Almost r - contact structures on the tangent bundle

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Abstract

The differential geometry of tangent bundles was studied by several authors, for example: D. E. Blair [1], E. T. Davies [3], P. Dombrowski [4], S. Ianus [5, 6], A. J. Ledger and K. Yano [7], V. Oproiu [8, 9], C. Udriste [10], Yano and Davies [11], Yano and Ishihara [13, 14] and among others. It is well known that different structures defined on a manifold M can be lifted to the same type of structures on its tangent bundle. Several authors cited herein obtained results in this direction. However, when we consider an almost contact structure not the same type of structure is obtained on the tangent bundle. In this case the base manifold must be of odd dimension while the tangent bundle is always of even dimension. Motivated by this fact, our goal is to see as to what kind of structure is defined on the tangent bundle $T(M)$ when we consider an almost r – contact structure on the base manifold.

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1 Introduction

Let M be an n-dimensional differentiable manifold of class C^∞ and let $T_p(M)$ be the tangent space of M at a point p of M. Then the set [14]

$$T(M) = \underset{p \varepsilon M}{\cup} T_p(M) \tag{1.1}$$

is called the tangent bundle over the manifold M. For any point $\tilde{p}$ of $T(M)$, the correspondence $\tilde{p} \to p$ determines the bundle projection $\pi : T(M) \to M$, Thus $\pi(\tilde{p}) = p$, where $\pi : T(M) \to M$ defines the bundle projection of $T(M)$ over M. The set $\pi^{-1}(p)$ is called the fibre over $p \; \varepsilon \; g$ and M the base space.

Suppose that the base space M is covered by a system of coordinate neighbourhoods $\{U; x^h\}$, where (x^h) is a system of local coordinates defined in the neighbourhood U of M. The open set $\pi^{-1}(U) \subset T(M)$ is naturally differentiably homeomorphic to the direct product $U \times R^n$, R^n being the n-dimensional vector space over the real field R, in such a way that a point $\tilde{p} \varepsilon T_p(M)$ $(p \; \varepsilon \; U)$is represented by an ordered pair (P, X) of the point $p \; \varepsilon \; U$, and a vector $X \; \varepsilon \; R^n$whose components are given by the cartesian coordinates (y^h) of $\tilde{p}$ in the tangent space $T_p\,(M)$ with respect to the natural

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base $\{\partial_h\}$, where $\partial_h = \frac{\partial}{\partial x^h}$. Denoting by (x^h) the coordinates of $p = \pi(\tilde{p})$ in U and establishing the correspondence $(x^h, y^h) \to \tilde{p} \ \varepsilon \ \pi^{-1}(U)$, we can introduce a system of local coordinates (x^h, y^h) in the open set $\pi^{-1}(U) \subset T(M)$. Here we call (x^h, y^h) the coordinates in $\pi^{-1}(U)$ induced from (x^h) or simply, the induced coordinates in $\pi^{-1}(U)$.

We denote by $\Im_s^r(M)$ the set of all tensor fields of class C^∞ and of type (r, s) in M. We now put $\Im(M) = \sum_{r,s=0}^{\infty} \Im_s^r(M)$, which is the set of all tensor fields in M. Similarly, we denote by $\Im_s^r(T(M))$ and $\Im(T(M))$ respectively the corresponding sets of tensor fields in the tangent bundle $T(M)$.

Vertical lifts

If f is a function in M, we write f^V for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \to M$ and $f : M \to R$,so that

$$f^V = f \ o \ \pi \tag{1.2}$$

Thus, if a point $\tilde{p} \ \varepsilon \ \pi^{-1}(U)$ has induced coordinates (x^h, y^h), then

$$f^V(\tilde{p}) = f^V(x, y) = f \ o \ \pi \ (\tilde{p}) \ = \ f \ (p) \ = \ f \ (x) \tag{1.3}$$

Thus the value of $f^V \ (\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^V the vertical lift of the function f.

Let $\tilde{X} \ \varepsilon \ \Im_0^1(T(M))$be such that $\tilde{X} f^V = 0$ for all $f \ \varepsilon \ \Im_0^0(M)$. Then we say that $\tilde{X}$ is a vertical vector field. Let $\begin{bmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{bmatrix}$ be components of $\tilde{X}$ with respect to the induced coordinates. Then $\tilde{X}$ is vertical if and only if its components in $\pi^{-1}(U)$ satisfy

$$\begin{bmatrix} \tilde{X}^h \\ X^{\bar{h}} \end{bmatrix} = \begin{bmatrix} 0 \\ X^{\bar{h}} \end{bmatrix} \tag{1.4}$$

Suppose that $X \ \varepsilon \ \Im_0^1(M)$, so that X is a vector field in M. We define a vector field X^V in $T(M)$ by

$$X^V(\iota \ \omega) = (\omega(X))^V \tag{1.5}$$

ω being an arbitrary 1-form in M. We call X^V the vertical lift of X.

