_i,\ E_{\bar{i}}\}$, where $\delta y^h := dy^h + y^b \Gamma_{ab}{}^h dx^a$.

By the definition of the adapted frame, we have the following

Lemma 1 *The Lie brackets of the adapted frame of TM satisfy the following identities:*

(1) $[E_j,\ E_i] = y^b K_{ijb}{}^a E_{\bar{a}}$,
(2) $[E_j,\ E_{\bar{i}}] = \Gamma_{ji}{}^a E_{\bar{a}}$,
(3) $[E_{\bar{j}},\ E_{\bar{i}}] = 0$.

Complete lift connection of ∇

Let $X = X^a \partial_a$ be a vector field on M. Then the *complete lift* X^C of X is defined by

$$X^C := X^a E_a + y^b \nabla_b X^a E_{\bar{a}}. \tag{2.2}$$

There exists a unique affine connection $\widetilde{\nabla}$ on TM which satisfies

$$\widetilde{\nabla}_{X^C} Y^C = (\nabla_X Y)^C \tag{2.3}$$

for any $X,\ Y \in \mathfrak{T}^1_0(M)$. This affine connection is called the *complete lift connection* of ∇ to TM. Then we have

$$\begin{aligned} &\widetilde{\nabla}_{E_j} E_i = \Gamma_{ji}{}^a E_a + y^b K_{bji}{}^a E_{\bar{a}}, \\ &\widetilde{\nabla}_{E_j} E_{\bar{i}} = \Gamma_{ji}{}^a E_{\bar{a}}, \\ &\widetilde{\nabla}_{E_{\bar{j}}} E_i = 0, \qquad \widetilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}} = 0. \end{aligned} \tag{2.4}$$

Adapted almost complex structure on TM

Let $X = X^a \partial_a$ be a vector field on M. Then the vertical lift X^V and the horizontal lift X^H of X with respect to ∇ are defined as follows:

$$X^H := X^a E_a \quad \text{and} \quad X^V := X^a E_{\bar{a}}. \tag{2.5}$$

We now define a tensor field J of type $(1,1)$ on TM by

$$JX^H := X^V, \qquad JX^V := -X^H \tag{2.6}$$

for any $X \in \mathfrak{T}^1_0(M)$, i.e.,

$$JE_i = E_{\bar{i}} \quad \text{and} \quad JE_{\bar{i}} = -E_i.$$

Then we obtain

$$J^2 = -I.$$

Therefore J is an almost complex structure on TM. This almost complex structure is called the *adapted almost complex structure*. It is known that J is integrable if and only if M is locally flat.

§3. Proofs of Theorems

Proof of Theorem 1.

Here we prove only the necessary condition because it is easy to prove the sufficient condition. Let $\widetilde{V}$ be an infinitesimal holomorphically projective transformation with the associated 1-form $\widetilde{\Omega}$ on TM.

$$(3.1)\qquad (L_{\widetilde{V}}\widetilde{\nabla})(\widetilde{X},\ \widetilde{Y}) = \widetilde{\Omega}(\widetilde{X})\widetilde{Y} + \widetilde{\Omega}(\widetilde{Y})\widetilde{X} - \widetilde{\Omega}(J\widetilde{X})J\widetilde{Y} - \widetilde{\Omega}(J\widetilde{Y})J\widetilde{X}$$

for any $\widetilde{X},\ \widetilde{Y} \in \mathfrak{T}^1_0(TM)$.

From $(L_{\widetilde{V}}\widetilde{\nabla})(E_{\bar{j}},\ E_{\bar{i}}) = \widetilde{\Omega}_{\bar{j}}E_{\bar{i}} + \widetilde{\Omega}_{\bar{i}}E_{\bar{j}} - \widetilde{\Omega}_j E_i - \widetilde{\Omega}_i E_j$, we obtain

$$(3.2)\qquad \partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^h = -\widetilde{\Omega}_j\delta^h_i - \widetilde{\Omega}_i\delta^h_j$$

and

$$(3.3)\qquad \partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^{\bar{h}} = \widetilde{\Omega}_{\bar{j}}\delta^h_i + \widetilde{\Omega}_{\bar{i}}\delta^h_j.$$

From (3.2), there exist $\varphi \in \mathfrak{T}^0_0(M)$, $\Psi = (\Psi_i) \in \mathfrak{T}^0_1(M)$, $B = (B^h) \in \mathfrak{T}^1_0(M)$ and $A = ({A_i}^h) \in \mathfrak{T}^1_1(M)$ satisfying

$$(3.4)\qquad \widetilde{\psi} = -\varphi + y^a\Psi_a,$$

$$(3.5)\qquad \widetilde{\Omega}_i = \partial_{\bar{i}}\widetilde{\psi} = \Psi_i$$

and

$$(3.6)\qquad \widetilde{V}^h = B^h + y^a {A_a}^h + 2\varphi y^h - y^a\Psi_a y^h,$$

where $\widetilde{\psi} := -\dfrac{1}{n+1}\partial_{\bar{a}}\widetilde{V}^a$.

Similarly, from (3.3), there exist $\psi \in \mathfrak{T}^0_0(M)$, $\Phi = (\Phi_i) \in \mathfrak{T}^0_1(M)$, $D = (D^h) \in \mathfrak{T}^1_0(M)$ and $C = ({C_i}^h) \in \mathfrak{T}^1_1(M)$ satisfying

$$(3.7)\qquad \widetilde{\varphi} = \psi + y^a\Phi_a,$$

$$(3.8)\qquad \widetilde{\Omega}_{\bar{i}} = \partial_{\bar{i}}\widetilde{\varphi} = \Phi_i$$

and

$$(3.9)\qquad \widetilde{V}^{\bar{h}} = D^h + y^a {C_a}^h + 2\psi y^h + y^a\Phi_a y^h,$$

where $\widetilde{\varphi} := \dfrac{1}{n+1}\partial_{\bar{a}}\widetilde{V}^{\bar{a}}$.

