Kahler-Einstein structures of general natural lifted type on the cotangent bundles

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Abstract. We study the conditions under which the cotangent bundle T^*M of a Riemannian manifold (M, g), endowed with a Kählerian structure (G, J) of general natural lift type (see [4]), is Einstein. We first obtain a general natural Kähler-Einstein structure on the cotangent bundle T^*M. In this case, a certain parameter, λ involved in the condition for (T^*M, G, J) to be a Kählerian manifold, is expressed as a rational function of the other two, the value of the constant sectional curvature, c, of the base manifold (M, g) and the constant ρ involved in the condition for the structure of being Einstein. This expression of λ is just that involved in the condition for the Kählerian manifold to have constant holomorphic sectional curvature (see [5]). In the second case, we obtain a general natural Kähler-Einstein structure only on T_0M, the bundle of nonzero cotangent vectors to M. For this structure, λ is expressed as another function of the other two parameters, their derivatives, c and ρ.

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Key words: cotangent bundle, Riemannian metric, general natural lift, Kähler-Einstein structure.

1 Introduction

A few natural lifted structures introduced on the cotangent bundle T^*M of a Riemannian manifold (M, g), have been studied in recent papers such as [1]–[5], [14], [17], [18], [20]–[26]. The similitude between some results from the mentioned papers and results from the geometry of the tangent bundle TM (e.g. [2], [6]-[8], [15], [16], [27]-[29], [13]), may be explained by the duality cotangent bundle – tangent bundle. The fundamental differences between the geometry of the cotangent bundle and that of the tangent bundle of a Riemannian manifold, are due to the different construction of lifts to T^*M, which cannot be defined just like in the case of TM (see [30]).

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Briefly speaking, a natural operator (in the sense of [10]–[12]) is a fibred manifold mapping, which is invariant with respect to the group of local diffeomorphisms of the base manifold.

The results from [10] and [11] concerning the natural lifts, and the classification of the natural vector fields on the tangent bundle of a pseudo-Riemannian manifold, made by Janyška in [9], allowed the present author to introduce in the paper [4], a general natural almost complex structure J of lifted type on the cotangent bundle T^*M, and a general natural lifted metric G defined by the Riemannian metric g on T^*M (see the paper [15] by Oproiu, for the case of the tangent bundle). The main result from [4] is that the family of general natural Kähler structures on T^*M depends on three essential parameters (one is a certain proportionality factor obtained from the condition for the structure to be almost Hermitian and the other two are coefficients involved in the definition of the integrable almost complex structure J on T^*M).

In the present paper we are interested in finding the conditions under which the cotangent bundle T^*M of a Riemannian manifold (M, g), endowed with a Kählerian structure (G, J) of general natural lift type (see [4]), is an Einstein manifold. To this aim, we have to study the vanishing conditions for the components of the difference between the Ricci tensor of (T^*M, G, J) and ρG, where ρ is a constant.

After some quite long computations with the RICCI package from the program Mathematica, we obtain two cases in which a general natural Kählerian manifold (T^*M, G, J) is Einstein. In the first case, (T^*M, G, J) is a Kähler-Einstein manifold if the proportionality factor λ, involved in the condition for the manifold to be Kählerian, is expressed as a rational function of the first two essential parameters, the value of the constant sectional curvature of the base manifold (M, g), the constant ρ, from the condition for the manifold to be Einstein, and the energy density. In this case the expression of λ leads to the condition obtained in [5] for (T^*M, G, J) to have constant holomorphic sectional curvature. In the second case, (G, J) is a Kähler-Einstein structure on the the bundle of nonzero cotangent vectors to M, T_0^*M, if and only if λ' is expressed as a certain function of λ, the other two parameters, their derivatives, the constant sectional curvature of the base manifold, and the energy density. The similar problem on tangent bundle TM of a Riemannian manifold (M, g) was treated by Oproiu and Papaghiuc in the paper [19].