Let $\tilde{\omega} \ \varepsilon \ \Im_1^0(T(M))$ be such that $\tilde{\omega}(X^V) = 0$ for all $X \ \varepsilon \ \Im_0^1(M)$. Then we say that $\tilde{\omega}$ is a vertical 1-form in $T(M)$. We define the vertical lift ω^V of the 1-form ω by

$$\omega^V = (\omega_i)^V (dx^i)^V \tag{1.6}$$

in each open set $\pi^{-1}(U)$, where $\{U; x^h\}$ is coordinate neighbourhood in M and ω is given by $\omega = \omega_i dx^i$ in U. The vertical lift $\omega^V$of ω with local expression $\omega = \omega_i \ dx^i$ has components of the form

$$\omega^V : (\omega^i, 0) \tag{1.7}$$

with respect to the induced coordinates in $T(M)$.

Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients by the conditions

$$(P \otimes Q)^V = P^V \otimes Q^V, \ (P + R)^V = \ P^V + R^V \tag{1.8}$$

P, Q and R being arbitrary elements of $\Im(M)$. The vertical lifts F^V of an element $F \ \varepsilon \ \Im_1^1 \ (M)$ with local components F_i^h has components of the form

$$F^V : \begin{pmatrix} 0 & 0 \\ F_i^h & 0 \end{pmatrix} \tag{1.9}$$

Complete lifts

If f is a function in M, we write f^C for the function in $T(M)$ defined by

$$f^C = \iota(df) \tag{1.10}$$

and call f^C the complete lift of the function f. The complete lift f^C of a function f has the local expression

$$f^C = y^i \partial_i f = \partial f \tag{1.11}$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X \ \varepsilon \ \Im_0^1 \ (M)$. We define a vector field X^C in $T(M)$ by

$$X^C f^C = (Xf)^C, \tag{1.12}$$

f being an arbitrary function in M and call X^C the complete lift of X in $T(M)$. The complete lift X^C of X with components x^h in M has components

$$X^C : \begin{bmatrix} x^h \\ \partial x^h \end{bmatrix} \tag{1.13}$$

with respect to the induced coordinates in$T(M)$.

Suppose that $\omega \ \varepsilon \ \Im_1^0(M)$. Then a 1-form ω^C in $T(M)$ defined by

$$\omega^C(X^C) = (\omega(X))^C \tag{1.14}$$

X being an arbitrary vector field in M. We call ω^C the complete lift of ω. The complete lift ω^C of ω with components ω_i in M has components of the form

$$\omega^C : \ (\partial \ \omega_i, \ \omega_i) \tag{1.15}$$

with respect to the induced coordinates in $T(M)$.

The complete lifts to a unique algebra isomorphism of the tensor algebra $J(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients, is given by the conditions

$$(P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C, \ (P + R)^C = P^C + R^C, \tag{1.16}$$

P, Q and R being arbitrary elements of $\Im(M)$.

The complete lifts F^C of an element F of $\Im_1^1(M)$ with local components F_i^h has components of the form

$$F^C : \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix} \tag{1.17}$$

Horizontal lifts

The horizontal lift f^H of $f \varepsilon \Im_0^0(M)$ to the tangent bundle $T(M)$ is given by

(1.18) $$f^H = f^C - \nabla_\gamma f$$

where

(1.19) $$\nabla_\gamma f =_\gamma (\nabla f),$$

Let $X \ \varepsilon \ \Im_0^1(M)$. Then the horizontal lift X^H of X defined by

(1.20) $$X^H = X^C - \nabla_\gamma X,$$

in $T(M)$, where

(1.21) $$\nabla_\gamma X = \gamma(\nabla X)$$

The horizontal lift $X^H$of X has the components

(1.22) $$X^H : \begin{pmatrix} x^h \\ -\Gamma_i^h x^i \end{pmatrix}$$

with respect to the induced coordinates in $T(M)$, where

(1.23) $$\Gamma_i^h = y^j \Gamma_{ji}^h$$

Let $\omega \ \varepsilon \ \Im_1^0(M)$ with affine connection ∇. Then the horizontal lift ω^H of ω is defined by

(1.24) $$\omega^H = \omega^C - \nabla_\gamma \omega$$

in $T(M)$, where $\nabla_\gamma \omega =_\gamma (\nabla \omega)$. The horizontal lift ω^H of ω has component of the form

(1.25) $$\omega^H : (\Gamma_i^h \omega_h, \omega_i)$$

with respect to the induced coordinates in $T(M)$.

