A criterion for dispersive dynamical systems on a topological manifold

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Abstract

In this paper we show that Auslander-Bhatia's-theorem [2], relative to a dynamical system on a metric space, can be extended to dynamical systems defined on topological manifolds.

M.S.C. 2000: 37C55, 54H20.
Key words: dispersive dynamical system, periodic orbit.

1 Preliminaries

Let X be a topological manifold over R^n, and let $\mathcal{A}$ be the atlas which gives the manifold structure, $\dim(X) = n$,

$$\mathcal{A} = \{h_a(\chi_a, U_a)\}_{a \in J},$$

where $U_a \subset X$, and $\chi_a : U_a \to \chi_a(U_a) \subset R^n$ are homeomorphisms for all $a \in J$.

Definition 1.1. [2] *A continuous dynamical system on a topological manifold X is defined by a triplet (X, R, Φ), where Φ is a transformation $\Phi : X \times R \to X$ satisfying the following properties:*
i) $\Phi(x, 0) = x,\ (\forall)\ x \in X$;
ii) $\Phi(\Phi(x, t), s) = \Phi(x, t + s)\ (\forall)\ x \in X,\ (\forall)\ t, s \in R$;
iii) Φ *is continuous.*

In the following we use the notation: $\Phi(x, t) \equiv xt$. The properties (i-ii) can be respectively rewritten:
i') $x0 = x,\quad (\forall)\ x \in X$;
ii') $xt(s) = x(t + s)\quad (\forall)\ x \in X,\quad (\forall)\ t, s \in R$.

We note that the property of continuity iii) is equivalent to
iii') If (x_n) and (t_n) are sequences in X, respectively in R, such that $x_n \to x, t_n \to t$, then $x_n t_n \to xt$.

In concordance with the above notation, if $M \subset X$ and $A \subset R$ we will write

$$MA = \{xt | x \in M,\quad t \in A\}.$$

In the sequel we recall the definitions of some notions involved in our approach. The trajectory of the dynamical system through the point $x \in X$ (the orbit through x) is

Differential Geometry - Dynamical Systems, Vol.7, 2005, pp. 11-14.

the set $\gamma(x) \overset{not}{=} xR = \{xt|\ t \in R\}$. The set $\gamma^+(x) = xR^+$, respectively $\gamma^-(x) = xR^-$, is called the positive semi-trajectory, respectively the negative semi-trajectory through x. A subset $M \subset X$ with the property that $xR \subset M$, $(\forall)\ x \in M$, is called *invariant set.*

The trajectory through $x \in X$ with the property that there exists some $\tau \in R$ such that $x(t+\tau) = xt$ for any $t \in R$, is called *periodic trajectory of period* τ.

A point $x \in X$ such that $xt = x$ for any $t \in R$ is called *rest point* or *equilibrium point.*

For any fixed $x \in X$, the application $\Phi_x : R \to X$ defined by $\Phi_x(t) = xt$ is called *the motion of* x.

2 Omega limit prolongation and prolongational limit set

Definition 2.1. $y \in X$ is a *positive (negative) limit point* of some $x \in X$ if there exists a sequence $(t_n), t_n \to \infty(-\infty)$, such that $xt_n \to y$.
The ω–limit set of a point $x \in X$ is denoted by
$\omega(x) = \{y \in X|(\exists)\ (t_n) \subset R^+,\ t_n \to \infty \text{ and } xt_n \to y\}$.

Definition 2.2 [2] The positive prolongation limit set of a point x, respectively the negative prolongation limit set is the set defined by
$D^+(x) = \{y \in X|(\exists)\ (x_n) \subset X \text{ and } (t_n) \subset R^+, \text{ such that } x_n \to x \text{ and } x_nt_n \to y\}$
$D^-(x) = \{y \in X|(\exists)\ (x_n) \subset X \text{ and } (t_n) \subset R^-, \text{ such that } x_n \to x \text{ and } x_nt_n \to y\}$

Definition 2.3 [2] The *positive prolongational limit set*, and the *negative prolongational limit set* of any $x \in X$ are the sets defined respectively by:
$J^+(x) = \{y \in X|(\exists)\ (x_n) \subset X \text{ and } (t_n) \subset R, \text{ such that } x_n \to x, y_n \to \infty \text{ and } x_nt_n \to y\}$
$J^-(x) = \{y \in X|(\exists)\ (x_n) \subset X \text{ and } (t_n) \subset R, \text{ such that } x_n \to x, y_n \to -\infty \text{ and } x_nt_n \to y\}$.

From these definitions it follows imediately:

Proposition 2.4. $\overline{\gamma^+(x)} \subset D^+(x)$ and $D^+(x) = \gamma^+(x) \cup J^+(x)$.