In 2001, Chaki introduced in [3] the notion of generalized quasi-Eistein manifolds (the most recent generalization for the Einstein manifolds), presented in the last years in papers like [22].

The present work could be extended at the study of the generalized quasi-Einstein Kähler manifolds of general natural lifted type on the tangent and cotangent bundles of a Riemannian manifold.

The manifolds, tensor fields and other geometric objects considered in the present paper are assumed to be differentiable of class C^∞ (i.e. smooth). The Einstein summation convention is used throughout this paper, the range of the indices h, i, j, k, l, m, r being always $\{1, \ldots, n\}$.

2 Preliminary results

The cotangent bundle of a smooth n-dimensional Riemannian manifold may be endowed with a structure of a $2n$-dimensional smooth manifold, induced from the structure of the base manifold. If (M, g) is a smooth Riemannian manifold of the dimension n, we denote its cotangent bundle by $\pi : T^*M \to M$. From every local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ on M, it is induced a local chart $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$, on T^*M, as follows. For a cotangent vector $p \in \pi^{-1}(U) \subset T^*M$, the first n local coordinates $q^1, \ldots, q^n$ are the local coordinates of its base point $x = \pi(p)$ in the local chart (U, φ) (in fact we have $q^i = \pi^* x^i = x^i \circ \pi,\ i = 1, \ldots n$). The last n local coordinates $p_1, \ldots, p_n$ of $p \in \pi^{-1}(U)$ are the vector space coordinates of p with respect to the natural basis $(dx^1_{\pi(p)}, \ldots, dx^n_{\pi(p)})$, defined by the local chart (U, φ), i.e. $p = p_i dx^i_{\pi(p)}$.

We recall the splitting of the tangent bundle to T^*M into the vertical distribution $VT^*M = \mathrm{Ker}\ \pi_*$ and the horizontal one determined by the Levi Civita connection $\dot\nabla$ of g:

$$TT^*M = VT^*M \oplus HT^*M. \tag{2.1}$$

If $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$ is a local chart on T^*M, induced from the local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$, the local vector fields $\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}$ on $\pi^{-1}(U)$ define a local frame for VT^*M over $\pi^{-1}(U)$ and the local vector fields $\frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n}$ define a local frame for HT^*M over $\pi^{-1}(U)$, where

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma^0_{ih} \frac{\partial}{\partial p_h}, \quad \Gamma^0_{ih} = p_k \Gamma^k_{ih},$$

and $\Gamma^k_{ih}(\pi(p))$ are the Christoffel symbols of g.

The set of vector fields $\{\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n}\}$ defines a local frame on T^*M, adapted to the direct sum decomposition (2.1).

We consider

$$t = \frac{1}{2}\|p\|^2 = \frac{1}{2} g^{-1}_{\pi(p)}(p, p) = \frac{1}{2} g^{ik}(x) p_i p_k, \quad p \in \pi^{-1}(U)$$

the energy density defined by g in the cotangent vector p. We have $t \in [0, \infty)$ for all $p \in T^*M$.

The computations will be done in local coordinates, using a local chart (U, φ) on M and the induced local chart $(\pi^{-1}(U), \Phi)$ on T^*M.

We shall use the following lemma, which may be proved easily.

Lemma 2.1. *If $n > 1$ and u, v are smooth functions on T^*M such that*

$$u g_{ij} + v p_i p_j = 0, \quad u g^{ij} + v g^{0i} g^{0j} = 0, \quad \text{or} \quad u \delta^i_j + v g^{0i} p_j = 0,\ \forall i, j = \overline{1, n},$$

on the domain of any induced local chart on T^*M, then $u = 0,\ v = 0$.