Next, from (3.1) we have

$$(L_{\widetilde{V}}\widetilde{\nabla})(E_{\bar{j}},\ E_i) = \Phi_j E_i + \Phi_i E_j + \Psi_j E_{\bar{i}} + \Psi_i E_{\bar{j}} \tag{3.10}$$

or

$$(L_{\widetilde{V}}\widetilde{\nabla})(E_j,\ E_{\bar{i}}) = \Phi_j E_i + \Phi_i E_j + \Psi_j E_{\bar{i}} + \Psi_i E_{\bar{j}},$$

from which, we get

$$\begin{aligned}
&(\Phi_j \delta_i^a + \Phi_i \delta_j^a)E_a + (\Psi_j \delta_i^a + \Psi_i \delta_j^a)E_{\bar{a}} \\
&= \{(\nabla_j A_i{}^a + 2\delta_i^a \partial_j \varphi) - y^b(\delta_b^a \nabla_j \Psi_i + \delta_i^a \nabla_j \Psi_b)\}E_a \\
&\quad + \{(K_{bji}{}^a B^b + \nabla_j C_i{}^a + 2\delta_i^a \partial_j \psi) \\
&\qquad + y^b(A_b{}^c K_{cji}{}^a - A_i{}^c K_{jbc}{}^a + 4\varphi K_{bji}{}^a + \delta_b^a \nabla_j \Phi_i + \delta_i^a \nabla_j \Phi_b) \\
&\qquad + y^c y^b(\Psi_i K_{jcb}{}^a - 2\Psi_c K_{bji}{}^a)\}E_{\bar{a}}.
\end{aligned} \tag{3.11}$$

Comparing both hands of the above equation, we obtain

$$\begin{aligned}
&\Phi_i = \partial_i \varphi, \qquad \nabla_j \Phi_i = 0, \\
&\Psi_i = \partial_i \psi, \qquad \nabla_j \Psi_i = 0, \\
&\nabla_j A_i{}^h = \Phi_i \delta_j^h - \Phi_j \delta_i^h, \\
&\nabla_j C_i{}^h = \Psi_i \delta_j^h - \Psi_j \delta_i^h - K_{aji}{}^h B^a, \\
&A_k{}^a K_{aji}{}^h = -2\varphi K_{kji}{}^h, \quad \Psi_l K_{kji}{}^h = 0.
\end{aligned} \tag{3.12}$$

Lastly, from $(L_{\widetilde{V}}\widetilde{\nabla})(E_j,\ E_i) = \Psi_j E_i + \Psi_i E_j - \Phi_j E_{\bar{i}} - \Phi_i E_{\bar{j}}$, we obtain

$$\begin{aligned}
&(\Psi_j \delta_i^a + \Psi_i \delta_i^a)E_a - (\Phi_j \delta_i^a + \Phi_i \delta_j^a)E_{\bar{a}} \\
&= (L_B \Gamma_{ji}{}^a)E_a + \{L_D \Gamma_{ji}{}^a \\
&\qquad + y^b(B^c \nabla_c K_{bji}{}^a - K_{bji}{}^c C_c{}^a + K_{cji}{}^a C_b{}^c + K_{bci}{}^a \nabla_j B^c + K_{bjc}{}^a \nabla_i B^c) \\
&\qquad + y^c y^b(\Phi_c K_{bji}{}^a + \Phi_c K_{bij}{}^a - \Phi_j K_{cib}{}^a - \Phi_i K_{cjb}{}^a \\
&\qquad\quad + 2\varphi \nabla_c K_{bji}{}^a + 2\varphi \nabla_j K_{cib}{}^a + A_c{}^d \nabla_d K_{bji}{}^a + A_c{}^d \nabla_j K_{dib}{}^a)\}E_{\bar{a}},
\end{aligned}$$

from which, we get the following important information:

$$L_B \Gamma_{ji}{}^h = \Psi_j \delta_i^h + \Psi_i \delta_j^h. \tag{3.13}$$

$$L_D \Gamma_{ji}{}^h = -\Phi_j \delta_i^h - \Phi_i \delta_j^h. \tag{3.14}$$

(That is, B and D are infinitesimal projective transformations on M.)

$$B^a \nabla_a K_{kji}{}^h = K_{kji}{}^a C_a{}^h - K_{aji}{}^h C_k{}^a - K_{kai}{}^h \nabla_j B^a - K_{kja}{}^h \nabla_i B^a. \tag{3.15}$$

$$\Phi_l K_{kji}{}^h = 0. \tag{3.16}$$

This completes the proof. □

Proof of Theorem 2.

Let $\widetilde{V}$ be a non-affine infinitesimal holomorphically projective transformation on TM. Using (3) in Theorem 1, we have $\nabla_i\|\Phi\|^2 = \nabla_i\|\Psi\|^2 = 0$. Therefore, $\|\Phi\|$ and $\|\Psi\|$ are constant on M. Suppose that M is not locally flat, then $\Phi = \Psi = 0$ by virtue of (9) in Theorem 1, that is, $\widetilde{V}$ is an infinitesimal affine transformation. This is a contradiction. Therefore M is locally flat.

In this case, TM is also locally flat, because the Riemannian curvature tensor of $\widetilde{\nabla}$ is the complete lift of K ([7, 8]). □

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