Multi-Time Lagrange Spaces

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Abstract

M. Neagu and C. Udriste in [6]-[9] introduced and studied the multi-time Lagrange spaces supplied with d-connection. V. Balan in [2] and [3] examined geodesics, paths, Jacobi fields in these spaces. Here the connection is generalized and the relations for torsion and curvature tensors appeare in widely achieved form. The introduced notation help us to write that complicated expressions in simpler form.

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1. Adapted bases

Let E be a $n+p+np$ dimensional C^∞ manifold. Some point u of E in some local chart (U,φ) has coordinates

$$(x^i, t^\alpha, x^i_\alpha), \quad i,j,k,\ldots=\overline{1,n}, \quad \alpha,\beta,\gamma,\delta,\ldots=\overline{1,p}.$$

If in some other chart (U',φ') the same point u has coordinates $(x^{i'}, t^{\alpha'}, x^{i'}_{\alpha'})$, then the allowable coordinate transformations are given by

$$\begin{aligned} x^{i'} &= x^{i'}(x^i) \Leftrightarrow x^i = x^i(x^{i'}) \\ t^{\alpha'} &= t^{\alpha'}(t^\alpha) \Leftrightarrow t^\alpha = t^\alpha(t^{\alpha'}) \\ x^{i'}_{\alpha'} &= B^{i'}_i B^\alpha_{\alpha'} x^i_\alpha, \quad B^{i'}_i = \frac{\partial x^{i'}}{\partial x^i}, \quad B^\alpha_{\alpha'} = \frac{\partial t^\alpha}{\partial t^{\alpha'}}. \end{aligned} \tag{1.1}$$

The transformations of (1.1) are regular, so exist inverse transformations and we have

$$B^{i'}_i B^i_{j'} = \delta^{i'}_{j'}, \quad B^i_{i'} B^{i'}_j = \delta^i_j, \quad B^{\alpha'}_\alpha B^\alpha_{\beta'} = \delta^{\alpha'}_{\beta'}, \quad B^\alpha_{\alpha'} B^{\alpha'}_\beta = \delta^\alpha_\beta. \tag{1.2}$$

Theorem 1.1. *The transformations of type (1.1) form a pseudo-group.*

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The natural basis $\bar{B}$ of $T(E)$ is

$$\bar{B} = \left\{ \partial_i = \frac{\partial}{\partial x^i}, \partial_\alpha = \frac{\partial}{\partial t^\alpha} \quad \partial_i^\alpha = \frac{\partial}{\partial x^i_\alpha} \right\}. \tag{1.3}$$

Under coordinate transformation (1.1) the elements of $\bar{B}$ are transforming in the following way:

$$\begin{aligned} \partial_{i'} &= B^i_{i'} \partial_i + B_{j'}{}^i{}_{i'} B^{\alpha'}_\alpha x^{j'}_{\alpha'} \partial^\alpha_i \\ \partial_{\alpha'} &= B^\alpha_{\alpha'} \partial_{\alpha'} + B^i_{i'} B_\alpha{}^{\gamma'}{}_\beta B^\beta_{\alpha'} x^{i'}_{\gamma'} \partial^\alpha_i \\ \partial^{\alpha'}_{i'} &= B^{\alpha'}_\alpha B^i_{i'} \partial^\alpha_i . \end{aligned} \tag{1.4}$$

The above equation in the matrix form can be written as follows

$$[\partial_{i'} \partial_{\alpha'} \partial^{\alpha'}_{i'}] = [\partial_i \partial_\alpha \partial^\alpha_i] \cdot \begin{bmatrix} B^i_{i'} & 0 & 0 \\ 0 & B^\alpha_{\alpha'} & 0 \\ B_{j'}{}^i{}_{i'} B^{\alpha'}_\alpha x^{j'}_{\alpha'} & B^i_{i'} B_\alpha{}^{\gamma'}{}_\beta B^\beta_{\alpha'} x^{i'}_{\gamma'} & B^i_{i'} B^{\alpha'}_\alpha \end{bmatrix} \tag{1.5}$$

The natural basis $\bar{B}^*$ of $T^*(E)$ is

$$\bar{B}^* = \{dx^i, dt^\alpha, dx^i_\alpha\}. \tag{1.6}$$

Under coordinate transformation (1.1) the elements of $\bar{B}^*$ are transforming as follows

$$\begin{aligned} dx^{i'} &= B^{i'}_i dx^i \\ dt^{\alpha'} &= B^{\alpha'}_\alpha dt^\alpha \\ dx^{i'}_{\alpha'} &= B_j{}^{i'}{}_i B^\alpha_{\alpha'} x^j_\alpha dx^i + B^{i'}_i B_{\alpha'}{}^\gamma{}_{\beta'} B^{\beta'}_\alpha x^i_\gamma dt^\alpha + B^{i'}_i B^\alpha_{\alpha'} dx^i_\alpha . \end{aligned} \tag{1.7}$$

(1.7) in the matrix form is given by following relation

$$\begin{bmatrix} dx^{i'} \\ dt^{\alpha'} \\ dx^{i'}_{\alpha'} \end{bmatrix} = \begin{bmatrix} B^{i'}_i & 0 & 0 \\ 0 & B^{\alpha'}_\alpha & 0 \\ B_j{}^{i'}{}_i B^\alpha_{\alpha'} x^j_\alpha & B^{i'}_i B_{\alpha'}{}^\gamma{}_{\beta'} B^{\beta'}_\alpha x^i_\gamma & B^{i'}_i B^\alpha_{\alpha'} \end{bmatrix} \begin{bmatrix} dx^i \\ dt^\alpha \\ dx^i_\alpha . \end{bmatrix} \tag{1.8}$$

Theorem 1.2. *The duality of $\bar{B}^*$ to $\bar{B}$ is coordinate invariant.*

Proof. Let us suppose that

$$\begin{bmatrix} dx^i \\ dt^\alpha \\ dx^i_\alpha \end{bmatrix} [\partial_k \partial_\gamma \partial^\gamma_k] = \begin{bmatrix} \delta^i_k & 0 & 0 \\ 0 & \delta^\alpha_\gamma & 0 \\ 0 & 0 & \delta^\gamma_\alpha \delta^i_k \end{bmatrix} \tag{1.9}$$

then using (1.8), (1.9) and (1.4) we have

$$\begin{bmatrix} dx^{i'} \\ dt^{\alpha'} \\ dx^{i'}_{\alpha'} \end{bmatrix} [\partial_{k'} \partial_{\gamma'} \partial^{\gamma'}_{k'}] =$$

$$\begin{bmatrix} B^{i'}_i & 0 & 0 \\ 0 & B^{\alpha'}_\alpha & 0 \\ B_j{}^{i'}{}_i B^\alpha_{\alpha'} x^j_\alpha & B^{i'}_i B_{\alpha'}{}^\gamma{}_{\beta'} B^{\beta'}_\alpha x^i_\gamma & B^{i'}_i B^\alpha_{\alpha'} \end{bmatrix} \begin{bmatrix} \delta^i_k & 0 & \\ 0 & \delta^\alpha_\gamma & 0 \\ 0 & 0 & \delta^\gamma_\alpha \delta^i_k \end{bmatrix} \cdot$$

$$\begin{bmatrix} B^k_{k'} & 0 & \\ 0 & B^\gamma_{\gamma'} & 0 \\ B_{h'}{}^k{}_{k'} B^{\gamma'}_\gamma x^{k'}_{\gamma'} & B^k_{k'} B_\alpha{}^{\delta'}{}_\kappa B^\kappa_{\gamma'} x^{k'}_{\delta'} & B^k_{k'} B^{\gamma'}_\gamma \end{bmatrix} =$$

$$\begin{bmatrix} B^{i'}_i & 0 & 0 \\ 0 & B^{\alpha'}_\alpha & 0 \\ B_j{}^{i'}{}_i B^\gamma_{\alpha'} x^j_\gamma & B^{i'}_i B_{\alpha'}{}^\gamma{}_{\beta'} B^{\beta'}_\alpha x^i_\gamma & B^{i'}_i B^\alpha_{\alpha'} \end{bmatrix} \begin{bmatrix} B^i_{k'} & 0 & 0 \\ 0 & B^\alpha_{\gamma'} & 0 \\ B_{h'}{}^i{}_{k'} B^{\gamma'}_\alpha x^{h'}_{\gamma'} & B^i_{k'} B_\alpha{}^{\delta'}{}_\kappa B^\kappa_{\gamma'} x^{k'}_{\delta'} & B^i_{k'} B^{\gamma'}_\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \delta^{i'}_{k'} & 0 & \\ 0 & \delta^{\alpha'}_{\gamma'} & 0 \\ \bar{A} & \bar{B} & \delta^{i'}_{k'} \delta^{\gamma'}_{\alpha'} \end{bmatrix},$$

where

(1.10) $$\bar{A} = B_j{}^{i'}{}_i B^i_{k'} B^\alpha_{\alpha'} x^j_\alpha + B^{i'}_i B^\alpha_{\alpha'} B_{h'}{}^i{}_{k'} B^{\gamma'}_\alpha x^{h'}_{\gamma'}$$

(1.11) $$\bar{B} = B^{i'}_i B_{\alpha'}{}^\gamma{}_{\delta'} B^{\delta'}_\alpha x^i_\gamma B^\alpha_{\gamma'} + B^{i'}_i B^\alpha_{\alpha'} B^i_{k'} B_\alpha{}^{\delta'}{}_\kappa B^\kappa_{\gamma'} x^{k'}_{\delta'}.$$

From $B^{i'}_i B^i_{k'} = \delta^{i'}_{k'}$ it follows

(1.12) $$B_i{}^{i'}{}_j B^i_{k'} + B^{i'}_i B_{k'}{}^i{}_{j'} B^{j'}_j = 0.$$

The substitution of the above equation into $\bar{A}$ results

(1.13) $$\bar{A} = -B^{i'}_i B_{k'}{}^i{}_{j'} B^{j'}_j B^\alpha_{\alpha'} x^j_\alpha + B^{i'}_i B^\alpha_{\alpha'} B_{j'}{}^i{}_{k'} B^{\gamma'}_\alpha x^{j'}_{\gamma'}.$$