In the paper [4], the present author considered the real valued smooth functions $a_1,\ a_2,\ a_3,\ a_4,\ b_1,\ b_2,\ b_3,\ b_4$ on $[0, \infty) \subset R$ and studied a general natural tensor of type $(1, 1)$ on T^*M, defined by the relations

$$\begin{cases} JX^H_p = a_1(t)(g_X)^V_p + b_1(t) p(X) p^V_p + a_4(t) X^H_p + b_4(t) p(X)(p^\sharp)^H_p, \\ J\theta^V_p = a_3(t)\theta^V_p + b_3(t) g^{-1}_{\pi(p)}(p, \theta) p^V_p - a_2(t)(\theta^\sharp)^H_p - b_2(t) g^{-1}_{\pi(p)}(p, \theta)(p^\sharp)^H_p, \end{cases} \tag{2.2}$$

in every point p of the induced local card $(\pi^{-1}(U), \Phi)$ on T^*M, $\forall\ X \in \mathcal{X}(M), \forall\ \theta \in \Lambda^1(M)$, where g_X is the 1-form on M defined by $g_X(Y) = g(X,Y)$, $\forall Y \in \mathcal{X}(M)$, $\theta^\sharp = g_\theta^{-1}$ is a vector field on M defined by $g(\theta^\sharp, Y) = \theta(Y)$, $\forall\ Y \in \mathcal{X}(M)$, the vector $p^\sharp$ is tangent to M in $\pi(p)$, p^V is the Liouville vector field on T^*M , and $(p^\sharp)^H$ is the similar horizontal vector field on T^*M.

The definition of the general natural lift given by (2.2), is based on the Janyška's classification of the natural vector fields on the tangent bundle, but the construction is different, being specific for the cotangent bundle.

Theorem 2.1. ([4]) *A natural tensor field J of type $(1,1)$ on T^*M, given by (2.2), defines an almost complex structure on T^*M, if and only if $a_4 = -a_3, b_4 = -b_3$ and the coefficients $a_1,\ a_2,\ a_3,\ b_1,\ b_2$ and b_3 are related by*

$$a_1a_2 = 1 + a_3^2\ , \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1 + (a_3 + 2tb_3)^2.$$

Studying the vanishing conditions for the Nijenhuis tensor field N_J, we may state:

Theorem 2.2. ([4]) *Let (M, g) be an $n(> 2)$-dimensional connected Riemannian manifold. The almost complex structure J defined by (2.2) on T^*M is integrable if and only if (M, g) has constant sectional curvature c and the coefficients $b_1,\ b_2,\ b_3$ are given by:*

$$b_1 = \frac{2c^2ta_2^2 + 2cta_1a_2' + a_1a_1' - c + 3ca_3^2}{a_1 - 2ta_1' - 2cta_2 - 4ct^2a_2'},\ b_2 = \frac{2ta_3'^2 - 2ta_1'a_2' + ca_2^2 + 2cta_2a_2' + a_1a_2'}{a_1 - 2ta_1' - 2cta_2 - 4ct^2a_2'},$$

$$b_3 = \frac{a_1a_3' + 2ca_2a_3 + 4cta_2'a_3 - 2cta_2a_3'}{a_1 - 2ta_1' - 2cta_2 - 4ct^2a_2'}.$$

Remark 2.3. The integrability conditions for the almost complex structure J on T^*M, may be expressed in the equivalent form

$$\text{(2.3)} \qquad \begin{cases} a_1' = \frac{1}{a_1+2tb_1}(a_1b_1 + c - 3ca_3^2 - 4cta_3b_3), \\ a_2' = \frac{1}{a_1+2tb_1}(2a_3b_3 - a_2b_1 - ca_2^2), \\ a_3' = \frac{1}{a_1+2tb_1}(a_1b_3 - 2ca_2a_3 - 2cta_2b_3). \end{cases}$$

In the paper [4], the author defined a Riemannian metric G of general natural lift type, given by the relations

$$\text{(2.4)} \qquad \begin{cases} G_p(X^H, Y^H) = c_1(t)g_{\pi(p)}(X,Y) + d_1(t)p(X)p(Y), \\ \\ G_p(\theta^V, \omega^V) = c_2(t)g^{-1}_{\pi(p)}(\theta,\omega) + d_2(t)g^{-1}_{\pi(p)}(p,\theta)g^{-1}_{\pi(p)}(p,\omega), \\ \\ G_p(X^H, \theta^V) = G_p(\theta^V, X^H) = c_3(t)\theta(X) + d_3(t)p(X)g^{-1}_{\pi(p)}(p,\theta), \end{cases}$$

$\forall\ X, Y \in \mathcal{X}(M)$, $\forall\ \theta, \omega \in \Lambda^1(M), \forall\ p \in T^*M$.

