General adapted linear connections in almost paracontact and contact geometries

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Dedicated to the memory of Professor dr. Vasile Oproiu (1941-2020)

Abstract. Given an almost paracontact structure (φ, ξ, η) on a pseudo-Riemannian manifold (M^{2n+1}, g) of signature $(n+1, n)$ we define a linear connection as being adapted if it parallelizes all its structural elements φ, η, ξ. We find the class of all adapted connections using the tools of derivations. The particular cases of para-Sasakian and para-Kenmotsu manifolds are detailed in order to compare with the Levi-Civita connection of g and with the canonical connection of S. Zamkovoy from [24]. Also, we unify our framework with the almost contact geometry by using a parameter ε corresponding to ± 1 and we find the class of linear connections which provide the general admissible triples of covariant derivatives for (φ, η, ξ); in particular the Matzeu-Oproiu linear connection is analyzed. We search applications of our computations to statistical and weak Frobenius structures.

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1 Introduction

The almost paracontact geometry introduced by Kaneyuki and co-workers, for example in [16], offers an interesting counterpart to the more known almost contact geometry. Now, the setting is provided by a pseudo-Riemannian manifold (M^{2n+1}, g) of signature $(n+1, n)$ instead of the Riemannian framework of almost contact geometry. Also, the prefix "para" corresponds to an almost paracomplex structure while the almost contact version gives an almost complex one. Similar to the almost contact case there are some particular remarkable geometries: para-cosymplectic, para-Sasakian and para-Kenmotsu.

In this paper we shall study a special class of linear connections D on an almost paracontact manifold $(M, g, \varphi, \eta, \xi)$ following the studies [3]-[5]. More precisely, we

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define D as being *adapted* if parallelizes the structural tensors (φ, η, ξ). A strong motivation for such a study comes from the appearance of paracontact structures in some physical theory e.g. para-Sasakian geometry in thermodynamic fluctuation theory of [6]. We find the family of all adapted connections following the technique from almost contact geometry developed in [18] and based on derivations. More precisely, we find a process to associate at any linear connection ∇ an adapted one D^∇ and a large part of our article concerns with a study of the corresponding D^g induced by the Levi-Civita connection ∇^g. For example, we compute explicitly the difference $A = D^g - \nabla^g \in \mathcal{T}^1_2(M)$ in para-Sasakian and para-Kenmotsu manifolds and also, we compare our D^g with the canonical connection introduced by S. Zamkovoy in [24]. An interesting result holds in para-Sasakian geometry: this canonical connection is a fixed point of the transformation $\nabla \to D^\nabla$. As usually, a main attention is devoted to the torsion and curvature of general adapted connections, again with a special view towards para-Sasakian and para-Kenmotsu structures. We discuss also the Schouten connection associated to an almost product structure τ, naturally provided by the underlying paracontact structure, and a generalization to almost r-paracontact manifolds, with $r \geq 2$.

Another direction of study is the unification of almost contact and almost paracontact geometries. We perform this in section 4 by introducing a parameter ε with $\varepsilon = -1$ corresponding to the almost contact case respectively $\varepsilon = +1$ to the almost paracontact situation. In this general framework we consider a triple $J^* = (\varphi^*, \eta^*, \xi^*) \in \mathcal{T}^1_2(M) \times \mathcal{T}^0_2(M) \times \mathcal{T}^1_1(M)$ and searching D^* satisfying $D^*\varphi = \varphi^*$, $D^*\eta = \eta^*$ and $D^*\xi = \xi^*$ it results the class of admissible triples J^* and the corresponding D^* also through a general map $\nabla \to D^{*,\nabla}$. Again, the ε-Sasakian case is a discussed example as well as some generalizations of admissible triples from the almost contact geometries. We finish this study searching for applications to statistical and weak Frobenius structures in almost paracontact setting.

2 Almost paracontact manifolds

Almost paracontact geometry appears in [16] and some important studies are [8], [9], [24]. Let M be a $(2n+1)$-dimensional smooth manifold, φ a tensor field of $(1,1)$-type called *the structural endomorphism*, ξ a vector field called *the characteristic vector field*, η a 1-form called *the paracontact form* and g a pseudo-Riemannian metric on M of signature $(n+1, n)$. We say that (φ, ξ, η, g) defines an *almost paracontact metric structure* on M if [24, p. 38], [8]:
1. $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$, 2. $\eta(\xi) = 1$, $\varphi^2 = I - \eta \otimes \xi$,
3. φ induces on the $2n$-dimensional distribution $\mathcal{D} := \ker \eta$ an almost paracomplex structure P i.e. $P^2 = 1$ and the eigensubbundles T^+, T^-, corresponding to the eigenvalues $1, -1$ of P respectively, have equal dimension n; hence $\mathcal{D} = T^+ \oplus T^-$,
4. $g(\varphi\cdot, \varphi\cdot) = -g + \eta \otimes \eta$.

For a list of examples of almost paracontact metric structures see [10, p. 666], [12], [13, p. 569], [15, p. 84] and [19]. From the definition it follows that the rank of φ is $2n$ and η is the g-dual of ξ i.e. $\eta(X) = g(X, \xi)$ for any $X \in \Gamma(TM) = \mathfrak{X}(M)$. Also, ξ is an unitary vector field:

$$g(\xi, \xi) = 1 \tag{2.1}$$

which means that it is space-like and φ is a g-skew-symmetric operator:

$$g(\varphi X, Y) = -g(X, \varphi Y). \tag{2.2}$$

The tensor field:

$$\omega(X, Y) := g(X, \varphi Y) \tag{2.3}$$

is skew-symmetric and:

$$\omega(\varphi X, Y) = -\omega(X, \varphi Y), \quad \omega(\varphi X, \varphi Y) = -\omega(X, Y). \tag{2.4}$$

The 2-form ω is called *the fundamental form* of the given geometry. Remark that the canonical distribution $\mathcal{D}$ is φ-invariant since $\mathcal{D} = Im\varphi$: if $X \in \mathfrak{X}(M)$ has the decomposition $X = X^+ + X^- + \eta(X)\xi$ with $X^* \in T^*$ (with $* \in \{+, -\}$) then $\varphi X = X^+ - X^-$. Moreover, ξ is orthogonal to $\mathcal{D}$ and therefore the tangent bundle splits orthogonally:

$$TM = \mathcal{D} \oplus \langle \xi \rangle. \tag{2.5}$$

Following [18, p. 267] we consider also *the vertical projector* $V := \eta \otimes \xi$ which satisfies:

$$\eta \circ V = \eta, \quad V^2 = V, \quad V \circ \varphi = \varphi \circ V = 0 \tag{2.6}$$

and which have ξ as eigenvector corresponding to the eigenvalue $+1$. The *horizontal projector* is as usual $H := I - V$.