3 Dispersive dynamical systems

Definition 3.1 [2] A dynamical system (X, R, Φ) is called *dispersive* if for any $x, y \in X$ there exist two neighborhoods U_x and U_y and a constant $T > 0$ such that $\Phi_t(U_x) \cap U_y = \varnothing$ for all $t \in R$, $|t| \geq T$.

Theorem 3.2 $J^+(x)$ *is a closed invariant set for all* $x \in X$.

Proof. We show that $J^+(x)$ is closed. For a sequence $(y_k) \subset J^+(x)$ such that $y_k \to y$, it follows that $y \in J^+(x)$. Indeed, for each $k \in N^*$ there exists the sequence $(t_n^k) \subset R; t_n^k \to \infty$ and $(x_n^k) \subset X;\ \ x_n^k \to x$ with $\lim\limits_{n\to\infty} x_n^k t_n^k = y_k$.

Consider $y \in U$, where U is the geometric zone of the chart $h(U, \chi) \in \mathcal{A}$. Then there exists an integer $n_0 \in N^*$ such that for each $k \geq n_0, y_k \in U$. For each $y_k \in U, k \geq n_0$ there exists n_k^0 such that for $n_k \in N^*, n_k \geq n_0^k$ we have that $t_{n_k}^k > k, x_{n_k}^k t_{n_k}^k \in U$, and

$$||\chi(x_{n_k}^k t_{n_k}^k) - \chi(y_k)|| < \frac{1}{k}$$

Consider now the sequences:

$$(t_k : t_k = t_{n_k}^k), \quad (x_k : x_k = x_{n_k}^k).$$

We observe that $t_k \to \infty$, $x_k \to x$, and $x_k t_k \to y$ because

$$||\chi(x_k t_k) - \chi(y)|| \leq ||\chi(x_k t_k) - \chi(y_k)|| + ||\chi(y_k) - \chi(y)|| \leq \frac{1}{k} + ||\chi(y_k) - \chi(y)||$$

Hence $y \in J^+(x)$.

In order to show that $J^+(x)$ is an invariant set we prove that for every $y \in J^+(x)$ we have $yR \subset J^+(x)$. For this, we take the sequences $(t_n) \subset R$ $t_n \to \infty$ and $(x_n) \subset X, x_n \to x, x_n t_n \to y \in J^+(x)$, and $\tau \in R$. We have $x_n(t_n + \tau) = (x_n t_n)\tau \to y\tau$ and this implies $y\tau \in J^+(x)$. q.e.d.

Recently we have proved [4] for a dynamical system on a topological space the following property:

Theorem 3.3 *A dynamical system (X, R, Φ) is dispersive if and only if for each $x \in X$, $J^+(x) = \varnothing$.*

By means of this result we extend the Auslander-Bhatia's theorem [2] which states that a continuous dynamical system on a metric space is dispersive if and only if the positive prolongation limit set and the positive semi-trajectory of each point are identically and the system does not exhibit rest points or periodic trajectories.

Theorem 3.4 *The dynamical system (X, R, Φ) is dispersive if and only if for each $x \in X$ the positive prolongation limit set of x coincides with the positive semi–trajectory through x, i.e. $D^+(x) = \gamma^+(x)$, and the system does not exhibit rest points or periodic trajectories.*

Proof. If (X, R, Φ) is dispersive then by Theorem 3.3, $J^+(x) = \varnothing$ $(\forall)$ $x \in X$. Using Proposition 2.4 we obtain $D^+(x) = \gamma^+(x), (\forall)$ $x \in X$. If x is a rest point or $\gamma(x)$ is a periodic trajectory, then $\gamma(x) \equiv \omega(x) \subset J^+(x)$, and this contradicts $J^+(x) = \varnothing$.

Conversely, suppose that $D^+(x) = \gamma^+(x)$ and that there are no rest points or periodic orbits. We must prove that $J^+(x) = \varnothing$.

From Proposition 2.4 we have $D^+(x) = \gamma^+(x) \cup J^+(x) = \gamma^+(x)$. This implies that $J^+(x) \subset \gamma^+(x)$; By Theorem 3.2, $J^+(x)$ is a closed invariant set and if $J^+(x)$ is not empty, we have that $\gamma(x) \subset J^+(x) \subset \gamma^+(x)$, i.e. $\gamma(x) = \gamma^+(x)$. This shows that if $t' < 0$, then there exists $t \geq 0$ such that $xt' = xt$, i.e.

$$x = x(t - t').$$

Since $t - t' > 0$, we conclude that the trajectory $\gamma(x)$ is closed and has the period $t - t'$, which is a contradiction. q.e.d.

References

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