From (1.1) and (1.2) it follows that

$$x^{j'}_{\gamma'} B^{\gamma'}_\alpha = B^{j'}_j B^\gamma_{\gamma'} B^{\gamma'}_\alpha x^j_\gamma = B^{j'}_j \delta^\gamma_\alpha x^j_\gamma = B^{j'}_j x^j_\alpha,$$

which together with (1.12) gives

(1.14) $$\bar{A} = 0.$$

As $B^\gamma_{\delta'} B^{\delta'}_\kappa = \delta^\gamma_\kappa$ we have

(1.15) $$B_{\delta'}{}^\gamma{}_{\alpha'} B^{\delta'}_\kappa + B^\gamma_{\delta'} B_\kappa{}^{\delta'}{}_\alpha B^\alpha_{\alpha'} = 0.$$

(1.1) and the substitution of the above equation into (1.11) result:

$$\bar{B} = -B^{i'}_i B^\gamma_{\delta'} B_\kappa{}^{\delta'}{}_\alpha B^\alpha_{\alpha'} x^i_\gamma B^\kappa_{\gamma'} +$$

$$B^{i'}_i B^\alpha_{\alpha'} B^i_{k'} B_\alpha{}^{\delta'}{}_\kappa B^\kappa_{\gamma'} x^{k'}_{\delta'},$$

i.e.

$$\bar{B} = B_{\kappa\ \alpha}^{\delta'} B_{\alpha'}^{\alpha} B_{\gamma'}^{\kappa} (-x_{\delta'}^{i'} + x_{\delta'}^{i'}) = 0 \qquad \Box. \tag{1.16}$$

Definition 1.1. *The adapted basis*

$$B = \{\delta_i, \delta_\alpha \delta_i^\alpha\}$$

of $T(E)$ is given by

$$\delta_i = \partial_i - N_{i\beta}^{\ k} \partial_k^\beta \tag{1.17}$$

$$\delta_\alpha = \partial_\alpha - M_{\alpha\beta}^{\ \ k} \partial_k^\beta$$

$$\delta_i^\alpha = \partial_i^\alpha.$$

We want, that elements of the adapted basis B under (1.1) transform as tensors i.e.

$$\delta_{i'} = B_{i'}^i \delta_i, \quad \delta_{\alpha'} = B_{\alpha'}^\alpha \delta_\alpha, \quad \delta_{i'}^{\alpha'} = B_\alpha^{\alpha'} B_{i'}^i \delta_i^\alpha. \tag{1.18}$$

Theorem 1.3. *The elements of the adapted basis B of $T(E)$ are transforming as tensors (i.e satisfy (1.18)), when the functions $N_{j\beta}^{\ k}$ and $M_{\alpha\beta}^{\ \ k}$ satisfy (1.19) and (1.20):*

$$N_{j\beta}^{\ k} = N_{j'\beta'}^{\ \ k'} B_j^{j'} B_{k'}^k B_\beta^{\beta'} + B_{k'}^k B_{h\ j}^{\ k'} x_\beta^h \tag{1.19}$$

$$M_{\alpha\beta}^{\ \ k} = M_{\alpha'\beta'}^{\ \ \ k'} B_\alpha^{\alpha'} B_{k'}^k B_\beta^{\beta'} + B_{\alpha'\ \beta'}^{\ \gamma} B_\alpha^{\alpha'} B_\gamma^{\beta'} x_\beta^k. \tag{1.20}$$

Proof. If we substitute (1.17) into (1.18) we get

$$N_{i\beta}^{\ k} B_{i'}^i = N_{i'\beta'}^{\ \ k'} B_\beta^{\beta'} B_{k'}^k - B_{j'\ i'}^{\ k} B_\beta^{\alpha'} x_{\alpha'}^{j'}. \tag{1.21}$$

$$M_{\alpha\beta}^{\ \ k} B_{\alpha'}^\alpha = M_{\alpha'\beta'}^{\ \ \ k'} B_\beta^{\beta'} B_{k'}^k - B_{i'}^k B_{\beta\ \gamma}^{\ \gamma'} B_{\alpha'}^\gamma x_{\gamma'}^{i'}. \tag{1.22}$$

From (1.12) and (1.15) we obtain

$$-B_{j'i'}^k B_\beta^{\alpha'} x_{\alpha'}^{j'} = B_{j'}^k B_{hi}^{j'} B_{i'}^i x_\beta^h. \tag{1.23}$$

$$-B_{i'}^k B_{\beta\gamma}^{\gamma'} B_{\alpha'}^\gamma x_{\gamma'}^{i'} = B_{\alpha'\beta'}^\gamma x_\beta^k B_\gamma^{\beta'}. \tag{1.24}$$

The substitution of (1.23) into (1.21) gives (1.19) and the substitution of (1.24) into (1.22) results (1.20).

Definition 1.2. *The adapted basis*

$$B^* = \{\delta x^i, \delta t^\alpha, \delta x_\alpha^i\} \tag{1.25}$$

of $T^*(E)$ is given by

$$\delta x^i = dx^i, \quad \delta t^\alpha = dt^\alpha \tag{1.26}$$

$$\delta x_\alpha^i = dx_\alpha^i + M_{\alpha\beta}^i dt^\beta + N_{\alpha j}^i dx^j.$$

The transformation on law of functions $M^i_{\alpha\beta}$ and $N^i_{\alpha j}$ should be determined in such a way that δx^i, δt^α, δx^i_α transform as tensors, i.e

$$\delta x^{i'} = B^{i'}_i \delta x^i, \quad \delta t^{\alpha'} = B^{\alpha'}_\alpha \delta t^\alpha, \quad \delta x^{i'}_{\alpha'} = B^{i'}_i B^\alpha_{\alpha'} x^i_\alpha. \tag{1.27}$$

Theorem 1.4. *The elements of the adapted basis B^* of $T^*(E)$ are transforming as tensors, when the functions $N^i_{\alpha j}$, $M^i_{\alpha\beta}$ satisfy (1.28) and (1.29)*

$$N^k_{\beta i} = N^{k'}_{\beta' i'} B^{i'}_i B^k_{k'} B^{\beta'}_\beta + B^{k'}_{ji} B^k_{k'} x^j_\beta. \tag{1.28}$$

$$M^k_{\beta\alpha} = M^{k'}_{\beta\alpha'} B^{\alpha'}_\alpha B^k_{k'} B^{\beta'}_\beta + B^\gamma_{\beta'\alpha'} B^{\beta'}_\beta B^{\alpha'}_\alpha x^k_\gamma. \tag{1.29}$$

Proof. The substitution of (1.26) into (1.27) gives

$$N^j_{\alpha i} B^\alpha_{\alpha'} B^{i'}_j = N^{i'}_{\alpha' j'} B^{j'}_i + B^{i'}_{ji} B^\alpha_{\alpha'} x^j_\alpha. \tag{1.30}$$

$$M^i_{\gamma\alpha} B^{i'}_i B^\gamma_{\alpha'} = M^{i'}_{\alpha'\gamma'} B^{\gamma'}_\alpha + B^{i'}_i B^\gamma_{\alpha'\beta'} x^i_\gamma B^{\beta'}_\alpha. \tag{1.31}$$

From (1.30), (1.31) and (1.2) it follows (1.28) and (1.29). □

From Theorem 1.4 it follows

$$N^k_{\beta i} = N^k_{\beta i}(x^j, t^\alpha, x^j_\alpha) \quad M^k_{\beta i} = M^k_{\beta i}(x^j, t^\alpha, x^j_\alpha). \tag{1.32}$$

Theorem 1.5. *The functions $N_{j\beta}^{\ \ k}$ and $N^k_{\beta j}$, further $M_{\alpha\beta}^{\ \ k}$ and $M^k_{\beta\alpha}$ have the same law of transformation.*

Proof. It follows from comparison of (1.19) with (1.28) further (1.20) with (1.29).

Theorem 1.6. *If the natural bases $\bar{B}^*$ and $\bar{B}$ are dual to each other, then the adapted bases B^* and B are dual to each other if and only if*

$$N_{j\beta}^{\ \ k} = N^k_{\beta j} \quad M_{\alpha\beta}^{\ \ k} = M^k_{\beta\alpha}. \tag{1.33}$$

Proof.

$$\begin{aligned}
\langle \delta x^i, \delta_j \rangle &= \langle dx^i, \partial_j - N_{j\beta}^{\ \ k} \partial^\beta_k \rangle = \delta^i_j \\
\langle \delta x^i, \delta_\alpha \rangle &= \langle dx^i, \partial_\alpha - M_{\alpha\beta}^{\ \ k} \partial^\beta_k \rangle = 0 \\
\langle \delta x^i, \delta^\alpha_i \rangle &= \langle dx^i, \partial^\alpha_i \rangle = 0 \\
\langle \delta t^\alpha, \delta_i \rangle &= \langle dt^\alpha, \partial_i - N_{i\beta}^{\ \ k} \partial^\beta_k \rangle = 0 \\
\langle \delta t^\alpha, \delta_\beta \rangle &= \langle dt^\alpha, \partial_\beta - M_{\beta\gamma}^{\ \ k} \partial^\gamma_k \rangle = \delta^\alpha_\beta \\
\langle \delta t^\alpha, \delta^\beta_i \rangle &= \langle dt^\alpha, \partial^\beta_i \rangle = 0 \\
\langle \delta x^i_\alpha, \delta_j \rangle &= \langle dx^i_\alpha + M^i_{\alpha\beta} dt^\beta + N^i_{\alpha k} dx^k, \partial_j - N_{j\beta}^{\ \ k} \partial^\beta_k \rangle = \\
&\quad -N_{j\beta}^{\ \ k} \delta^i_k \delta^\beta_\alpha + N^i_{\alpha k} \delta^k_j = 0 \Rightarrow N_{j\alpha}^{\ \ i} = N^i_{\alpha j} \\
\langle \delta x^i_\alpha, \delta_\beta \rangle &= \langle dx^i_\alpha + M^i_{\alpha\gamma} dt^\gamma + N^i_{\alpha k} dx^k, \partial_\beta - M_{\beta\gamma}^{\ \ k} \partial^\gamma_k \rangle = \\
&\quad -M_{\beta\gamma}^{\ \ k} \delta^i_k \delta^\gamma_\alpha + M^i_{\alpha\gamma} \delta^\gamma_\beta = 0 \Rightarrow M_{\beta\alpha}^{\ \ i} = M^i_{\alpha\beta}. \square
\end{aligned}$$

2. Different kinds of connections on $T(E)$

Let us denote by T_0, T_1, T_2 the subspaces of $T(E)$ generated by $\{\delta_i\}$, $\{\delta_\alpha\}$, $\{\delta_i^\alpha\}$ respectively and by T_0^*, T_1^*, T_2^* the subspaces of $T^*(E)$ generated by $\{\delta x^i\}$, $\{\delta t^\alpha\}$ and $\{\delta x^i_\alpha\}$ respectively.