The almost paracontact structure of $ap(M) := (M, \varphi, \eta, \xi)$ yields an almost paracomplex structure J on the product $M \times \mathbb{R}$ and $ap(M)$ is called *normal* if J is integrable, [24, p. 39]. Also $ap(M)$ is called *paracontact metric manifold* if:

$$2\omega(X, Y) = 2d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) = (\nabla^g_X \eta)Y - (\nabla^g_Y \eta)X. \tag{2.7}$$

where ∇^g is the Levi-Civita connection of g. On a paracontact metric manifold we have:

$$\nabla^g_X \xi = -\varphi X + \varphi h X. \tag{2.8}$$

where:

$$h = \frac{1}{2}\mathcal{L}_\xi \varphi \tag{2.9}$$

with $\mathcal{L}$ the Lie derivative. In a paracontact metric manifold the tensor field h vanishes if and only if $ap(M)$ is K-*paracontact* i.e. ξ is a Killing vector field with respect to g.

Another important class of almost para-contact geometries is provided by *para-Kenmotsu manifolds* satisfying [25]:

$$(\nabla^g_X \varphi)Y = -\eta(Y)\varphi X - \omega(X, Y)\xi. \tag{2.10}$$

In [4] there are studied two types of linear connections ∇, namely that which parallelize simultaneous g and η respectively parallelize the triple (g, η, ξ). We introduce now another suitable type of linear connections as aim of our study:

Definition 2.1. The linear connection D is *adapted* to the almost paracontact geometry $ap(M) = (M, \varphi, \eta, \xi)$ if all structural fields are covariant constant: $D\varphi = 0$, $D\eta = 0$, $D\xi = 0$.

The tool to determine the set of adapted linear connections consists in derivations to which we devote the next section.

3 Derivations and initial data

We present the general theory of derivations following [18, p. 288]. Let $\mathcal{T}(M) = \otimes_{r,s}\mathcal{T}^r_s(M)$ be the tensorial algebra of M.

Definition 3.1. A linear endomorphism ∂ of $\mathcal{T}(M)$ is called *derivation* if the following properties hold:
i) is type preserving, i.e. ∂ maps $\mathcal{T}^r_s(M)$ into itself,
ii) satisfies a Leibniz rule: $\partial(A \otimes B) = \partial A \otimes B + A \otimes \partial B$,
iii) commutes with any contraction.
The set of all derivations is a $C^\infty(M)$-module denoted $Der(M)$.

A technical characterization is:

Proposition 3.1. *Fix $\mu \in \mathfrak{X}(M)$ and the additive map $\Phi : \mathcal{T}^1_0(M) \to \mathcal{T}^1_0(M)$ satisfying:*

$$\Phi(fX) = \mu(f)X + f\Phi(X). \tag{3.1}$$

Then there exists a unique $\partial \in Der(M)$ with $\partial|_{C^\infty(M)} = \mu$ and $\partial|_{\mathcal{T}^1_0(M)} = \Phi$.

We introduce then the notations $\partial = \{\mu, \Phi\}$, $\mu = res_0\partial$ and $\Phi = res_1\partial$ and we remark that $\mu = 0$ means that $\Phi \in \mathcal{T}^1_1(M)$. Also, we point out that the action of $\partial = \{\mu, \Phi\}$ on $F \in \mathcal{T}^1_1(M)$, $\omega \in \mathcal{T}^0_1(M) = \Omega^1(M)$ is:

$$\partial(F) = \Phi \circ F - F \circ \Phi, \quad \partial(\omega) = \mu \circ \omega - \omega \circ \Phi. \tag{3.2}$$

We return now to an almost paracontact manifold $ap(M)$ and we introduce the second main notion of this work:

Definition 3.2. A triple $J = (\varphi^*, \eta^*, \xi^*) \in \mathcal{T}^1_1(M) \times \Omega^1(M) \times \mathfrak{X}(M)$ is called *system of $ap(M)$-initial data* if:

$$\varphi^* \circ \xi + \varphi \circ \xi^* = 0, \quad \eta^* \circ \varphi + \eta \circ \varphi^* = 0, \quad \eta^*(\xi) + \eta(\xi^*) = 0, \quad \varphi^* \varphi + \varphi \varphi^* = -V^* \tag{3.3}$$

with the endomorphism $V^* \in \mathcal{T}^1_1(M)$ given by: $V^* := \eta^* \otimes \xi + \eta \otimes \xi^*$.

Direct consequences of this definition are some relations similar to (2.2) of [18, p. 269]:

$$\eta^* \circ V + \eta \circ V^* = \eta^*, V^*(\xi) + V(\xi^*) = \xi^*, V^*V + VV^* = V^*, \varphi^* V + \varphi V^* = V^* \varphi + V \varphi^* = 0. \tag{3.4}$$

The set $\mathcal{ID}(ap(M))$ of all systems of $ap(M)$-initial data is a $C^\infty(M)$-submodule of the $C^\infty(M)$-module $\mathcal{T}^1_1(M) \times \mathcal{T}^0_1(M) \times \mathcal{T}^1_0(M)$.

A motivation for the introduction of systems of $ap(M)$-initial data is provided by the following result:

Proposition 3.2. *If $\partial \in Der(M)$ then $J_\partial := (\partial(\varphi), \partial(\eta), \partial(\xi) = \Phi(\xi)) \in \mathcal{ID}(ap(M))$.*

Hence we have a $C^\infty(M)$-linear map:

$$K : Der(M) \to \mathcal{ID}(ap(M)), \quad K(\partial) := J_\partial \tag{3.5}$$

and a natural problem is the surjectivity of it:

Definition 3.3. Fix $J \in \mathcal{ID}(ap(M))$. A derivation $\partial \in Der(M)$ is called *J-adapted* if: $K(\partial) = J$.

Remark 3.4. Suppose that ∂ is $(0,0,0)$-adapted. Then:

$$\partial V = \partial(I - \varphi^2) = -\partial\varphi^2 = -(\partial\varphi \circ \varphi + \varphi \circ \partial\varphi) = 0, \quad \partial H = \partial(I - V) = 0. \tag{3.6}$$

Then the associated *almost product structure* $\tau = V - H$ is also a zero of ∂. The adapted derivations to almost product structures are studied in [17]. $\square$

In order to obtain the set of all J-adapted derivations we introduce another $C^\infty(M)$-linear map:

$$L : \mathcal{ID}(ap(M)) \to Der(M), L(J) = \partial_J := \{0, \frac{1}{2}(-\varphi\varphi^* - VV^* + \eta\otimes\xi^* - \eta^*\otimes\xi)\}. \tag{3.7}$$

A main property of L is exactly the answer to the problem raised above:

Proposition 3.3. *$K \circ L : \mathcal{ID}(ap(M)) \to \mathcal{ID}(ap(M))$ is the identity map and hence K is a surjection and L is an injection.*

We introduce now another $C^\infty(M)$-linear map:

Definition 3.5. The application $C : Der(M) \to Der(M)$ given by:

$$C := Id - 2LK \tag{3.8}$$

is called *$ap(M)$-conjugation of derivations*.