The conditions for G to be positive definite are assured if

$$c_1 + 2td_1 > 0, \quad c_2 + 2td_2 > 0, \quad (c_1 + 2td_1)(c_2 + 2td_2) - (c_3 + 2td_3)^2 > 0.$$

The author proved the following result:

Theorem 2.3. ([4]) *The family of Riemannian metrics G of general natural lifted type on T^*M such that (T^*M, G, J) is an almost Hermitian manifold, is given by (2.4), provided that the coefficients $c_1,\ c_2,\ c_3,\ d_1,\ d_2$, and d_3 are related to the coefficients $a_1,\ a_2,\ a_3,\ b_1,\ b_2$, and b_3 by the following proportionality relations*

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \frac{c_3}{a_3} = \lambda, \qquad \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \frac{c_3 + 2td_3}{a_3 + 2tb_3} = \lambda + 2t\mu,$$

where the proportionality coefficients $\lambda > 0$ and $\lambda + 2t\mu > 0$ are functions depending on t.

Considering the two-form Ω defined by the almost Hermitian structure (G, J) on T^*M, given by $\Omega(X, Y) = G(X, JY)$, for any vector fields X, Y on T^*M, we may formulate the main results from [4]:

Theorem 2.4. ([4]) *The almost Hermitian structure (T^*M, G, J) is almost Kählerian if and only if*

$$\mu = \lambda'.$$

Theorem 2.5. *A general natural lifted almost Hermitian structure (G, J) on T^*M is Kählerian if and only if the almost complex structure J is integrable (see Theorem 2.2) and $\mu = \lambda'$.*

Examples of such structures may be found in [21], [24].

3 General natural Kähler-Einstein structures on cotangent bundles

The Levi-Civita connection ∇ of the Riemannian manifold (T^*M, G) is obtained from the Koszul formula, and it is characterized by the conditions

$$\nabla G = 0,\ T = 0,$$

where T is the torsion tensor of ∇.

In the case of the cotangent bundle T^*M we may obtain the explicit expression of ∇.

The symmetric $2n \times 2n$ matrix associated to the metric G in the adapted frame, has the inverse H with the entries

$$H^{kl}_{(1)} = e_1 g^{kl} + f_1 g^{0k} g^{0l}, \quad H^{(2)}_{kl} = e_2 g_{kl} + f_2 p_k p_l, \quad H3^k_l = e_3 \delta^k_l + f_3 g^{0k} p_l.$$

Here g^{kl} are the components of the inverse of the matrix (g_{ij}), $g^{0k} = p_i g^{ik}$, and $e_1,\ f_1,\ e_2,\ f_2,\ \ e_3,\ f_3 : [0, \infty) \to \mathbf{R}$, some real smooth functions. In the paper [5], by using Lemma 2.1, we got $e_1,\ e_2,\ e_3$ as functions of $c_1,\ c_2,\ c_3$ and $f_1,\ f_2,\ f_3$ as functions of $c_1,\ c_2,\ c_3,\ d_1,\ d_2,\ d_3,\ e_1,\ e_2,\ e_3$, and next we obtained the expression of the Levi Civita connection of the Riemannian metric G on T^*M.