Some tensor T on $T_0^* \otimes T_0 \otimes T_1^* \otimes T_1 \otimes T_2^* \otimes T_2$ is given by

$$T = T_{i\ \alpha\ \ k\gamma}^{\ j\ \beta\ \delta\ h} \delta x^i \otimes \delta_j \otimes \delta t^\alpha \otimes \delta_\beta \otimes \delta x^k_\delta \otimes \delta^\gamma_h.$$

If in another chart $(U', \varphi')T$ has the form

$$T = T_{i'\ \alpha'\ \ k'\gamma'}^{\ j'\ \beta'\ \delta'\ h'} \delta x^{i'} \otimes \delta_{j'} \otimes \delta t^{\alpha'} \otimes \delta_{\beta'} \otimes \delta x^{k'}_{\delta'} \otimes \delta^{\gamma'}_{h'},$$

then using the transformation law of adapted bases vectors, which span T_0, T_1, T_2, T_1^*, T_1^*, T_2^* ((1.18) and (1.27)) we get

$$T_{i\ \alpha\ \ k\gamma}^{\ j\ \beta\delta\ h} = T_{i'\ \alpha'\ \ k'\gamma'}^{\ j'\ \beta'\delta'\ h'} B^{i'j\alpha'\beta\delta k'h\gamma'}_{ij'\alpha\beta'\delta'kh'\gamma}, \tag{2.1}$$

where

$$B^{i'j\alpha'\beta\delta k'h\gamma'}_{ij'\alpha\beta'\delta'kh'\gamma} = B^{i'}_i B^j_{j'} B^{\alpha'}_\alpha B^\beta_{\beta'} B^\delta_{\delta'} B^{k'}_k B^h_{h'} B^{\gamma'}_\gamma.$$

Let us suppose that on $T^*(E) \otimes T^*(E)$ the metric tensor G is given in normal form:

$$G = g_{ij}\delta x^i \otimes \delta x^j + g_{\alpha\beta}\delta t^\alpha \otimes \delta t^\beta + g^{\alpha\beta}_{ij}\delta x^i_\alpha \delta x^j_\beta. \tag{2.2}$$

In this case T_0, T_1 and T_2 are mutually orthogonal with respect to G.

Theorem 2.1. *The following equations are coordinate invariant*

$$\begin{aligned} g_{ij}\delta x^i = \delta_j &\Leftrightarrow g^{ij}\delta_j = \delta x^i \\ g_{\alpha\beta}\delta t^\alpha = \delta_\beta &\Leftrightarrow g^{\alpha\beta}\delta_\beta = \delta t^\alpha \\ g^{\alpha\beta}_{ij}\delta x^i_\alpha = \delta^\beta_j &\Leftrightarrow g^{ij}_{\alpha\beta}\delta^\beta_j = \delta x^i_\alpha \end{aligned} \tag{2.3}$$

where

$$g_{ij}g^{jk} = \delta^k_i, \quad g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha, \quad g^{\alpha\beta}_{ij} g^{jk}_{\beta\delta} = \delta^k_i \delta^\alpha_\delta. \tag{2.4}$$

Proof. Using (1.18), (1.27) and (2.1) we have

$$g_{i'j'}\delta x^{i'} = B^{i\ j}_{i'j'} g_{ij} B^{i'}_k \delta x^k = g_{kj} B^j_{j'} \delta x^k = B^j_{j'}\delta_j = \delta_{j'},$$

$$g_{\alpha'\beta'}\delta t^{\beta'} = B^{\alpha\ \beta}_{\alpha'\beta'} g_{\alpha\beta} B^{\beta'}_\gamma \delta t^\gamma = g_{\alpha\gamma} B^\alpha_{\alpha'}\delta t^\gamma = B^\alpha_{\alpha'}\delta_\alpha = \delta_{\alpha'},$$

$$g^{\alpha'\beta'}_{i'j'}\delta x^{i'}_{\alpha'} = g^{\alpha\beta}_{ij} B^{\alpha'\ \beta' i\ j\ i'\delta}_{\alpha\ \beta i'j'k\alpha'} \delta x^k_\delta = g^{\delta\beta}_{kj} B^{\beta'\ j}_{\beta\ j'} \delta x^k_\delta = B^{\beta' j}_{\beta j'} \delta^\beta_j = \delta^{\beta'}_{j'}. \ \square$$

If the relations (2.3) are valid, using the metric tensor, we can change the place of indices in tensors in the following way:

$$\begin{aligned} &T_i^{\ j}\delta x^i \otimes \delta_j = T_i^{\ j} g^{ik}\delta_k \otimes \delta_j = T^{kj}\delta_k \otimes \delta_j, \\ &T_i^{\ j}\delta x^i \otimes \delta_j = T_i^{\ j}\delta x^i \otimes g_{kj}\delta x^k = T_{ik}\delta x^i \otimes \delta x^k, \\ &T^{\alpha\ j}_{i\ \beta}\delta x^i_\alpha \otimes \delta^\beta_j = T^{\alpha\ j}_{i\ \beta} g^{i\ k}_{\alpha\ \varepsilon}\delta^\varepsilon_k \otimes \delta^\beta_j = T^{kj}_{\varepsilon\beta}\delta^\varepsilon_k \otimes \delta^\beta_j, \ldots \end{aligned} \tag{2.5}$$

Theorem 2.2. *In tensors the indices can change their place using the metric tensor only, when*

$$g_{ij} = g_{ij}(x^k, t^\gamma, x^h_\delta), \quad g_{\alpha\beta} = g_{\alpha\beta}(x^k, t^\gamma, x^h_\gamma), \quad g^{\alpha\beta}_{ij} = g^{\alpha\beta}_{ij}(x^k, t^\gamma, x^h_\delta). \tag{2.6}$$

Proof. As δ_i is the function of $N^{\ k}_{i\beta}$, δ_β of $M^{\ k}_{\beta\gamma}$ and δx^i_α is the function of $N^{\ k}_{i\beta}$ and $M^{\ k}_{\beta\gamma}$ and (1.32) is valid, from (2.4) we conclude (2.6).

Definition 2.1. *The linear connection* $\nabla : (T(E) \times T(E) \to T(E)$, $\nabla : (X, Y) \to \nabla_X Y$ *for* $\forall X, Y \in T(E)$ *is defined by*

$$\begin{aligned}
\nabla_{\delta_i}\delta_j &= \underline{\Gamma_{j\ i}^{\ k}\delta_k} + \Gamma_{j\ i}^{\ \gamma}\delta_\gamma + \Gamma_{j\gamma i}^{\ \ k}\delta^\gamma_k,\\
\nabla_{\delta_i}\delta_\alpha &= \Gamma_{\alpha\ i}^{\ k}\delta_k + \underline{\Gamma_{\alpha\ i}^{\ \gamma}\delta_\gamma} + \Gamma_{\alpha\gamma i}^{\ \ k}\delta^\gamma_k,\\
\nabla_{\delta_i}\delta^\beta_j &= \Gamma_{j\ \ i}^{\beta k}\delta_k + \Gamma_{j\ \ i}^{\beta\gamma}\delta_\gamma + \underline{\Gamma_{j\gamma\ i}^{\beta k}\delta^\gamma_k},\\
\nabla_{\delta_\alpha}\delta_j &= \underline{\Gamma_{j\ \alpha}^{\ k}\delta_k} + \Gamma_{j\ \alpha}^{\ \gamma}\delta_\gamma + \Gamma_{j\ \gamma\alpha}^{\ k}\delta^\gamma_k,\\
\nabla_{\delta_\alpha}\delta_\beta &= \Gamma_{\beta\ \alpha}^{\ k}\delta_k + \underline{\Gamma_{\beta\ \alpha}^{\ \gamma}\delta_\gamma} + \Gamma_{\beta\gamma\alpha}^{\ \ k}\delta^\gamma_k,\\
\nabla_{\delta_\alpha}\delta^\beta_j &= \Gamma_{j\ \alpha}^{\beta k}\delta_k + \Gamma_{j\ \ \alpha}^{\beta\gamma}\delta_\gamma + \underline{\Gamma_{j\gamma\ \alpha}^{\beta k}\delta^\gamma_k},\\
\nabla_{\delta^\alpha_i}\delta_j &= \underline{\Gamma_{j\ i}^{\ k\alpha}\delta_k} + \Gamma_{j\ \ i}^{\ \gamma\alpha}\delta_\gamma + \Gamma_{j\gamma i}^{\ \ k\alpha}\delta^\gamma_k,\\
\nabla_{\delta^\alpha_i}\delta_\beta &= \Gamma_{\beta\ \ i}^{\ k\alpha}\delta_k + \underline{\Gamma_{\beta\ \ i}^{\ \gamma\ \alpha}\delta_\gamma} + \Gamma_{\beta\gamma i}^{\ \ k\alpha}\delta^\gamma_k,\\
\nabla_{\delta^\alpha_i}\delta^\beta_j &= \Gamma_{j\ \ i}^{\beta k\alpha}\delta_k + \Gamma_{j\ \ \ i}^{\beta\gamma\alpha}\delta_\gamma + \underline{\Gamma_{j\gamma\ i}^{\beta k\alpha}\delta^\gamma_k}.
\end{aligned} \tag{2.7}$$

Definition 2.2. *The linear connection defined by (2.7) is d-connection if on the right hand side in (2.7) only the underlined terms remain, the other are equal to zero.*

It can be seen, that for the d-connection if Y belongs to one of T_0, T_1 or T_2, then $\nabla_X Y$ belongs to the same subspace of $T(E)$ for every $X \in T(E)$.