Its main properties are as following:

Proposition 3.4. *i) C is an involution i.e. $C^2 = Id$ and $res_0 \circ C = res_0$, $KC = -K$ which means $(C\partial)(\varphi, \eta, \xi) = -(\partial\varphi, \partial\eta, \partial\xi)$. ii) $(0,0,0)$-adapted derivations are exactly the fixed points of C.*

Hence, the set of all $(0,0,0)$-adapted derivation is the image $Im(\chi)$ where:

$$\chi : Der(M) \to Der(M), \quad \chi := \frac{1}{2}(Id + C). \tag{3.9}$$

A straightforward computation gives the explicit action of C and χ on a fixed $\partial = \{\mu, \Phi\}$:

$$\begin{cases} C(\partial) = \{\mu, \varphi\Phi\varphi + V\Phi V + \partial\eta \otimes \xi - \eta \otimes \Phi(\xi)\}, \\ \chi(\partial) = \{\mu, \frac{1}{2}\left(\Phi + \varphi\Phi\varphi + V\Phi V + \partial\eta \otimes \xi - \eta \otimes \Phi(\xi)\right)\}. \end{cases} \tag{3.10}$$

We obtain now the general set of a J-adapted derivations:

Theorem 3.5. *Fix $J \in \mathcal{ID}(ap(M))$. Then the class of all J-adapted derivations is the space $\partial_J + Im(\chi)$.*

Proof. The result is immediately from the remark that ∂ is J-adapted if and only if $\partial - \partial_J$ is $(0,0,0)$-adapted. We point out also that χ is a projector: $\chi^2 = \chi$. $\square$

4 Adapted linear connections

In order to use the results of the previous section let us point out that given a vector field X and a linear connection ∇ if ∇_X is $(0,0,0)$-adapted then there exists $\partial_X \in Der(M)$ such that $\nabla_X = \chi(\partial_X)$. Since $res_0(\nabla_X) = X$ it results $res_0\partial_X = X$ and the correspondence $X \to \partial_X$ is actually a linear connection. It follows then the first main result of this section:

Theorem 4.1. *A linear connection D is adapted if and only if there exists a linear connection ∇ such that:*

$$D_X = \chi(\nabla_X) \tag{4.1}$$

for every vector field X. More precisely, denoting D^∇ the right hand side of (4.1) *we get:*

$$D^\nabla_X Y = \nabla_X Y - \eta(Y)\nabla_X\xi - \frac{1}{2}(\nabla_X\varphi)(\varphi Y) + \frac{1}{2}[(\nabla_X\eta)(Y) + \eta(Y)\eta(\nabla_X\xi)]\xi. \tag{4.2}$$

Its torsion is:

$$(T^D - T^\nabla)(X,Y) = \eta(X)\nabla_Y\xi - \eta(Y)\nabla_X\xi + \frac{1}{2}[(\nabla_Y\varphi)(\varphi X) - (\nabla_X\varphi)(\varphi Y)]+$$

$$+\frac{1}{2}[(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X) + \eta(Y)\eta(\nabla_X\xi) - \eta(X)\eta(\nabla_Y\xi)]\xi. \tag{4.3}$$

If ∇ is symmetric then:

$$T^D(X,Y) = \eta(X)\nabla_Y\xi - \eta(Y)\nabla_X\xi + \frac{1}{2}[(\nabla_Y\varphi)(\varphi X) - (\nabla_X\varphi)(\varphi Y)]+$$

$$+\frac{1}{2}[2d\eta(X,Y) + \eta(Y)\eta(\nabla_X\xi) - \eta(X)\eta(\nabla_Y\xi)]\xi. \tag{4.4}$$

If ∇ is metrical then:

$$(T^D - T^\nabla)(X,Y) = \eta(X)\nabla_Y\xi - \eta(Y)\nabla_X\xi + \frac{1}{2}[(\nabla_Y\varphi)(\varphi X) - (\nabla_X\varphi)(\varphi Y)]+$$

$$+\frac{1}{2}[(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X)]\xi. \tag{4.5}$$

The covariant derivative of the metric g with respect to D^∇ is:

$$2(D^\nabla_X g - \nabla_X g)(Y,Z) = g((\nabla_X\varphi)(\varphi Y), Z) + g(Y, (\nabla_X\varphi)(\varphi Z)) - 2\eta(Y)\eta(Z)\eta(\nabla_X\xi)+$$

$$+\eta(Y)[g(\nabla_X\xi, Z) - (\nabla_X g)(Z,\xi)] + \eta(Z)[g(\nabla_X\xi, Y) - (\nabla_X g)(Y,\xi)]. \tag{4.6}$$

and hence if ∇ is a metrical connection then D^∇ is also a metrical connection. The covariant derivative of the fundamental form and $d\eta$ are given by, respectively:

$$2(D^\nabla_X\omega - \nabla_X\omega)(Y,Z) = \omega((\nabla_X\varphi)(\varphi Y), Z) + \omega(Y, (\nabla_X\varphi)(\varphi Z))+$$

$$+2\eta(Y)\omega(\nabla_X\xi, Z) + 2\eta(Z)\omega(Y, \nabla_X\xi), \tag{4.7$_1$}$$

$$(D^\nabla_X d\eta - \nabla_X d\eta)(Y,Z) = \eta(Y)d\eta(\nabla_X\xi, Z) + \eta(Z)d\eta(Y, \nabla_X\xi)+$$

$$+\frac{1}{2}[d\eta((\nabla_X\varphi)\varphi Y, Z) + d\eta(Y, (\nabla_X\varphi)\varphi Z)]-$$

$$-\frac{1}{2}[(\nabla_X\eta)Y + \eta(Y)\eta(\nabla_X\xi)]d\eta(\xi, Z) - \frac{1}{2}[(\nabla_X\eta)Z + \eta(Z)\eta(\nabla_X\xi)]d\eta(Y,\xi). \tag{4.7$_2$}$$

Proof. The formula (4.1) means:

$$2D^{\nabla}_X Y = \nabla_X Y + \varphi(\nabla_X \varphi Y) + V(\nabla_X VY) - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi. \tag{4.8}$$

but we have:

$$\nabla_X VY = X(\eta(Y))\xi + \eta(Y)\nabla_X \xi, \quad V(\nabla_X VY) = [X(\eta(Y)) + \eta(Y)\eta(\nabla_X \xi)]\xi \tag{4.9}$$

and we obtain:

$$2D^{\nabla}_X Y = \nabla_X Y + \varphi(\nabla_X \varphi Y) - \eta(Y)\nabla_X \xi + [2X(\eta(Y)) + \eta(Y)\eta(\nabla_X \xi) - \eta(\nabla_X Y)]\xi.$$

Also from:

$$(\nabla_X \varphi)(\varphi Y) = \nabla_X(\varphi^2 Y) - \varphi(\nabla_X \varphi Y)$$

we get the final (4.2). The computations of the torsion and covariant derivative of g are straightforward. A direct remark from (4.2) is in accord with ii) of proposition 3.4: if the initial ∇ is adapted then $D^{\nabla} = \nabla$. □

A first natural choice for ∇ in Theorem 4.1 is the Levi-Civita connection ∇^g of g. It follows then the metrical and adapted connection D^g given by:

$$D^g_X Y = \nabla^g_X Y - \eta(Y)\nabla^g_X \xi - \frac{1}{2}(\nabla^g_X \varphi)(\varphi Y) + \frac{1}{2}(\nabla^g_X \eta)Y \cdot \xi \tag{4.10}$$

since (2.1) gives that $\nabla^g_X \xi \in \mathcal{D}$. S. Zamkovoy [24, p. 49] defined on an almost paracontact metric manifold a connection $\tilde{\nabla}$ using the Levi-Civita connection ∇^g of the structure:

$$\tilde{\nabla}_X Y := \nabla^g_X Y + \eta(X)\varphi Y - \eta(Y)\nabla^g_X \xi + (\nabla^g_X \eta)Y \cdot \xi \tag{4.11}$$

and called it *canonical paracontact connection*. It is a metrical linear connection making parallel only the 1-form η and its dual vector field ξ. According to Proposition 4.2 of [24, p. 49] in a paracontact metric manifold this linear connection is adapted if and only if:

$$(\nabla^g_X \varphi)Y = \eta(Y)(X - hX) - g(X - hX, Y)\xi. \tag{4.12}$$

We derive now the second main result of this section:

Proposition 4.2. *i) Suppose that $ap(M)$ is K-paracontact. Then $\tilde{\nabla}$ is adapted if and only if:*

$$(\nabla^g_X \varphi)Y = \eta(Y)X - g(X, Y)\xi \tag{4.13}$$

which means that $ap(M)$ is a para-Sasakian manifold. Hence on a para-Sasakian manifold we have:

$$D^g Y = \nabla^g Y + \eta(Y)\varphi + \omega(\cdot, Y)\xi, \quad D^g Y = \tilde{\nabla} Y - \eta \otimes \varphi(Y). \tag{4.14}$$

ii) Suppose that $ap(M)$ is paracontact metric manifold. Then:

$$D^g_X Y = \nabla^g_X Y + \eta(Y)(\varphi X - \varphi hX) - \frac{1}{2}(\nabla^g_X \varphi)(\varphi Y) + \frac{1}{2}\omega(X - hX, Y)\xi. \tag{4.15}$$

iii) Suppose that $ap(M)$ is a para-Kenmotsu manifold. Then:

$$D^g_X Y = \nabla^g_X Y - \eta(Y)X + g(X, Y)\xi, \quad D^g Y = \tilde{\nabla} Y - \eta \otimes \varphi(Y). \tag{4.16}$$

Proof. i) The para-Sasakian condition (4.13) yields in (4.10):

$$D^g_X Y = \nabla^g_X Y - \eta(Y)\nabla^g_X \xi + \frac{1}{2}\omega(X,Y)\xi + \frac{1}{2}(\nabla^g_X \eta)(Y)\xi. \tag{4.17}$$

With $Y = \xi$ in (4.13) we get:

$$-\varphi(\nabla^g_X \xi) = X - \eta(X)\xi \tag{4.18}$$

and we apply φ to obtain that in a para-Sasakian geometry:

$$\nabla^g_X \xi = -\varphi(X). \tag{4.19}$$

Also:

$$(\nabla^g_X \eta)Y = X(g(Y,\xi)) - g(\xi, \nabla^g_X Y) = g(\nabla^g_X \xi, Y) = g(-\varphi X, Y) = \omega(X,Y) \tag{4.20}$$

and we get the first part of (4.14). Plugging the above computations in (4.11) gives:

$$\tilde{\nabla}_X Y = \nabla^g_X Y + \eta(X)\varphi(Y) + \eta(Y)\varphi(X) + \omega(X,Y)\xi \tag{4.21}$$

and hence we derive the second part of (4.14).
ii) The paracontact metric condition (2.7) gives in (4.10):

$$D^g_X Y = \nabla^g_X Y + \eta(Y)(\varphi X - \varphi h X) - \frac{1}{2}(\nabla^g_X \varphi)(\varphi Y) + \frac{1}{2}(\nabla^g_X \eta)(Y)\xi \tag{4.22}$$

and a similar computation to (4.20) yields:

$$(\nabla^g_X \eta)Y = \omega(X - hX, Y). \tag{4.23}$$

The formula (4.15) follows directly.
iii) The para-Kenmotsu condition (2.10) gives:

$$(\nabla^g_X \varphi)(\varphi Y) = [\eta(X)\eta(Y) - g(X,Y)]\xi, \quad \nabla^g_X \xi = X - \eta(X)\xi$$
$$(\nabla^g_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) = (\nabla^g_Y \eta)X \to d\eta = 0. \tag{4.24}$$

and the claimed (4.16) follows. We remark that η being closed it results that the distribution $\mathcal{D}$ is integrable and (4.24) can be expressed in a simpler form, $\nabla^g \eta = g - \eta \odot \eta$, with $\odot$ the *symmetric product* on 1-forms:

$$\alpha \odot \beta(X,Y) := \frac{1}{2}(\alpha(X)\beta(Y) + \alpha(Y)\beta(X)).$$

□

Remark 4.1. i) The second natural choice for ∇ in Theorem 4.1 is exactly the canonical paracontact connection $\tilde{\nabla}$. The resulting metrical and adapted connection $\tilde{D} = \chi(\tilde{\nabla})$ will be called *canonical-adapted connection* and its expression is:

$$\tilde{D}_X Y = \tilde{\nabla}_X Y - \frac{1}{2}(\tilde{\nabla}_X \varphi)(\varphi Y) = \tilde{\nabla}_X Y - \frac{1}{2}(\nabla^g_X \varphi)(\varphi Y) - \frac{1}{2}\omega(X - hX, Y)\xi \tag{4.25}$$

with the torsion:

$$2(T^{\tilde{D}} - T^{\tilde{\nabla}})(X,Y) = (\nabla^g_Y\varphi)(\varphi X) - (\nabla^g_X\varphi)(\varphi Y) + [\omega(Y - hY, X) - \omega(X - hX, Y)]\xi. \tag{4.26}$$

A remarkable result holds in the para-Sasakian geometry: $\tilde{D} = \tilde{\nabla}$, which means that for this geometry the derivations $\tilde{\nabla}_X$ are fixed points of the map χ. In the para-Kenmotsu case:

$$\tilde{D}_X Y = \tilde{\nabla}_X Y - \frac{1}{2}[\eta(X)\eta(Y) - g(X,Y) + \omega(X,Y)]\xi \tag{4.27}$$

since again $h = 0$.
ii) The second part of (4.14) and (4.16) means that the canonical paracontact connection on a para-Sasakian or para-Kenmotsu manifold is given by:

$$\tilde{\nabla}_X = \chi(\nabla^g_X) + \eta(X)\varphi. \tag{4.28}$$

In the para-Sasakian setting:

$$\tilde{\nabla}_X = \chi(\tilde{\nabla}_X) \tag{4.29}$$

and hence:

$$\varphi = \chi(\tilde{\nabla}_\xi - \nabla^g_\xi). \tag{4.30}$$

Indeed, a direct computation from the para-Sasakian and para-Kenmotsu properties gives:

$$\tilde{\nabla}_\xi Y = \nabla^g_\xi Y + \varphi(Y). \tag{4.31}$$

iii) In the para-Sasakian case the torsion of D^g is:

$$\begin{cases} T^{D^g}(X,Y) = \eta(Y)\varphi(X) - \eta(X)\varphi(Y) + 2\omega(X,Y)\xi = \eta(Y)\varphi(X) - \eta(X)\varphi(Y) + N_\varphi(X,Y), \\ T^{D^g}(X,\xi) = \varphi(X), \quad \eta \circ T^{D^g} = 2\omega \end{cases} \tag{4.32}$$

where N_φ is the Nijenhuis tensor field of φ; in a para-Sasakian manifold $N_\varphi = 2\omega \otimes \xi$ while the general expression is (5.20) from the following section. The para-Kenmotsu linear connection (4.16) is a semi-symmetric one since its torsion is:

$$T^{D^g}(X,Y) = \eta(X)Y - \eta(Y)X \in \mathcal{D}, \quad T^{D^g}(X,\xi) = \eta(X)\xi - X. \tag{4.33}$$