Suppose there is given a tensor field

(1.26) $$S = S_{k \ldots j}^{i \ldots h} \frac{\partial}{\partial x^i} \otimes \ ... \ \otimes \frac{\partial}{\partial x^h} \otimes \partial x^k \otimes \ ... \ \otimes \partial x^j$$

in M with affine connection ∇, and in $T(M)$ a tensor field $\nabla_\gamma S$ defined by

(1.27) $$\nabla_\gamma S = (y^l \nabla_l S_{k \ldots j}^{i \ldots h}) \frac{\partial}{\partial y^i} \otimes ... \otimes \frac{\partial}{\partial y^h} \otimes \partial x^k \otimes \ ... \ \otimes \partial x^j$$

with respect to the induced coordinates (x^h, y^h) in $\pi^{-1}(U)$.

The horizontal lift S^H of a tensor field S of arbitrary type in M to $T(M)$ is defined by

(1.28) $$S^H = S^C - \nabla_\gamma S$$

For any $P, \ Q \ \varepsilon \ \Im(M)$. We have

(1.29) $$\nabla_\gamma (P \otimes Q) = (\nabla_\gamma P) \otimes Q^V + P^V \otimes (\nabla_\gamma Q)$$

or $(P \otimes Q)^H = P^H \otimes Q^V + P^V \otimes Q^H$

2 Complete lifts of almost complex structure and almost r-contact structure in the tangent bundle

Let $\bar{M}$ be an $2n+r$ dimensional differentiable manifold of Class C^∞ and $T(\bar{M})$ denotes the tangent bundle of $\bar{M}$. Suppose there is given in $\bar{M}$, a tensor field $F(1,1), r$ vector fields U_α and r 1-forms ω^α (r some finite integer and $\alpha = 1, 2, ...r$) satisfying

$$F^2 = -I + \sum_{\alpha=1}^{r} U_\alpha \otimes \omega^\alpha \tag{2.1}$$

where

$$FU_a = 0 \tag{2.2}$$

$$\omega_\alpha oF = 0 \tag{2.3}$$

$$\omega^\alpha(U_\beta) = \delta_\beta^\alpha \tag{2.4}$$

where $\alpha, \beta = 1, 2, ...r$ and δ_β^α denotes kronecker delta.

Thus the manifold $\bar{M}$ satisfying conditions (2.1) and (2.2) will be said to possess an almost r-contact structure [2].

Now we will prove the following two theorems:

Theorem 2.1. *Let $\bar{M}$ be a differentiable Manifold endowed with almost r-contact structure $(F,\ U_\alpha,\ \omega^\alpha)$, then*
$\tilde{J} = F^C + \sum_{\alpha=1}^{r} \left(U_\alpha^V \otimes \omega^{\alpha V} - U_\alpha^C \otimes \omega^{\alpha C}\right)$ is almost complex structure on $T(\bar{M})$.

Proof **:** From (2.1) and (2.2), we have

$$\left(F^C\right)^2 = -I + \sum_{\alpha=1}^{r} U_\alpha^V \otimes w^{\alpha C} + U_\alpha^C \otimes \omega^{\alpha V} \tag{2.5}$$

and

$$F^C U_\alpha^V = 0, \quad F^C U_\alpha^C = 0 \tag{2.6}$$

$$\omega^{\alpha V} oF^C = 0, \omega^{\alpha C} oF^V = 0 \quad \omega^{\alpha C} oF^C = 0 \tag{2.7}$$

$$\omega^{\alpha V}\left(U_\beta^V\right) = 0, \quad \omega^{\alpha V}\left(U_\beta^C\right) = \delta_\beta^\alpha, \quad \omega^{\alpha C}\left(U_\beta^V\right) = \delta_\beta^\alpha, \omega^{\alpha c}(U_\beta^C) = 0 \tag{2.8}$$

Let us define an element $\tilde{J}$ of $J_1^1(T(\bar{M}))$ by

$$\tilde{J} = F^C + \sum_{\alpha=1}^{r} \left(U_\alpha^V \otimes \omega^{\alpha V} - U_\alpha^C \otimes \omega^{\alpha C}\right) \tag{2.9}$$

then we find by using (2.5), (2.6) and (2.9) that

$$\tilde{J}^2 = -I \tag{2.10}$$

Thus $\tilde{J}$ is an almost complex structure in $T(\bar{M})$.