Remark. In papers [6], [7], [8], [9] d-connection is considered.

Definition 2.3. *The linear connection is called strongly distinguished connection (s.d. connection) if it is defined by*

$$\nabla_{\delta_i}\delta_j = \Gamma_{j\ i}^{\ k}\delta_k, \nabla_{\delta_\alpha}\delta_\beta = \Gamma_{\beta\ \alpha}^{\ \gamma}\delta_\gamma, \nabla_{\delta^\alpha_i}\delta^\beta_j = \Gamma_{j\gamma i}^{\beta k\alpha}\delta^\gamma_k, \tag{2.8}$$

the other connection coefficients are equal to zero.

For the s.d. connection X, Y and $\nabla_X Y$ belong to the same subspace T_0 or T_1 or T_2.

Proposition 2.1. *The components of the linear connection ∇ under transformations given by (1.1) behave as follows:*

(2.9)
$$(a)\quad \Gamma_{j'}{}^{k'}{}_{i'} = \Gamma_{j}{}^{k}{}_{i} B^{i\,j\,k'}_{i'j'k} + B^{k}_{j'i'} B^{k'}_{k}$$
$$(b)\quad \Gamma_{j'}{}^{\alpha'}{}_{i'} = \Gamma_{j}{}^{\alpha}{}_{i} B^{i\,j\,\alpha'}_{i'j'\alpha}$$
$$(c)\quad \Gamma_{j'\beta'}{}^{k'}{}_{i'} = \Gamma_{j\beta}{}^{k}{}_{i} B^{j\,k'\beta\,i}_{j'k\beta'i'}$$

(2.10)
$$(a)\quad \Gamma_{\alpha'}{}^{k'}{}_{i'} = \Gamma_{\alpha}{}^{k}{}_{i} B^{\alpha k' i}_{\alpha'\,k\,i'}$$
$$(b)\quad \Gamma_{\alpha'}{}^{\beta'}{}_{i'} = \Gamma_{\alpha}{}^{\beta}{}_{i} B^{\alpha\,\beta' i}_{\alpha'\beta\,i'}$$
$$(c)\quad \Gamma_{\alpha'\beta'}{}^{k'}{}_{i'} = \Gamma_{\alpha\beta}{}^{k}{}_{i} B^{\alpha\,k'\beta\,i}_{\alpha'k\beta'i'}$$

(2.11)
$$(a)\quad \Gamma^{\beta'k'}_{j'}{}_{i'} = \Gamma^{\beta k}_{j}{}_{i} B^{j\,\beta'k'i}_{j'\beta\,k\,i'}$$
$$(b)\quad \Gamma^{\beta'\gamma'}_{j'}{}_{i'} = \Gamma^{\beta\gamma}_{j}{}_{i} B^{j\,\beta'\gamma'i}_{j'\beta\,\gamma\,i'}$$
$$(c)\quad \Gamma^{\beta'k'}_{j'\gamma'i'} = \Gamma^{\beta k}_{j\gamma}{}_{i} B^{j\,\beta'k'\gamma\,i}_{j'\beta\,k\,\gamma'i'} + \delta^{\beta'}_{\gamma'} B_{j'}{}^{k}{}_{i'} B^{k'}_{k}$$

(2.12)
$$(a)\quad \Gamma_{j'}{}^{k'}{}_{\alpha'} = \Gamma_{j}{}^{k}{}_{\alpha} B^{j\,k'\alpha}_{j'k\,\alpha'}$$
$$(b)\quad \Gamma_{j'}{}^{\gamma'}{}_{\alpha'} = \Gamma_{j}{}^{\gamma}{}_{\alpha} B^{j\,\gamma'\alpha}_{j'\gamma\,\alpha'}$$
$$(c)\quad \Gamma_{j'\gamma'}{}^{k'}{}_{\alpha'} = \Gamma_{j\gamma}{}^{k}{}_{\alpha} B^{j\,k'\gamma\,\alpha}_{j'k\,\gamma'\alpha'}$$

(2.13)
$$(a)\quad \Gamma_{\beta'}{}^{k'}{}_{\alpha'} = \Gamma_{\beta}{}^{k}{}_{\alpha} B^{\beta\,k'\alpha}_{\beta'k\,\alpha'}$$
$$(b)\quad \Gamma_{\beta'}{}^{\gamma'}{}_{\alpha'} = \Gamma_{\beta}{}^{\gamma}{}_{\alpha} B^{\beta\,\gamma'\alpha}_{\beta'\gamma\,\alpha'} + B_{\beta'}{}^{\gamma}{}_{\alpha'} B^{\gamma'}_{\gamma}$$
$$(c)\quad \Gamma_{\beta'\gamma'}{}^{k'}{}_{\alpha'} = \Gamma_{\beta\gamma}{}^{k}{}_{\alpha} B^{\beta\,k'\gamma\,\alpha}_{\beta'k\,\gamma'\alpha'}$$

(2.14)
$$(a)\quad \Gamma^{\beta'k'}_{j'}{}_{\alpha'} = \Gamma^{\beta k}_{j}{}_{\alpha} B^{j\,\beta'k'\alpha}_{j'\beta\,k\,\alpha'}$$
$$(b)\quad \Gamma^{\beta'\gamma'}_{j'}{}_{\alpha'} = \Gamma^{\beta\gamma}_{j}{}_{\alpha} B^{j\,\beta'\gamma'\alpha}_{j'\beta\,\gamma\,\alpha'}$$
$$(c)\quad \Gamma^{\beta'k'}_{j'\gamma'\alpha'} = \Gamma^{\beta k}_{j\gamma\alpha} B^{i\,\beta'k'\gamma\,\alpha}_{j'\beta\,k\,\gamma'\alpha'} + \delta^{k'}_{j'} B^{\beta'}_{\alpha\gamma} B^{\alpha}_{\alpha'} B^{\gamma}_{\gamma'}$$

(2.15)
$$(a)\quad \Gamma_{j'}{}^{k'\alpha'}_{\;\;i'} = \Gamma_{j}{}^{k\alpha}_{\;i} B^{j\;k'\alpha' i}_{j'k\;\alpha\; i'}$$
$$(b)\quad \Gamma_{j'}{}^{\gamma'\alpha'}_{\;\;i'} = \Gamma_{j}{}^{\gamma\;\alpha}_{\;\;i} B^{j\;\gamma'\alpha' i}_{j'\gamma\;\alpha\; i'}$$
$$(c)\quad \Gamma_{j'\gamma' i'}^{\;k'\alpha'} = \Gamma_{j\gamma\; i}^{\;k\;\alpha} B^{j\;k'\gamma\;\alpha' i}_{j'k\;\gamma'\alpha\; i'}$$

(2.16)
$$(a)\quad \Gamma_{\beta'}{}^{k'\alpha'}_{\;\;i'} = \Gamma_{\beta}{}^{k\alpha}_{\;i} B^{\beta\;k'\alpha' i}_{\beta'k\;\alpha\; i'}$$
$$(b)\quad \Gamma_{\beta'}{}^{\gamma'\alpha'}_{\;\;i'} = \Gamma_{\beta}{}^{\gamma\alpha}_{\;i} B^{\beta\;\gamma'\alpha' i}_{\beta'\gamma\;\alpha\; i'}$$
$$(c)\quad \Gamma_{\beta'}{}^{k'\alpha'}_{\gamma' i'} = \Gamma_{\beta\gamma}{}^{k\alpha}_{\;i} B^{\beta\;k'\gamma\alpha' i}_{\beta'k\;\gamma'\alpha\; i'}$$

(2.17)
$$(a)\quad \Gamma^{\beta'k'\alpha'}_{j'\;\;\;\;i'} = \Gamma^{\beta\gamma\alpha}_{j\;\;\;i} B^{j\;\beta'k'\alpha' i}_{j'\beta\;k\;\alpha\; i'}$$
$$(b)\quad \Gamma^{\beta'\gamma'\alpha'}_{j'\;\;\;\;i'} = \Gamma^{\beta\gamma\alpha}_{j\;\;\;i} B^{j\;\beta'\gamma'\alpha' i}_{j'\beta\;\gamma\;\alpha\; i'}$$
$$(c)\quad \Gamma^{\beta'k'\alpha'}_{j'\gamma' i'} = \Gamma^{\beta k\alpha}_{j\gamma i} B^{j\;\beta'k'\gamma\;\alpha' i}_{j'\beta\;k\;\gamma'\alpha\; i'}$$

Proof. From

$$\nabla_{\delta_{i'}}\delta_{j'} = \nabla_{B^i_{i'}\delta_i} B^j_{j'}\delta_j = B^i_{i'}(\delta_i B^j_{j'})\delta_j + B^{i\;j}_{i'j'}\nabla_{\delta_i}\delta_j,$$

$$\nabla_{\delta_{i'}}\delta_{j'} = \Gamma_{j'}{}^{k'}{}_{i'}\delta_{k'} + \Gamma_{j'}{}^{\alpha'}{}_{i'}\delta_{\alpha'} + \Gamma_{j'}{}^{\;k'}_{\beta' i}\delta^{\beta'}_{k'},$$

(1.17) and (2.7) we obtain (2.9a), (2.9b) and (2.9c). The other relations can be obtained on the similar way.