With the Definition 2.1 and notation of [1, p. 287] we remark that in the para-Kenmotsu case the connection D^{∇^g} has *a vectorial torsion* given by the vector field $V = \xi$. The $(0,3)$-variant of T^{D^g} obtained by the contraction with g is:

$$T^{D^g}(X,Y,Z) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z). \tag{4.34}$$

Modulo a different convention for the exterior derivative in [1] we get for the para-Sasakian case *the totally skew-symmetric torsion* of $(0,3)$-type $T^{D^g} = \eta \wedge d\eta$, similar to the Sasakian geometry as presented in [1, p. 295].
We can express the above torsions in a more compact form using the exterior covariant derivative d^∇ induced by a linear connection ∇:

$$(d^\nabla\varphi)(X,Y) := (\nabla_X\varphi)Y - (\nabla_Y\varphi)X. \tag{4.35}$$

Then a straightforward computation gives:

$$para-Sasakian: T^{D^g} = \varphi \circ d^{\nabla^g}\varphi + 2\omega \otimes \xi, para-Kenmotsu: T^{D^g} = \varphi \circ d^{\nabla^g}\varphi. \tag{4.36}$$

The curvature in para-Sasakian case is:

$$(R^{D^g} - R^g)(X,Y)Z = \omega(Z,Y)\varphi(X) - \omega(Z,X)\varphi(Y) + \eta(Z)[\eta(Y)X - \eta(X)Y] \in \mathcal{D} \tag{4.37}$$

while for para-Kenmotsu geometry is:

$$(R^{D^g} - R^g)(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{4.38}$$

iv) Recall the almost product structure $\tau = V - H$ of Remark 3.4. Similar to the process $\nabla \to D^{\nabla}$ of Theorem 4.1 any linear connection ∇ yields a linear connection ∇^S making P as parallel endomorphism. ∇^S is called *the Schouten connection associated to* ∇ and its expression is ([17], [22, p. 32]):

$$\nabla^S_X Y = V(\nabla_X VY) + H(\nabla_X HY). \tag{4.39}$$

Using (4.8) we derive:

$$\nabla^S_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + [(\nabla_X \eta)(Y) + 2\eta(Y)\eta(\nabla_X \xi)]\xi \tag{4.40}$$

and then $\nabla^S_X \xi = \eta(\nabla_X \xi)\xi = V(\nabla_X \xi)$. A straightforward computation gives that in both para-Sasakian and para-Kenmotsu cases we have $D^g = (\nabla^g)^S$.
v) In the para-Sasakian case the covariant derivative of the fundamental form is:

$$(D^g_X \omega)(Y,Z) = (\nabla^g_X \omega)(Y,Z) + \eta(Y)g(X,Z) - \eta(Z)g(X,Y) \tag{4.41}$$

while for the para-Kenmotsu case:

$$(D^g_X)\omega(Y,Z) = (\nabla^g_X \omega)(Y,Z) + \eta(Y)\omega(X,Z) - \eta(Z)\omega(X,Y). \tag{4.42}$$

Also the exterior covariant derivative of V with respect to ∇^g is:

$$(d^{\nabla^g}V)(X,Y) = \eta(Y)\nabla^g_X \xi - \eta(X)\nabla^g_Y \xi + 2d\eta(X,Y)\xi \tag{4.43}$$

and then in para-Sasakian geometry:

$$(d^{\nabla^g}V)(X,Y) = \eta(X)\varphi(Y) - \eta(Y)\varphi(X) + 2\omega(X,Y)\xi \tag{4.44}$$

while for the para-Kenmotsu case:

$$(d^{\nabla^g}V)(X,Y) = \eta(Y)X - \eta(X)Y = -T^{D^g}(X,Y). \tag{4.45}$$

vi) An adapted connection D preserves the bundle decomposition (2.5) and hence D restricts to linear connections in both vector bundles $\mathcal{D}$ and $\langle \xi \rangle$. For D^{∇} of (4.2) we have:

$$\begin{cases} 2D^{\nabla}_X Y = \nabla_X Y + \varphi(\nabla_X \varphi Y) - \eta(\nabla_X Y)\xi, \quad Y \in \mathcal{D}, \\ D^{\nabla}_X (f\xi) = X(f)\xi, \quad f \in C^{\infty}(M). \end{cases} \tag{4.46}$$

In particular, if the initial connection ∇ restricts to $\mathcal{D}$ then the restriction of D^{∇} to $\mathcal{D}$ is:

$$2(D^{\nabla}|_{\mathcal{D}})_X Y = \nabla_X Y + \varphi(\nabla_X \varphi Y). \tag{4.47}$$

For a para-Sasakian geometry the relation (4.46_1) becomes:

$$2D^g_X Y = 2\nabla^g_X Y + [\omega(X,Y) - \eta(\nabla^g_X Y)]\xi, \tag{4.48}$$

while for a para-Kenmotsu manifold the same formula is:

$$2D^g_X Y = 2\nabla^g_X Y + [g(X,Y) - \eta(\nabla^g_X Y)]\xi. \tag{4.49}$$

□

Example 4.2. Now we restrict to the dimension $2n+1=3$ for which the metric is a Lorentz one and the normality is equivalent with, [2, p. 119]:

$$\begin{cases} \nabla^g_X \xi = \alpha(X - \eta(X)\xi) + \beta\varphi(X), \\ (\nabla^g_X \varphi)Y = \alpha(g(\varphi X, Y)\xi - \eta(Y)\varphi X) + \beta(g(X,Y)\xi - \eta(Y)X). \end{cases} \tag{4.50}$$

where $\alpha = \frac{1}{2} div\xi$ and $\beta = \frac{1}{2} trace(\varphi\nabla\xi)$. The almost paracontact metric manifold $(M^3, \varphi, \xi, \eta, g)$ is:
1) *quasi-para-Sasakian* if $\alpha = 0$ and $\beta \neq 0$; in particular, for $\beta = -1$ the manifold is para-Sasakian;
2) *α-para-Kenmotsu* if $\beta = 0$ and $\alpha \neq 0$; in particular, for $\alpha = 1$ the manifold is para-Kenmotsu.
It results:

$$(\nabla^g_X \eta)(Y) = \alpha[g(X,Y) - \eta(X)\eta(Y)] - \beta\omega(X,Y) \tag{4.51}$$

and hence from (4.10) we have:

$$D^g_X Y = \nabla^g_X Y - \alpha\eta(Y)X - \beta\eta(Y)\varphi(X) + [\alpha g(X,Y) - \beta\omega(X,Y)]\xi \tag{4.52}$$

with:

$$T^{D^g}(X,Y) = \alpha[\eta(X)Y - \eta(Y)X] + \beta[\eta(X)\varphi(Y) - \eta(Y)\varphi(X)] - 2\beta\omega(X,Y)\xi. \tag{4.53}$$