In view of equation (2.9), we have

$$\tilde{J}X^V = (FX)^V - (\omega^\alpha(X))^V U_\alpha^C \tag{2.11}$$

$$\tilde{J}X^C = (FX)^C + (\omega^\alpha(X))^V U_\alpha^C - (\omega^\alpha(X))^C U_\alpha^C \tag{2.12}$$

In particular, we have

$$\tilde{J}X^V = (FX)^V, \quad \tilde{J}X^C = (FX)^C \tag{2.13}$$

$$JU_\alpha^V = -\delta_\beta^\alpha U_\alpha^C = -U_\beta^C; \tilde{J}U_\alpha^C = \delta_\beta^\alpha U_\alpha^C = U_\beta^C; \alpha\beta = 1, 2, ...r. \tag{2.14}$$

X being an arbitrary vector field in M such that $\omega^\alpha(X) = 0$

Theorem 2.2. *Let the tangent bundle $T(M)$ of the manifold M admits $\tilde{J}$ defined in (2.5), then for vector fields X, Y such that $\omega^\alpha(Y) = 0$, we have*

$$\left(L_X V\tilde{J}\right) Y^V = 0, \tag{2.15}$$

$$(L_X V\tilde{J}) Y^C = ((L_X F) Y)^V - ((L_X \omega^\alpha) Y)^V U_\alpha^C \tag{2.16}$$

$$(L_X V\tilde{J}) U_\alpha^V = -(L_X U_\alpha)^V \tag{2.17}$$

$$(L_X V\tilde{J}) U_\alpha^C = ((L_X F) U_\alpha)^V - ((L_X \omega^\alpha)(U_\alpha))^V U_\alpha^C \tag{2.18}$$

and

$$(L_X C\tilde{J}) Y^V = ((L_X F) Y)^V - ((L_X \omega^\alpha)(Y))^V U_\alpha^C \tag{2.19}$$

$$(L_X C\tilde{J}) Y^C = ((L_X F) Y)^C + ((L_X \omega^\alpha)(Y))^V U_\alpha - ((L_X \omega^\alpha)(Y))^C U_\alpha^C \tag{2.20}$$

$$(L_X C\tilde{J}) U_\alpha^V = ((L_X F) U_\alpha^C - [X, U_\alpha]^C - ((L_X \omega^\alpha)(U_\alpha)^V U_\alpha^C \tag{2.21}$$

$$(L_X C\tilde{J}) U_\alpha^C = ((L_X F) U_\alpha)^C + [X, U_\alpha]^V + ((L_X \omega^\alpha)(U_\alpha))^V U_a \tag{2.22}$$

$$- ((L_X \omega^\alpha)(U_\alpha))^C U_\alpha^C \tag{2.23}$$

Proof : The proof follows in an obvious manner on using (2.6), (2.12) and (2.14).

3 Horizontal lifts of an almost $r-$ contact structure

Let $(F, U_\alpha, \omega^\alpha)$ be an almost r-contact structure in $\bar{M}$ with an affine connection then in view of (2.18) and (2.22) we have

$$\left(F^H\right)^2 = \left(-I + \sum_{r=1}^{r} U_\alpha \otimes \omega^\alpha\right)^H \tag{3.1}$$

$$\left(F^H\right)^2 = -I + \sum_{\alpha=1}^{r} \left(U_\alpha \otimes \omega^\alpha\right)^H \tag{3.2}$$

$$\left(F^H\right)^2 = -I + \sum_{\alpha=1}^{r} \left(U_\alpha^H \otimes \omega^{\alpha V} + U_\alpha^V \otimes \omega^{\alpha H}\right) \tag{3.3}$$

Also,

$$F^H U_\alpha^H = 0, \quad F^H U_\alpha^V = 0 \tag{3.4}$$

$$\omega^{\alpha H}\left(U_\beta^H\right) = 0, \quad \omega^{\alpha H}(U_\alpha^V) = \delta_\beta^\alpha, \quad \omega^{\alpha V}\left(U_\beta^H\right) = \delta_\beta^\alpha \tag{3.5}$$

$$\omega^{\alpha H} \text{ of }^H = 0, \omega^{\alpha V} \text{ of }^H = 0. \tag{3.6}$$

Let us define a tenser field $\tilde{J}^*$ of type (1,1) in $T(\bar{M})$ by

$$\tilde{J}^* = F^H + \sum_{r=1}^{r} \left(U_\alpha^V \otimes \omega^{\alpha V} - U_\alpha^H \otimes \omega^{\alpha H}\right) \tag{3.7}$$

then it is easy to show that

$$\tilde{J}^{*^2} = -I \tag{3.8}$$

Thus J^* is an almost complex structure in$T(\bar{M})$ and we have

Theorem 3.1. *Let $(F, U_\alpha, \omega^\alpha)$ be an almost r-contact structure in $\bar{M}$ with an affine connection ∇. Then $\tilde{J}^*$ is an almost complex in $T(\bar{M})$.*

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