From the above we have

Theorem 2.3. *All coefficients of linear connection* ∇ *defined by (2.7) are tranasforming as tensors except* $\Gamma_{j}{}^{k}{}_{i}$, $\Gamma_{\beta}{}^{\gamma}{}_{\alpha}$, $\Gamma^{\beta\;j}_{j\;\gamma\;\alpha}$, $\Gamma^{\beta k}_{j\beta i}$ *(no summation over* j *and* β*).*

They have the following law of transformation:

(2.18)
$$\Gamma_{j'}{}^{k'}{}_{i'} = \Gamma_{j}{}^{k}{}_{i} B^{i\;j\;k'}_{i'j'k} + B_{j'}{}^{k}{}_{i'} B^{k'}_{k}$$

(2.19)
$$\Gamma_{\beta'}{}^{\gamma'}{}_{\alpha'} = \Gamma_{\beta}{}^{\gamma}{}_{\alpha} B^{\beta\;\gamma'\alpha}_{\beta'\gamma\;\alpha'} + B_{\beta'}{}^{\gamma}{}_{\alpha'} B^{\gamma'}_{\gamma}.$$

(2.20)
$$\Gamma^{\beta'k'}_{j'\beta' i'} = \Gamma^{\beta k}_{j\gamma i} B^{j\;\beta'k'\gamma\;i}_{j'\beta\;k\;\beta' i'} + B_{j'}{}^{k}{}_{i'} B^{k'}_{k}$$

(2.21)
$$\Gamma^{\beta' j'}_{j'\gamma'\alpha'} = \Gamma^{\beta k}_{j\gamma\alpha} B^{j\;\beta' j'\gamma\;\alpha}_{j'\beta\;k\;\gamma'\alpha'} + B_{\alpha}{}^{\beta'}{}_{\gamma} B^{\alpha}_{\alpha'} B^{\gamma}_{\gamma'}$$

(j' and β' are fixed).

Definition 2.4. *The action of the linear connection ∇ on the $T^*(E)$, $\nabla : T(E) \times T^*(E) \to T^*(E)$, $\nabla : (X, \omega) \to \nabla_X \omega \in T^*(E)$, $X \in T(E)$, $\omega \in T^*(E)$ is defined by*

$$\nabla_{\delta_i}\delta x^j = \bar\Gamma^j{}_{ki}\delta x^k + \bar\Gamma^j{}_{\gamma i}\delta t^\gamma + \bar\Gamma^{j\gamma}{}_{ki}\delta x^k_\gamma \tag{2.22}$$

$$\nabla_{\delta_i}\delta t^\beta = \bar\Gamma^\beta{}_{ki}\delta x^k + \bar\Gamma^\beta{}_{\gamma i}\delta t^\gamma + \bar\Gamma^{\beta\gamma}{}_{ki}\delta x^k_\gamma$$

$$\nabla_{\delta_i}\delta x^h_\delta = \bar\Gamma^h_{\delta ki}\delta x^k + \bar\Gamma^h_{\delta\gamma i}\delta t^\gamma + \bar\Gamma^{h\gamma}_{\delta ki}\delta x^k_\gamma$$

$$\nabla_{\delta_\alpha}\delta x^j = \bar\Gamma^j{}_{k\alpha}\delta x^k + \bar\Gamma^j{}_{\gamma\alpha}\delta t^\gamma + \bar\Gamma^{j\gamma}{}_{k\alpha}\delta x^k_\gamma$$

$$\nabla_{\delta_\alpha}\delta t^\beta = \bar\Gamma^\beta{}_{k\alpha}\delta x^k + \bar\Gamma^\beta{}_{\gamma\alpha}\delta t^\gamma + \bar\Gamma^{\beta\gamma}{}_{k\alpha}\delta x^k_\gamma$$

$$\nabla_{\delta_\alpha}\delta x^h_\delta = \bar\Gamma^h_{\delta k\alpha}\delta x^h + \bar\Gamma^h_{\delta\gamma\alpha}\delta t^\gamma + \bar\Gamma^{h\gamma}_{\delta\ k\alpha}\delta x^k_\gamma$$

$$\nabla_{\delta^\alpha_i}\delta x^j = \bar\Gamma^{j\ \alpha}_{ki}\delta x^k + \bar\Gamma^{j\ \alpha}_{\gamma i}\delta t^\gamma + \bar\Gamma^{j\gamma\alpha}_{k\ i}\delta x^k_\gamma$$

$$\nabla_{\delta^\alpha_i}\delta t^\beta = \bar\Gamma^{\beta\ \alpha}_{k\ i}\delta x^k + \bar\Gamma^{\beta\ \alpha}_{\gamma\ i}\delta t^\gamma + \bar\Gamma^{\beta\gamma\alpha}_{k\ i}\delta x^k_\gamma$$

$$\nabla_{\delta^\alpha_i}\delta x^j_\beta = \bar\Gamma^{j\ \ \alpha}_{\beta ki}\delta x^k + \bar\Gamma^{j\ \ \alpha}_{\beta\gamma i}\delta t^\gamma + \bar\Gamma^{j\gamma\alpha}_{\beta ki}\delta x^k_\gamma.$$

Theorem 2.4. *The connection coefficients Γ and $\bar\Gamma$ are connected by:*

$$\bar\Gamma^j{}_{ki} = -\Gamma^{\ j}_{k\ i}, \quad \bar\Gamma^j{}_{\gamma i} = -\Gamma^{\ j}_{\gamma\ i}, \quad \bar\Gamma^{j\gamma}{}_{ki} = -\Gamma^{\gamma j}_{k\ \ i} \tag{2.23}$$

$$\bar\Gamma^\beta{}_{ki} = -\Gamma^{\ \beta}_{k\ i}, \quad \bar\Gamma^\beta{}_{\gamma i} = -\Gamma^{\ \beta}_{\gamma\ i}, \quad \bar\Gamma^{\beta\gamma}{}_{k\ i} = -\Gamma^{\gamma\beta}_{k\ \ i}$$

$$\bar\Gamma^h_{\delta ki} = -\Gamma^{\ \ h}_{k\delta i}, \quad \bar\Gamma^h_{\delta\ \gamma i} = -\Gamma^{\ \ h}_{\gamma\delta i}, \quad \bar\Gamma^{h\gamma}_{\delta k\ i} = -\Gamma^{\gamma h}_{k\delta\ i}.$$

All formulae in (2.23) are valied if everywhere the index i is substituted by α or by ${}^\alpha_i$.

Proof. From the duality of the adapted bases B and B^* we obtain

$$\langle\delta x^h, \delta_j\rangle = \delta^h_j, \quad \langle\delta x^h, \delta_\beta\rangle = 0, \quad \langle\delta x^h, \delta^\gamma_k\rangle = 0 \Rightarrow$$

$$\langle\nabla_{\delta_i}\delta x^h, \delta_j\rangle = \langle\bar\Gamma^h_{ki}\delta x^k + \bar\Gamma^h_{\gamma i}\delta t^\gamma + \bar\Gamma^{j\gamma}_{k\ i}\delta x^k_\gamma, \delta_j\rangle =$$

$$-\langle\delta x^h, \nabla_{\delta_i}\delta_j\rangle = -\langle\delta x^h, \Gamma^{\ k}_{j\ i}\delta_h + \Gamma^{\ \beta}_{j\ i}\delta_\beta + \Gamma^{\ \ k}_{j\gamma i}\delta^\gamma_k\rangle \Rightarrow$$

$$\bar\Gamma^h{}_{ji} = -\Gamma^{\ h}_{j\ i} \tag{2.24}$$

$$\langle\nabla_{\delta_i}\delta x^h, \delta_\beta\rangle = \langle\bar\Gamma^h{}_{k\ i}\delta x^k + \bar\Gamma^h{}_{\gamma\ i}\delta t^\gamma + \bar\Gamma^{h\gamma}_{k\ i}\delta x^k_\gamma, \delta_\beta\rangle =$$

$$-\langle\delta x^h, \nabla_{\delta_i}\delta_\beta\rangle = -\langle\delta x^h, \Gamma^{\ k}_{\beta\ i}\delta_k + \Gamma^{\ \gamma}_{\beta\ i}\delta_\gamma + \Gamma^{\ \ k}_{\beta\gamma\ i}\delta^\gamma_k\rangle \Rightarrow$$

$$\bar\Gamma^h{}_{\beta i} = -\Gamma^{\ h}_{\beta\ i}. \tag{2.25}$$

$$\langle \nabla_{\delta_i}\delta x^h, \delta_k^\gamma\rangle = \langle \bar\Gamma^h_{\ ki}\delta x^k + \bar\Gamma^h_{\ \gamma i}\delta t^\gamma + \bar\Gamma^{h\delta}_{\ \ j\ i}\delta x^j_\delta, \delta^\gamma_k\rangle =$$

$$-\langle \delta x^h, \nabla_{\delta_i}\delta^\gamma_k\rangle = -\langle \delta x^h, \Gamma^{\gamma j}_{k\ \ i}\delta_j + \Gamma^{\gamma\beta}_{k\ \ i}\delta_\beta + \Gamma^{\gamma h}_{k\ \delta\ i}\delta^\delta_h\rangle \Rightarrow$$

$$\bar\Gamma^{h\gamma}_{\ \ ki} = -\Gamma^{\gamma h}_{k\ \ i}. \tag{2.26}$$

(2.24), (2.25) and (2.26) are the first three equations in (2.23). The other can be proved on the similar way. □

Theorem 2.5. *If the linear connection* ∇ *acts on* $T(E)$ *as d-connection, then its action on* $T^*(E)$ *is given by*

$$\nabla_{\delta_i}\delta x^j = -\Gamma^{\ j}_{k\ i}\delta x^k, \quad \nabla_{\delta_i}\delta t^\alpha = -\Gamma^{\ \alpha}_{\gamma\ i}\delta t^\gamma, \nabla_{\delta_i}\delta x^h_\delta = -\Gamma^{\gamma h}_{k\delta\ i}\delta x^k_\gamma \tag{2.27}$$

$$\nabla_{\delta_\alpha}\delta x^j = -\Gamma^{\ j}_{k\ \alpha}\delta x^k, \quad \nabla_{\delta_\alpha}\delta t^\beta = -\Gamma^{\ \beta}_{\gamma\ \alpha}\delta t^\gamma, \quad \nabla_{\delta_\alpha}\delta x^h_\delta = -\Gamma^{\gamma h}_{k\delta\ \alpha}\delta x^k_\gamma$$

$$\nabla_{\delta^\alpha_i}\delta x^j = -\Gamma^{\ j\ \alpha}_{k\ \ i}\delta x^k, \quad \nabla_{\delta^\alpha_i}\delta t^\beta = -\Gamma^{\ \beta\alpha}_{\gamma\ \ i}\delta t^\gamma, \quad \nabla_{\delta^\alpha_i}\delta x^h_\delta = -\Gamma^{\gamma\ h\ \alpha}_{k\ \delta\ i}\delta x^k_\gamma.$$

Theorem 2.6. *If the linear connection* ∇ *on* $T(E)$ *as s.d. connection, then its action on* $T^*(E)$ *is given by*

$$\nabla_{\delta_i}\delta x^j = -\Gamma^{\ j}_{k\ i}\delta x^k, \tag{2.28}$$

$$\nabla_{\delta_\alpha}\delta t^\beta = -\Gamma^{\ \beta}_{\gamma\ \alpha}\delta t^\gamma,$$

$$\nabla_{\delta^\alpha_i}\delta x^j_\beta = -\Gamma^{\gamma\ j\ \alpha}_{k\ \beta\ i}\delta x^k_\gamma,$$

the other connection coefficients are equal to zero.