□

Due to the interest in totally skew-symmetric connections ([1], [23, p. 42]) we present the following characterization:

Proposition 4.3. *Suppose the dimension is 3 and $\beta \neq 0$. Then the adapted connection D^g has a totally skew-symmetric torsion if and only if the manifold is quasi-para-Sasakian.*

Proof. From (4.53) we must have totally skew-symmetry of the expression:

$$A(X,Y,Z) := \alpha\eta(X)g(Y,Z) - \alpha\eta(Y)g(X,Z).$$

The equality $A(X,Y,Z) = -A(X,Z,Y)$ means:

$$2\alpha\eta(X)g(Y,Z) = \alpha\eta(Y)g(X,Z) + \alpha\eta(Z)g(X,Y)$$

and replacing $Z = \xi$ gives:

$$\alpha[g(X,Y) - \eta(X)\eta(Y)] = 0$$

with the unique possibility $\alpha = 0$. □ □

A natural generalization of our setting is provided by *almost r-paracontact structures*, where r is a positive integer. We give now the pair (φ, g) as well as r pairs $(\xi_i, \eta_i)_{1\leq i\leq r}$ with [8]:
1. $\varphi(\xi_i) = 0$, $\eta^i \circ \varphi = 0$, $\eta^i(X) = g(X, \xi_i)$
2. $\eta^i(\xi_j) = \delta^i_j$, $\varphi^2 = I - V$ where $V = \sum_{i=1}^r \eta^i \otimes \xi_i$,
3. $g(\varphi\cdot, \varphi\cdot) = -g + \sum_{i=1}^r \eta^i \otimes \eta^i$.

Again an adapted connection makes parallel the endomorphism φ and all pairs (η^i, ξ_i). The generalization of Theorem 4.1 is that any linear connection ∇ gives an adapted linear connection D^∇ with:

$$2D^\nabla_X Y = \nabla_X Y + \varphi(\nabla_X \varphi Y) -$$

$$-\sum_{i=1}^r \eta^i(Y)\nabla_X \xi_i + \sum_{i=1}^r [2X(\eta^i(Y)) + \eta^i(\sum_{j=1}^r \eta^j(Y)\nabla_X \xi_j - \nabla_X Y)]\xi_i. \tag{4.54}$$

The r-paracontact version with Riemannian metric instead of a pseudo-Riemannian one is treated in [7] and the r-contact version is discussed in [14].

5 A generalization of adapted linear connections

Firstly we unify the settings of almost contact and almost paracontact by using a parameter $\varepsilon \in \{-1, +1\}$ and adapting the method of [22]. More precisely, we put:

$$\varphi^2 = \varepsilon(I - V), \quad g(\varphi\cdot, \varphi\cdot) = -\varepsilon(g - \eta \otimes \eta) \tag{5.1}$$

and $\varepsilon = -1$ corresponds to the almost contact case while $\varepsilon = +1$ to the almost paracontact case. The relations (3.10) become:

$$\begin{cases} C_\varepsilon(\partial) = \{\mu, \varepsilon\varphi\Phi\varphi + V\Phi V + \partial\eta \otimes \xi - \eta \otimes \Phi(\xi)\}, \\ \chi_\varepsilon(\partial) = \{\mu, \frac{1}{2}(\Phi + \varepsilon\varphi\Phi\varphi + V\Phi V + \partial\eta \otimes \xi - \eta \otimes \Phi(\xi))\} \end{cases} \tag{5.2}$$

The *canonical* ε*-connection* is:

$$\tilde{\nabla}^\varepsilon_X Y := \nabla^g_X Y + \varepsilon\eta(X)\varphi Y - \eta(Y)\nabla^g_X \xi + (\nabla^g_X \eta)Y \cdot \xi \tag{5.3}$$

The ε-Sasakian case is given by ([22, p. 38]):

$$\nabla^g_X \xi = -\varphi X, \quad (\nabla^g_X \eta)Y = \omega(X, Y), \quad (\nabla^g_X \varphi)Y = \varepsilon[\eta(Y)X - g(X, Y)\xi]. \tag{5.4}$$

Remark that the first two relations do not depend on ε while the first equation implies that ξ is a Killing vector field. Its linear connection D^g and canonical ε-connection are:

$$D^g_X Y = \nabla^g_X Y + \eta(Y)\varphi X + \omega(X, Y)\xi, \tag{5.5}$$

$$\tilde{\nabla}^\varepsilon_X Y = \nabla^g_X Y + \eta(Y)\varphi X + \varepsilon\eta(X)\varphi Y + \omega(X, Y)\xi. \tag{5.6}$$

Remark that D^g does not depends on ε and their difference is:

$$(\tilde{\nabla}^\varepsilon - D^g)_X Y = \varepsilon\eta(X)\varphi Y. \tag{5.7}$$

Secondly, we generalize the class of linear connections studied in the previous section. Following [18, p. 272] let us fix a triple $J^* = (\varphi^*, \eta^*, \xi^*) \in \mathcal{T}^1_2(M) \times \mathcal{T}^0_2(M) \times \mathcal{T}^1_1(M)$. We generalize the notion of adapted connection to:

Definition 5.1. The linear connection ∇ on $ap(M)$ is called J^*-*adapted* if:

$$\nabla\varphi = \varphi^*, \quad \nabla\eta = \eta^*, \quad \nabla\xi = \xi^*. \tag{5.8}$$

A direct remark is that if there exists a J^*-adapted connection then for a fixed vector field X we have that $J^*(X) = (\varphi^*(X,\cdot), \eta^*(X,\cdot), \xi^*(X)) \in \mathcal{ID}(ap(M))$ and hence from (3.3) – (3.4) we get:

$$\eta^*(X,\xi) + \eta(\xi^*(X)) = 0, \quad \varphi^*(X,\xi) + \varphi(\xi^* X) = 0, \quad \eta^*(X,\varphi Y) + \eta(\varphi^*(X,Y)) = 0,$$

$$\varphi^*(X,\varphi Y) + \varphi(\varphi^*(X,Y)) = -\varepsilon[\eta^*(X,Y)\xi + \eta(Y)\xi^*(X)]. \tag{5.9}$$

A triple J^* satisfying these conditions will be called *admissible*. Conversely, from Proposition 3.2 it follows that if these equations are satisfied then there exist J^*-adapted linear connections. More precisely, following (3.7) we define:

$$\begin{cases} V^*(X,Y) = \eta^*(X,Y)\xi + \eta(Y)\xi^*(X), \\ 2A_{J^*}(X,Y) := -\varepsilon\varphi(\varphi^*(X,Y)) - V(V^*(X,Y)) + \eta(Y)\xi^*(X) - \eta^*(X,Y)\xi. \end{cases} \tag{5.10}$$

Using (5.9) we deduce:

$$2A_{J^*}(X,Y) = 2\eta(Y)\xi^*(X) + \varepsilon\varphi^*(X,\varphi Y) - [\eta^*(X,Y) + \eta(Y)\eta(\xi^*(X))]\xi. \tag{5.11}$$

Hence, the generalization of Theorem 4.1 is:

Theorem 5.1. *A linear connection D is J^*-adapted if and only if there exists a linear connection ∇ such that:*

$$D_X = A_{J^*}(X,\cdot) + \chi(\nabla_X) \tag{5.12}$$

More precisely, denoting $D^{*,\nabla}$ the right hand side of (5.12) *we get:*

$$D^{*,\nabla}_X Y = D^{\nabla}_X Y + \eta(Y)\xi^*(X) + \frac{\varepsilon}{2}\varphi^*(X,\varphi Y) - \frac{1}{2}[\eta^*(X,Y) + \eta(Y)\eta(\xi^*(X))]\xi. \tag{5.13}$$

Its torsion is:

$$(T^{*,\nabla} - T^{D^\nabla})(X,Y) = \eta(Y)\xi^*(X) - \eta(X)\xi^*(Y) +$$

$$+\frac{\varepsilon}{2}[\varphi^*(X,\varphi Y) - \varphi^*(Y,\varphi X)] + \frac{1}{2}[\eta^*(Y,X) - \eta^*(X,Y) + \eta(X)\eta(\xi^*(Y)) - \eta(Y)\eta(\xi^*(X))]\xi. \tag{5.14}$$

The covariant derivative of the metric is:

$$(D^{*,\nabla}_X g - D^{\nabla}_X g)(Y,Z) = \eta(Y)\eta(Z)\eta(\xi^* X) - \eta(Y)g(\xi^* X, Z) - \eta(Z)g(Y, \xi^* X) -$$

$$-\frac{\varepsilon}{2}[g(\varphi^*(X,Y),Z) + g(Y,\varphi^*(X,Z))] + \frac{1}{2}[\eta(Z)\eta^*(X,Y) + \eta(Y)\eta^*(X,Z)]. \tag{5.15}$$

Example 5.2. i) A triple $(\varphi^*, \eta^* = \alpha g, \xi^* = \beta\varphi)$ with non-zero scalars α, β can not be admissible.
ii) Let us search for (5.9) applied to a triple $J^* = (\varphi^*, \eta^*, \xi^*) = (\alpha\omega\otimes\xi, \beta g, \gamma V = \gamma\eta\otimes\xi)$ with non-zero scalars α, β and γ. We obtain the unique solution $\alpha = -\beta = \gamma = 1$

independent of ε and then the triple $J^* = (\varphi^*, \eta^*, \xi^*) = (\omega \otimes \xi, -g, V = \eta \otimes \xi)$ is admissible. Its J^*-adapted connection induced by ∇ is:

$$D^{*;\nabla}_X Y = D^{\nabla}_X Y + g(X,Y)\xi, \quad T^{*,\nabla} = T^{D^{\nabla}}. \tag{5.16}$$

Hence, the parameter ε occurs only in the expression of D^{∇}.
iii) In the ε-Sasakian case let $\nabla = \nabla^g$. The associated J^*-adapted connection for an arbitrary admissible J^* is:

$$D^{*;g}_X Y = \nabla^g_X Y + \eta(Y)(\xi^* + \varphi)(X) + \frac{\varepsilon}{2}\varphi^*(X, \varphi Y) + [(\omega - \frac{\eta^*}{2})(X,Y) - \frac{1}{2}\eta(Y)\eta(\xi^* X)]\xi. \tag{5.17}$$

iv) Following the almost contact case of [20] we consider the ε-triple J^*:

$$\begin{cases} \xi^*(X) = -\varphi X - \frac{1}{2}d\eta(X,\xi)\xi, \quad \eta^*(X,Y) = \frac{1}{2}d\eta(X,Y) - \frac{\varepsilon}{2}d\eta(\varphi X, \varphi Y), \\ \varphi^*(X,Y) = \varepsilon\eta(Y)HX + \frac{1}{2}d\eta(\varphi X, HY)\xi - \frac{1}{2}d\eta(X, \varphi Y)\xi. \end{cases} \tag{5.18}$$

which is admissible. We call it *the normality triple* of the given ε-geometry since the almost contact structure is normal if and only if there exists a torsion-free J^*-adapted connection. In the ε-Sasakian case exactly the Levi-Civita connection is such a normality-adapted connection since (4.18) reduces to $J^* = (\varphi^* = \varepsilon I \otimes \eta - \varepsilon g \otimes \xi, \eta^* = \omega, \xi^* = -\varphi)$ and we compare it with (5.4).
v) Inspired by [23, p. 84] a triple J^* is called *conical* if $\xi^* = I$. Then the equations (5.9) are solved by $\eta^* = -\eta \otimes \eta$ and $\varphi^*(X,Y) = -\eta(Y)\varphi X$ and then (5.13) reduces to:

$$D^{*;\nabla}_X Y = D^{\nabla}_X Y + \eta(Y)X. \tag{5.19}$$

Comparing with (4.33) it results that for a symmetric connection D^{∇} the linear connection $D^{*,\nabla}$ has a vectorial torsion given by the vector field $V = -\xi$.
vi) Also, a J^* can be called *Nijenhuis* if $\varphi^* = N_\varphi$ which is expressed through any symmetric connection, particularly ∇^g, as:

$$(N_\varphi - 2\varepsilon d\eta \otimes \xi)(X,Y) = (\nabla^g_{\varphi X}\varphi)Y - (\nabla^g_{\varphi Y}\varphi)X + (\nabla^g_X \varphi)\varphi Y - (\nabla^g_Y \varphi)\varphi X +$$

$$+\varepsilon[\eta(Y)\nabla^g_X \xi - \eta(X)\nabla^g_Y \xi]. \tag{5.20}$$

Hence the general expression of torsion for the linear connection D^g of (4.10) is:

$$(2T^{D^g} + N_\varphi + 4d\eta \otimes \xi)(X,Y) = \eta(X)\nabla^g_Y \xi - \eta(Y)\nabla^g_X \xi + (\nabla^g_{\varphi X}\varphi)Y - (\nabla^g_{\varphi Y}\varphi)X. \tag{5.21}$$

□

We finish this section by discussing the admissibility of a triple appearing in the geometry of pseudo-convex CR-structures, following the terminology and notations of [21], adapted to our setting. Fix a pseudo-convex CR-structure $(M, H(M))$ and an associated almost contact structure (φ, η, ξ). The Theorem 3.1 of the cited paper proves the existence and uniqueness of a symmetric linear connection D, called *the canonical torsion-free connection*, and satisfying:

$$D(\varphi, \eta, \xi) = J^*, \quad Dd\eta = 0 \tag{5.22}$$

for:

$$\varphi^*(X,Y) = 2\eta(X)hY - d\eta(X,\varphi Y)\xi, \quad \eta^* = d\eta, \quad \xi^* = 0 \tag{5.23}$$

and using the following assumptions:

$$h(\xi) = 0, \quad \eta \circ h = 0, \quad \varphi \circ h + h \circ \varphi = 0, \quad d\eta(hX,Y) + d\eta(X,hY) = 0. \tag{5.24}$$