Theorem 2.7. *If* $X, Y \in T(E)$, *i.e.*

$$X = X^A\delta_A = X^i\delta_i + X^\alpha\delta_\alpha + X^i_\alpha\delta^\alpha_i$$

$$Y = Y^B\delta_B = Y^j\delta_j + Y^\beta\delta_\beta + Y^j_\beta\delta^\beta_j,$$

then

$$\begin{aligned} \nabla_Y X \quad = \quad & X^A_{\ |B}Y^B\delta_A = \\ & X^i_{\ |j}Y^j\delta_i + X^\alpha_{\ |j}Y^j\delta_\alpha + X^i_{\alpha|j}Y^j\delta^\alpha_i + \\ & X^i_{\ |\beta}Y^\beta\delta_i + X^\alpha_{\ |\beta}Y^\beta\delta_\alpha + X^i_{\alpha|\beta}Y^\beta\delta^\alpha_i + \\ & X^i|^\beta_j Y^j_\beta\delta_i + X^\alpha|^\beta_j Y^j_\beta\delta_\alpha + X^i_\alpha|^\beta_j Y^j_\beta\delta^\alpha_i, \end{aligned} \tag{2.29}$$

where

$$X^v_{\ |u} = \delta_u X^v + \Gamma^{\ v}_{k\ u}X^k + \Gamma^{\ v}_{\gamma\ u}X^\gamma + \Gamma^{\gamma v}_{k\ \ u}X^k_\gamma, \tag{2.30}$$

$$u \in \{j, \beta, {}^\beta_j\}, \quad v \in \{i, \alpha, {}^i_\alpha\}.$$

For instance we have

$$X^i_{|j} = \delta_j X^i + \Gamma^{\ i}_{k\ j} X^j + \Gamma^{\ i}_{\gamma\ j} X^\gamma + \Gamma^{\gamma i}_{k\ j} X^k_\gamma,$$

$$X^i_\alpha|^\beta_j = \delta^\beta_j X^i_\alpha + \Gamma^{\ i\ \beta}_{k\alpha j} X^k + \Gamma^{\ i\beta}_{\gamma\alpha j} X^\gamma + \Gamma^{\gamma i\beta}_{k\alpha j} X^k_\gamma, \ldots$$

Theorem 2.8. *If $\omega \in T^*(E)$, i.e.*

$$\omega = \omega_i \delta x^i + \omega_\alpha \delta t^\alpha + \omega^\alpha_i \delta x^i_\alpha,$$

then

$$(2.31) \qquad \begin{aligned} \nabla_Y \omega \ = \ & \omega_{i|j} Y^j \delta x^i + \omega_{\alpha|j} Y^j \delta t^\alpha + \omega^\alpha_{i|j} Y^j \delta x^i_\alpha + \\ & \omega_{i|\beta} Y^\beta \delta x^i + \omega_{\alpha|\beta} Y^\beta \delta t^\alpha + \omega^\alpha_{i|\beta} Y^\beta \delta x^i_\alpha + \\ & \omega_i|^\beta_j Y^j_\beta \delta x^i + \omega_\alpha|^\beta_j Y^j_\beta \delta t^\alpha + \omega^\alpha_i|^\beta_j Y^j_\beta \delta x^i_\alpha, \end{aligned}$$

where

$$(2.32) \qquad \omega_{v|u} = \delta_u \omega_v - \Gamma^{\ k}_{v\ u} \omega_k - \Gamma^{\ \gamma}_{v\ u} \omega_\gamma - \Gamma^{\ k}_{v\gamma u} \omega^\gamma_k$$

$$u \in \{j, \beta, {}^\beta_j\}, \quad v \in \{i, \alpha, {}^\alpha_i\}.$$

For instance

$$\omega^\alpha_{i|\beta} = \delta_\beta \omega^i_\alpha - \Gamma^{\alpha k}_{i\ \beta} \omega_k - \Gamma^{\alpha\gamma}_{i\ \beta} \omega_\gamma - \Gamma^{\alpha k}_{i\gamma\beta} \omega^\gamma_k,$$

$$\omega^\alpha_i|^\beta_j = \delta^\beta_j \omega^\alpha_i - \Gamma^{\alpha k\beta}_{i\ j} \omega_k - \Gamma^{\alpha\gamma\beta}_{i\ j} \omega_\gamma - \Gamma^{\alpha k\beta}_{i\gamma j} \omega^\gamma_k, \ldots.$$

If the function $f = f(x^k, t^\gamma, x^k_\gamma)$ is defined on E, then

$$(2.33) \qquad \nabla_Y f = f_{|j} Y^j + f_{|\beta} Y^\beta + f|^\beta_j Y^j_\beta,$$

where

$$(2.34) \qquad \begin{aligned} f_{|j} &= \delta_j f = (\partial_j - N^k_{\gamma j} \partial^\gamma_k) f \\ f_{|\beta} &= \delta_\beta f = (\partial_\beta - M^k_{\gamma\beta} \partial^\gamma_k) f \\ f|^\beta_j &= \partial^\beta_j f. \end{aligned}$$

Theorem 2.9. *For the d-connection (2.29) is valid, but now we have*

$$(2.35) \qquad \begin{aligned} X^i_{|u} &= \delta_u X^i + \Gamma^{\ i}_{j\ u} X^j \\ X^\alpha_{|u} &= \delta_u X^\alpha + \Gamma^{\ \alpha}_{\beta\ u} X^\beta \\ X^i_{\alpha|u} &= \delta_u X^i_\alpha + \Gamma^{\gamma i}_{k\alpha u} X^k_\gamma \\ u &\in \{j, \beta, {}^\beta_j\} \end{aligned}$$

For the s.d. connection we have

$$\nabla_X Y = X^i_{\ |j} Y^j \delta_i + X^\alpha_{\ |\beta} Y^\beta \delta_\alpha + X^i_\alpha |^\beta_j Y^j_\beta \delta^\alpha_i, \tag{2.36}$$

and

$$X^i_{\ |j} = \delta_j X^i + \Gamma^{\ i}_{k\ j} X^k, \tag{2.37}$$

$$X^\alpha_{\ |\beta} = \delta_\beta X^\alpha + \Gamma^{\ \alpha}_{\gamma\ \beta} X^\gamma,$$

$$X^i_\alpha |^\beta_j = \delta^\beta_j X^i_\alpha + \Gamma^{\gamma i \beta}_{k \alpha j} X^k_\gamma.$$

Definition 2.5. *The s.d. connection is called recurrent connection if for the metric tensor*

$$g = g_{ij} \delta x^i \otimes \delta x^j + g_{\alpha\beta} \delta t^\alpha \otimes \delta t^\beta + g^{\alpha\beta}_{ij} \delta x^i_\alpha \otimes \delta x^j_\beta$$

the following relations are valid:

$$g_{ij|k} = \delta_k g_{ij} - g_{hj} \Gamma^{\ h}_{i\ k} - g_{ih} \Gamma^{\ h}_{j\ k} = \lambda_k g_{ij} \tag{2.38}$$

$$g_{\alpha\beta|\gamma} = \delta_\gamma g_{\alpha\beta} - g_{\delta\beta} \Gamma^{\ \delta}_{\alpha\ \gamma} - g_{\alpha\delta} \Gamma^{\ \delta}_{\beta\ \gamma} = \lambda_\gamma g_{\alpha\beta}$$

$$g^{\alpha\beta}_{ij} |^\gamma_k = \delta^\gamma_k g^{\alpha\beta}_{ij} - g^{\delta\beta}_{hj} \Gamma^{\alpha h \gamma}_{i \delta k} - g^{\alpha\delta}_{ih} \Gamma^{\beta h \gamma}_{j \delta k} = \lambda^\gamma_k g^{\alpha\beta}_{ij},$$

where

$$\lambda = \lambda_k \delta x^h + \lambda_\gamma \delta t^\gamma + \lambda^\gamma_k \delta x^k_\gamma \tag{2.39}$$

is a 1*-form field. If in (2.38)*

$$\lambda_h = 0, \quad \lambda_\gamma = 0, \quad \lambda^\gamma_h = 0$$

the connection is called metric connection.

Using the usual procedure the connection coefficients can be obtained as functions of the metric tensors and the one form field λ.

3. The torsion tensor

The torsion tensor in the space E is given as usual by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Proposition 3.1. *The Lie brackets on* $T(E)$ *are given by*

$$[\delta_i, \delta_j] = K_{j\delta i}^{\ \ h} \delta^\delta_h, \tag{3.1}$$

$$K_{j\delta i}^{\ \ h} = -[(\partial_i N^h_{\delta j} - N^k_{\gamma i} \partial^\gamma_k N^h_{\delta j}) - (i/j)]$$

$$[\delta_i, \delta_\beta] = K_{\beta\delta i}^{\ \ h} \delta^\delta_h,$$

$$
\begin{aligned}
K_{\beta\delta i}^{\ \ \ h} &= -[(\partial_i M_{\delta\beta}^h - N_{\gamma i}^k \partial_k^\gamma M_{\delta\beta}^h) + \\
&\quad (\partial_\beta N_{\delta i}^h - M_{\gamma\beta}^k \partial_k^\gamma N_{\delta i}^h)] \\
[\delta_i, \delta_j^\beta] &= K_{j\delta i}^{\beta h} \delta_h^\delta, \quad K_{j\delta i}^{\beta h} = \partial_j^\beta N_{\delta i}^k \\
[\delta_\alpha, \delta_\beta] &= K_{\beta\delta\alpha}^{\ \ \ h} \delta_h^\delta, \\
K_{\beta\delta\alpha}^{\ \ \ h} &= [(\partial_\alpha M_{\delta\beta}^h - M_{\gamma\alpha}^k \partial_k^\gamma M_{\delta\beta}^h) - (\alpha/\beta)] \\
[\delta_\alpha, \delta_j^\beta] &= K_{j\ \delta\alpha}^{\beta h} \delta_h^\delta, \quad K_{j\ \ \delta\alpha}^{\beta\ h} = \partial_j^\beta M_{\delta\alpha}^h \\
[\delta_i^\alpha, \delta_j^\beta] &= 0.
\end{aligned}
$$

From the above it follows that all Lie brackets of the basis vectors of $T(E)$ lie in T_2, generated by $\{\delta_h^\delta\}$.

The components of the torsion tensors are:

$$T(\delta_i, \delta_j) = T_{j\ i}^{\ h} \delta_h + T_{j\ \ i}^{\ \alpha} \delta_\alpha + T_{j\ \ \delta i}^{\ \ h} \delta_h^\delta, \tag{3.2}$$

where

$$
\begin{aligned}
T_{j\ i}^{\ k} &= \Gamma_{j\ i}^{\ k} - \Gamma_{ij}^{\ k} \\
T_{j\ \ i}^{\ \alpha} &= \Gamma_{j\ \ i}^{\ \alpha} - \Gamma_{i\ \ j}^{\ \alpha} \\
T_{j\ \ \delta i}^{\ \ h} &= \Gamma_{j\ \ \delta i}^{\ \ h} - \Gamma_{i\ \ \delta j}^{\ \ h} - K_{j\ \ \delta i}^{\ \ h}.
\end{aligned}
$$

$$T(\delta_i, \delta_\alpha) = T_{\alpha\ i}^{\ k} + T_{\alpha\ \ i}^{\ \beta} \delta_\beta + T_{\alpha\delta i}^{\ \ \ h} \delta_h^\delta, \tag{3.3}$$

where

$$
\begin{aligned}
T_{\alpha\ \ i}^{\ k} &= \Gamma_{\alpha\ i}^{\ k} - \Gamma_{i\ \alpha}^{\ k} \\
T_{\alpha\ \ \ i}^{\ \ \beta} &= \Gamma_{\alpha\ i}^{\ \beta} - \Gamma_{i\ \ \alpha}^{\ \beta} \\
T_{\alpha\ \delta\ i}^{\ \ h} &= \Gamma_{\alpha\delta\ i}^{\ \ \ h} - \Gamma_{i\delta\alpha}^{\ \ \ h} - K_{\alpha\delta i}^{\ \ \ h},
\end{aligned}
$$

$$T(\delta_i, \delta_j^\beta) = T_{j\ \ \ i}^{\beta k} \delta_k + T_{j\ \ \ i}^{\beta\alpha} \delta_\alpha + T_{j\ \delta i}^{\beta h} \delta_h^\delta, \tag{3.4}$$

where

$$
\begin{aligned}
T_{j\ \ \ i}^{\beta k} &= \Gamma_{j\ \ \ i}^{\beta k} - \Gamma_{i\ \ j}^{\ k\beta} \\
T_{j\ \ \ i}^{\beta\alpha} &= \Gamma_{j\ \ \ i}^{\beta\alpha} - \Gamma_{i\ \ j}^{\ \alpha\beta} \\
T_{j\ \ \delta\ i}^{\beta\ h} &= \Gamma_{j\ \ \delta\ i}^{\beta\ h} - \Gamma_{i\ \delta\ j}^{\ h\beta} - K_{j\ \delta i}^{\beta h},
\end{aligned}
$$

$$T(\delta_\alpha, \delta_\beta) = T_{\beta\ \alpha}^{\ k}\delta_k + T_{\beta\ \alpha}^{\ \gamma}\delta_\gamma + T_{\beta\delta\alpha}^{\ h}\delta_k^\delta, \tag{3.5}$$

where

$$T_{\beta\ \alpha}^{\ k} = \Gamma_{\beta\ \alpha}^{\ k} - \Gamma_{\alpha\ \beta}^{\ k}$$

$$T_{\beta\ \alpha}^{\ \gamma} = \Gamma_{\beta\ \alpha}^{\ \gamma} - \Gamma_{\beta\ \alpha}^{\ \gamma}$$

$$T_{\beta\ \delta\alpha}^{\ h} = \Gamma_{\beta\delta\alpha}^{\ h} - \Gamma_{\alpha\delta\beta}^{\ h} - K_{\beta\delta\alpha}^{\ h}$$

$$T(\delta_\alpha, \delta_j^\beta) = T_{j\ \ \alpha}^{\beta k}\delta_k + T_{j\ \ \alpha}^{\beta\gamma}\delta_\alpha + T_{j\ \delta\alpha}^{\beta h}\delta_h^\delta, \tag{3.6}$$

where

$$T_{j\ \ \alpha}^{\beta k} = \Gamma_{j\ \ \alpha}^{\beta k} - \Gamma_{\alpha\ \beta}^{\ kj}$$

$$T_{j\ \ \alpha}^{\beta\gamma} = \Gamma_{j\ \ \alpha}^{\beta\gamma} - \Gamma_{\alpha\ \ j}^{\ \gamma\beta}$$

$$T_{j\ \delta\alpha}^{\beta h} = \Gamma_{j\ \delta\alpha}^{\beta h} - \Gamma_{\alpha\delta j}^{\ h\beta} - K_{j\ \delta\alpha}^{\beta h}$$

$$T(\delta_i^\alpha \delta_j^\beta) = T_{j\ \ i}^{\beta k\alpha}\delta_h + T_{j\ \ i}^{\beta\gamma\alpha}\delta_\gamma + T_{j\ \delta i}^{\beta h\alpha}\delta_h^\delta, \tag{3.7}$$

where

$$T_{j\ \ i}^{\beta k\alpha} = \Gamma_{j\ \ i}^{\beta k\alpha} - \Gamma_{i\ \ j}^{\alpha k\beta}$$

$$T_{j\ \ i}^{\beta\gamma\alpha} = \Gamma_{j\ \ i}^{\beta\gamma\alpha} - \Gamma_{i\ \ j}^{\alpha\gamma\beta}$$

$$T_{j\ \delta\ i}^{\beta h\alpha} = \Gamma_{j\ \delta\ i}^{\beta h\alpha} - \Gamma_{i\ \delta\ j}^{\alpha h\beta}$$

From the above we obtain

Theorem 3.1. *The torsion tensor in $T(E)$ for the linear connection ∇ is given by*

$$T(\delta_A, \delta_B) = T_{B\ A}^{\ k}\delta_k + T_{B\ \ A}^{\ \gamma}\delta_\gamma + T_{B\delta A}^{\ k}\delta_k^\delta,$$

$$A \in \{i, \alpha, {}^\alpha_i\}, \quad B \in \{j, \beta, {}^\beta_j\},$$

where

$$T_{B\ A}^{\ k} = \Gamma_{B\ A}^{\ k} - \Gamma_{A\ B}^{\ k}$$

$$T_{B\ \ A}^{\ \gamma} = \Gamma_{B\ \ A}^{\ \gamma} - \Gamma_{A\ \ B}^{\ \gamma}$$

$$T_{B\delta A}^{\ h} = \Gamma_{B\delta A}^{\ h} - \Gamma_{A\delta B}^{\ h} - K_{A\delta B}^{\ h}.$$

If we put $X = X^A\delta_A$, $Y = Y^B\delta_B$, then using the bilinearity of T we get

$$T(X, Y) = T_{B\ A}^{\ C}X^AY^B\delta_C,$$

where

$$C \in \{k, \gamma, {}^{\delta}_{h}\}$$

and the summation is going over all three types of indices. (If $C = {}^{\delta}_{h}$ *appeares as upper index it becomes* ${}^{h}_{\delta}$*).*

Theorem 3.2. *For the d-connection the torsion tensor has the form*

$$\text{(A) (a)}\quad T(\delta_i, \delta_j) = T_{j\ i}^{\ k}\delta_k - K_{j\delta i}^{\ k}\delta_h^\delta, \quad T_{j\ i}^{\ k} = \Gamma_{j\ i}^{\ k} - \Gamma_{i\ j}^{\ k}$$

$$\text{(b)}\quad T(\delta_i, \delta_\alpha) = T_{\alpha\ i}^{\ \beta}\delta_\beta - K_{\alpha\delta i}^{\ h}\delta_h^\delta, \quad T_{\alpha\ i}^{\ \beta} = \Gamma_{\alpha\ i}^{\ \beta}$$

$$\text{(c)}\quad T(\delta_i, \delta_j^\beta) = T_{j\ \delta i}^{\beta k}\delta_k^\delta, \quad T_{j\ \delta i}^{\beta k} = \Gamma_{j\ \delta i}^{\beta k} - K_{j\ \delta i}^{\beta k}$$

$$\text{(d)}\quad T(\delta_\alpha, \delta_\beta) = T_{\beta\ \alpha}^{\ \gamma}\delta_\gamma - K_{\beta\delta\alpha}^{\ h}\delta_h^\delta, \quad T_{\beta\ \alpha}^{\ \gamma} = \Gamma_{\beta\ \alpha}^{\ \gamma} - \Gamma_{\beta\ \alpha}^{\ \gamma}$$

$$\text{(e)}\quad T(\delta_\alpha, \delta_j^\beta) = T_{j\ \delta\alpha}^{\beta k}\delta_k^\delta, \quad T_{j\ \delta\alpha}^{\beta h} = \Gamma_{j\ \delta\alpha}^{\beta h} - K_{j\ \delta\alpha}^{\beta h}$$

$$\text{(f)}\quad T(\delta_i^\alpha, \delta_j^\beta) = T_{j\ \delta i}^{\beta k\alpha}\delta_k^\delta, \quad T_{j\ \delta i}^{\beta k\alpha} = \Gamma_{j\ \delta i}^{\beta k\alpha} - \Gamma_{i\ \delta j}^{\alpha k\beta}.$$