We analyze these conditions by comparing with (5.9) for J^* given by (5.23). Namely, (5.9_1) means $d\eta(X,\xi) = 0$ which together with $\eta(\xi) = 1$ are the defining equations for the Reeb vector field ξ. The conditions (5.9_2), (5.9_3) and (5.9_4) correspond exactly to (5.24_1), (5.24_2) and (5.24_3) in this order. Remark that (5.24_2) means that h is $H(M)$-valued and that the conditions (5.22_2) and (5.24_4) are used to obtain the uniqueness of D. For $X,Y \in H(M) = Ker(\eta)$ the expression of φ^* is in relationship with *the Levi form* L_η of $H(M)$:

$$\varphi^*(X,Y) = -d\eta(X,\varphi Y)\xi = -L_\eta(X,Y)\xi = \frac{1}{2}\eta([X,\varphi Y])\xi. \tag{5.25}$$

Another important remark is that from (5.22_2) the contact form η is covariant constant under the iterated covariant derivative D: $D(D\eta) = 0$. With the discussion above it results that (5.23) is an admissible triple and a direct computation gives this fact even for the almost paracontact case. Replacing this J^* in (5.13) we get D is a linear connection $D^{*,\nabla}$ with:

$$D^{*,\nabla}_X Y = D^{\nabla}_X Y - \eta(X)h\varphi Y - d\eta(X,Y)\xi \tag{5.26}$$

and a straightforward computation reveals that D^{∇} is exactly the Tanaka-Webster connection D^{TW}. By the way, the Tanaka-Webster connection D^{TW} is exactly the adapted connection D^g following the approach of section 4 for the almost contact case.

6 Applications to statistical and weak Frobenius structures

Recall after [5] that the triple (M,g,∇) is *a statistical manifold* if ∇g is totally symmetric:

$$(\nabla_X g)(Y,Z) = (\nabla_Y g)(Z,X) (= (\nabla_Z g)(X,Y)). \tag{6.1}$$

From (4.5) it follows that (M,g,D^{∇}) is a statistical manifold if and only if ∇ satisfies:

$$\begin{aligned} 2(\nabla_X g)(Y,Z) + g((\nabla_X\varphi)(\varphi Y),Z) + g(Y,(\nabla_X\varphi)(\varphi Z)) - 2\eta(Y)\eta(Z)\eta(\nabla_X\xi) + \\ +\eta(Y)[g(\nabla_X\xi,Z) - (\nabla_X g)(Z,\xi)] + \eta(Z)[g(\nabla_X\xi,Y) - (\nabla_X g)(Y,\xi)] = \\ = 2(\nabla_Y g)(Z,X) + g((\nabla_Y\varphi)(\varphi Z),X) + g(Z,(\nabla_Y\varphi)(\varphi X)) - 2\eta(Z)\eta(X)\eta(\nabla_Y\xi) + \\ +\eta(Z)[g(\nabla_Y\xi,X) - (\nabla_Y g)(X,\xi)] + \eta(X)[g(\nabla_Y\xi,Z) - (\nabla_Y g)(Z,\xi)] \end{aligned} \tag{6.2}$$

for all X, Y, Z. In particular, if ∇ is metrical this condition reduces to:

$$g((\nabla_X\varphi)(\varphi Y),Z) + g(Y,(\nabla_X\varphi)(\varphi Z)) + \eta(Y)g(\nabla_X\xi,Z) + \eta(Z)g(\nabla_X\xi,Y) =$$

$$g((\nabla_Y \varphi)(\varphi Z), X) + g(Z, (\nabla_Y \varphi)(\varphi X)) + \eta(Z) g(\nabla_Y \xi, X) + \eta(X) g(\nabla_Y \xi, Z). \quad (6.3)$$

For $Z = \xi$ this relation is satisfied and is an open problem to solve the resulting equation for $Z \perp \xi$.

The second application concerns with weak Frobenius structures introduced in [11, p. 7]. A triple $(M, g, A \in \mathcal{T}^1_2(M))$ is called *weak Frobenius structure* if $g^\sharp \circ A$ is totally symmetric i.e. for all vector fields X, Y, Z:

$$g(A(X,Y), Z) = g(A(Y,Z), X)(= g(A(Z,X), Y)). \quad (6.4)$$

With $Z = \xi$ it follows:

$$\eta(A(X,Y)) = g(A(Y,\xi), X). \quad (6.5)$$

We search for $A^g = D^g - \nabla^g$ and using (4.10) we get:

$$\eta(D^g_X Y - \nabla^g_X Y) = -g(\nabla^g_Y \xi, X) \quad (6.6)$$

which means:

$$\eta[\nabla^g_X (Y - \eta(Y)\xi)] - (\nabla^g_X \eta) Y = 2g(\nabla^g_Y \xi, X) \quad (6.7)$$

or equivalently:

$$-(\nabla^g_X \eta) Y = g(\nabla^g_Y \xi, X) \quad (6.8)$$

We remark that the same relation (6.8) corresponds to a second choice: $\tilde{A}^g = \tilde{D} - \nabla^g$ and this relation is satisfied in para-Sasakian geometry, both sides being $\omega(X,Y)$. In para-Kenmotsu geometry the relation (6.8) reduces to $g = \eta \otimes \eta$ which is the impossible relation $\nabla^g \eta = 0 = \nabla^g \xi$.

Returning with (6.8) in the general condition (6.4) we arrive at:

$$g((\nabla^g_Y \varphi)\varphi Z, X) - g((\nabla^g_X \varphi)\varphi Y, Z) = \eta(X)(\nabla^g_Y \eta) Z - 2\eta(Y)(\nabla^g_Z \eta) X + \eta(Z)(\nabla^g_X \eta) Y \quad (6.9)$$

which yields:

Proposition 6.1. *In para-Sasakian geometry the triples* (M, g, A^g) *and* $(M, g, \tilde{A}^g)$ *are not weak Frobenius structures.*

Proof. With (4.17) and (4.24) the condition (6.9) means:

$$\eta(X)\omega(Y,Z) + \eta(Y)\omega(X,Z) = 0 \quad (6.10)$$

and for $Y = \xi$ this relation is $\omega = 0$. □ □

The same negative answer holds for a third choice, namely $A^c = \tilde{\tilde{\nabla}} - D^g$, in the para-Sasakian and para-Kenmotsu settings since (6.4) reads for $A^c(X,Y) = \eta(X)\varphi Y$ as follows:

$$\eta(X)\omega(Z,Y) = \eta(Y)\omega(X,Z) \quad (6.11)$$

and we apply the same argument as in the above proof.

A linear connection which gives both structures of this section was introduced in [5]. Let λ be a 1-form and the linear connection:

$$\nabla^\lambda = \nabla^g + \lambda \otimes I + I \otimes \lambda + g \circ \lambda^{\sharp_g}. \quad (6.12)$$

Then ∇^λ is a statistical structure and $A^\lambda = \nabla^\lambda - \nabla^g$ provides a weak Frobenius structure with:

$$g(A^\lambda(X,Y)) = 2\sum_{cyclic}[\lambda(X)g(Y,Z)]. \tag{6.13}$$

Hence, for our study a natural problem is if ∇^λ is adapted. For $\lambda = \eta$ we have:

$$\nabla^\eta_X Y = \nabla^g_X Y + \eta(X)Y + \eta(Y)X + g(X,Y)\xi \tag{6.14}$$

and then $\nabla^\eta \xi = 0$ means:

$$\nabla^g_X \xi = -X - 2\eta(X)\xi. \tag{6.15}$$

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