Theorem 3.3. *For the s.d. connection we have*

$$\text{(B) (a)}\quad T(\delta_i, \delta_j) = T_{j\ i}^{\ k}\delta_k - K_{j\delta i}^{\ k}\delta_h^\delta, \quad T_{j\ i}^{\ k} = \Gamma_{j\ i}^{\ k} - \Gamma_{i\ j}^{\ k} - \Gamma_{ij}^{k}$$

$$\text{(b)}\quad T(\delta_i, \delta_\alpha) = -K_{\alpha\delta i}^{\ h}\delta_h^\delta$$

$$\text{(c)}\quad T(\delta_i, \delta_j^\beta) = -K_{j\ \delta i}^{\beta k}\delta_k^\delta$$

$$\text{(d)}\quad T(\delta_\alpha, \delta_\beta) = T_{\beta\ \alpha}^{\ \gamma}\delta_\gamma - K_{\beta\delta\alpha}^{\ h}\delta_h^\delta, \quad T_{\beta\ \alpha}^{\ \gamma} = \Gamma_{\beta\ \alpha}^{\ \gamma} - \Gamma_{\alpha\ \beta}^{\ \gamma}$$

$$\text{(e)}\quad T(\delta_\alpha, \delta_j^\beta) = -K_{j\ \delta\alpha}^{\beta k}\delta_k^\delta$$

$$\text{(f)}\quad T(\delta_i^\alpha, \delta_j^\beta) = T_{j\ \delta i}^{\beta k\alpha}\delta_k^\delta, \quad T_{j\ \delta i}^{\beta k\alpha} = \Gamma_{j\ \delta i}^{\beta k\alpha} - \Gamma_{i\ \delta j}^{\alpha k\beta}.$$

4. The curvature tensor

In the space E the curvature tensor $R(X,Y)Z$ is defined as usually by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{4.1}$$

If we use the notations

$$X = X^A\delta_A = X^i\delta_i + X^\alpha\delta_\alpha + X_\alpha^i\delta_i^\alpha \tag{4.2}$$

$$Y = Y^B\delta_B = Y^j\delta_j + Y^\beta\delta_\beta + Y_\beta^j\delta_j^\beta$$

$$Z = Z^C\delta_C = Z^h\delta_h + Z^\gamma\delta_\gamma + Z_\gamma^h\delta_h^\gamma$$

and the fact that R is linear with respect to all three components we get

$$R(X^A\delta_A, Y^B\delta_B)Z^C\delta_C = R(\delta_A,\delta_B)\delta_C X^A Y^B Z^C = R^D_{CBA} X^A Y^B Z^C \delta_D, \tag{4.3}$$

where

$$R^{\ D}_{C\ BA}\delta_D = R(\delta_A,\delta_B)\delta_C, \tag{4.4}$$

$$A \in \{i, \alpha, {}^{i}_{\alpha}\}, \quad B \in \{j, \beta, {}^{j}_{\beta}\} \qquad C \in \{k, \gamma, {}^{k}_{\gamma}\}, \quad D \in \{k, \delta, {}^{k}_{\delta}\}. \tag{4.5}$$

In (4.3) the summation is going over all three types of indices. From this fact it follows that on the right hand side of (4.3) there are 3^4 types of summands as:

$$R^{\ k}_{h\ ji} X^i Y^j Z^h \delta_k + R^{\ k}_{h\ j\alpha} X^\alpha Y^j Z^k \delta_k + \cdots + R^{\gamma k\beta\alpha}_{h\delta ji} X^i_\alpha Y^j_\beta Z^h_\gamma \delta^\delta_k.$$

If we use the shorter representation of (2.7) we can write:

$$\nabla_{\delta_B}\delta_C = \Gamma^{\ D}_{C\ B}\delta_D,$$

$$\nabla_{\delta_A}\nabla_{\delta_B}\delta_C = \{\delta_A \Gamma^{\ D}_{C\ B} + \Gamma^{\ E}_{C\ B}\Gamma^{\ D}_{E\ A}\}\delta_D.$$

We denote

$$K^{\ D}_{C\ BA}\delta_D = \nabla_{\delta_A}\nabla_{\delta_B}\delta_C - \nabla_{\delta_B}\nabla_{\delta_A}\delta_C = [(\delta_A \Gamma^{\ D}_{C\ B} + \Gamma^{\ E}_{C\ B}\Gamma^{\ D}_{E\ A}) - (A/B)]\delta_D. \tag{4.6}$$

From (3.1) it follows

$$[\delta_A, \delta_B] = K^{\ h}_{B\delta A}\delta^\delta_h \Rightarrow$$

$$\nabla_{[\delta_A,\delta_B]}\delta_C = K^{\ h}_{B\delta A}\Gamma^{\ D\ \delta}_{C\ \ h}\delta_D. \tag{4.7}$$

From (4.4), (4.6) and (4.7) it follows

Theorem 4.1. *For the linear connection ∇ in space E the curvature tensor has the form*

$$R^{\ D}_{C\ BA} = K^{\ D}_{C\ BA} - K^{\ h}_{B\delta A}\Gamma^{\ D\delta}_{C\ \ h}. \tag{4.8}$$

For indices A, B, C, D, (4.5) is valid.

As was mentioned before, that there are 3^4 types of curvature tensors; if $A = {}^{i}_{\alpha}$, $B = {}^{j}_{\beta}$ then $K^{\beta h\alpha}_{j\delta\ i} = 0$.

For the d-connection in (4.6) the indices C, D and E have to be of the same kind, so it follows

Theorem 4.2. *For the d-connection we have the following types of curvature tensor*

(4.9)
$$R_{h\ BA}^{\ k} = K_{h\ BA}^{\ k} - K_{B\varepsilon A}^{\ l}\Gamma_{h\ \ l}^{\ k\varepsilon}$$
$$R_{\gamma\ BA}^{\ \delta} = K_{\gamma\ BA}^{\ \delta} - K_{B\varepsilon A}^{\ l}\Gamma_{\gamma\ \ l}^{\ \delta\varepsilon}$$
$$R_{h\delta BA}^{\gamma k} = K_{h\delta BA}^{\gamma k} - K_{B\varepsilon A}^{\ l}\Gamma_{h\ \delta\ l}^{\gamma k\ \varepsilon},$$

where

$$K_{h\ BA}^{\ k} = [(\delta_A\Gamma_{h\ B}^{\ k} + \Gamma_{h\beta B}^{\ l}\Gamma_{l\ A}^{\ k}) - (A/B)]$$
$$K_{\gamma\ BA}^{\ \delta} = [(\delta_A\Gamma_{\gamma\ B}^{\ \delta} + \Gamma_{\gamma\ B}^{\ \varepsilon}\Gamma_{\varepsilon\ A}^{\ \delta}) - (A/B)]$$
$$K_{h\ \delta BA}^{\gamma k} = [(\delta_A\Gamma_{h\ \delta B}^{\gamma\ k} + \Gamma_{h\ \varepsilon B}^{\gamma\ l}\Gamma_{l\ \delta A}^{\varepsilon\ k}) - (A/B)]$$
$$A \in \{i, \alpha, {}^{i}_{\alpha}\}, \quad B \in \{j, \beta, {}^{j}_{\beta}.$$

From the above it is obvious that for the *d*-connection there are $3 \cdot 3^2$ types of curvature tensors. When the indices A and B belong to T_2, then the second terms on the right hand side in (4.8) are equal to zero.

Theorem 4.3. *For the s.d. connection we have the following types of curvature tensors:*

(4.10)
$$R_{h\ ji}^{\ k} = K_{h\ ji}^{\ k}$$
$$R_{\gamma\ \beta\alpha}^{\ \delta} = K_{\gamma\ \beta\alpha}^{\ \delta}$$
$$R_{h\ \delta ji}^{\gamma k\beta\alpha} = K_{h\ \delta ji}^{\gamma k\beta\alpha}.$$

where

$$K_{h\ ji}^{\ k} = (\delta_i\Gamma_{h\ j}^{\ k} + \Gamma_{h\ j}^{\ l}\Gamma_{l\ h}^{\ k}) - (i/j)$$
$$K_{\gamma\ \beta\alpha}^{\ \delta} = (\delta_\alpha\Gamma_{\gamma\ \beta}^{\ \delta} + \Gamma_{\gamma\ \beta}^{\ \varepsilon}\Gamma_{\varepsilon\ \alpha}^{\ \delta}) - (\alpha/\beta)$$
$$K_{h\ \delta ji}^{\gamma k\beta\alpha} = (\delta_i^\alpha\Gamma_{h\ \delta j}^{\gamma k\beta} + \Gamma_{h\ \varepsilon j}^{\gamma l\beta}\Gamma_{l\ \delta i}^{\varepsilon k\alpha}) - ({}^{\alpha}_{i}/{}^{\beta}_{j}).$$

For the s.d. connection all three indices in $\Gamma_{A\ C}^{\ B}$ are of the same type, the other are equal to zero and $K_{B\varepsilon A}^{\ l} = 0$, when A and B belong to T_2.

If in the space E the metric tensor is given by (2.2) and $g_{ij} = g_{ij}(x)$, $g_{\alpha\beta} = g_{\alpha\beta}(t)$, $g_{i\ j}^{\alpha\beta} = g_{i\ j}^{\alpha\beta}(x_\gamma^b)$, then the connection coefficients

$$\Gamma_{i\ j}^{\ k}, \quad \Gamma_{\alpha\ \beta}^{\ \gamma}, \quad \Gamma_{i\ \beta\ k}^{\alpha\ j\gamma}$$

can be obtained as Cristoffel symbols of the corresponding metric tensors. In this case the connection coefficients are symmetric.

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