Theorem 3.1. ([5]) *The Levi-Civita connection ∇ of G has the following expression in the local adapted frame $\{\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j}\}_{i,j=1,\dots,n}$*

$$\begin{cases} \nabla_{\frac{\partial}{\partial p_i}} \dfrac{\partial}{\partial p_j} = Q^{ij}{}_h \dfrac{\partial}{\partial p_h} + \widetilde{Q}^{ijh} \dfrac{\delta}{\delta q^h}, & \nabla_{\frac{\delta}{\delta q^i}} \dfrac{\partial}{\partial p_j} = (-\Gamma^j_{ih} + \widetilde{P}_i{}^j{}_h) \dfrac{\partial}{\partial p_h} + P_i{}^{jh} \dfrac{\delta}{\delta q^h}, \\ \nabla_{\frac{\partial}{\partial p_i}} \dfrac{\delta}{\delta q^j} = P_j{}^{ih} \dfrac{\delta}{\delta q^h} + \widetilde{P}_j{}^i{}_h \dfrac{\partial}{\partial p_h}, & \nabla_{\frac{\delta}{\delta q^i}} \dfrac{\delta}{\delta q^j} = (\Gamma^h_{ij} + \widetilde{S}_{ij}{}^h) \dfrac{\delta}{\delta p_h} + S_{ijh} \dfrac{\partial}{\partial p_h}, \end{cases}$$

where Γ^h_{ij} are the Christoffel symbols of the Levi-Civita connection $\dot{\nabla}$ of g, and the coefficients which appear in the right hand side are the M-tensor fields on T^*M, whose explicit expressions may be obtained from the Koszul formula for ∇.

The curvature tensor field K of the connection ∇ is defined by

$$K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \Gamma(TM).$$

By using the local adapted frame $\{\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j}\}_{i,j=1,\dots,n} = \{\delta_i, \partial^j\}_{i,j=1,\dots,n}$ we obtained in [5] the horizontal and vertical components of the curvature tensor field, for example:

$$K(\delta_i, \delta_j)\delta_k = QQQQ_{ijk}{}^h \delta_h + QQQP_{ijkh} \partial^h,$$
$$K(\delta_i, \delta_j)\partial^k = QQPQ_{ij}{}^{kh} \delta_h + QQPP_{ij}{}^k{}_h \partial^h,$$

where the coefficients are the M-tensor fields denoted by sequences of Q and P, to indicate horizontal or vertical argument on a certain position. Their expressions have been given in [5], and they depend on the components of the Levi-Civita connection, their first order partial derivatives with respect to the cotangential coordinates p_i, and the curvature of the base manifold.

In the following, we shall obtain the conditions under which the general natural Kählerian manifold (T^*M, G, J) is an Einstein manifold. The components of the Ricci tensor $Ric(Y,Z) = trace(X \to K(X,Y)Z)$ of the Kälerian manifold (T^*M, G, J) are given by the formulas:

$$RicQQ_{jk} = Ric(\delta_j, \delta_k) = QQQQ_{hjk}{}^h + PQQP^h_{jkh},$$

$$RicPP^{jk} = Ric(\partial_j, \partial_k) = PPPP^{hjk}{}_h - PQPQ^j{}_h{}^{kh},$$

$$RicQP_j{}^k = Ric(\delta_j, \partial^k) = RicPQ^k{}_j = Ric(\partial^k, \delta_j) = PQPP^h_j{}^k{}_h + QQPQ_{hj}{}^{kh}.$$

The conditions for the general natural Kählerian manifold (T^*M, G, J) to be Einstein, are

$$\begin{cases} RicQQ_{jk} - \rho G^{(1)}_{jk} = 0, \\ RicPP^{jk} - \rho G^{jk}_{(2)} = 0, \\ RicQP_j{}^k - \rho G3^k_j = 0, \end{cases}$$

where ρ is a constant.

After a straightforward computation, using the RICCI package from Mathematica, the three differences which we have to study, become of the next forms:

$$\begin{cases} RicQQ_{jk} - \rho G^{(1)}_{jk} = (\lambda + 2\lambda' t)^2 [\lambda(\lambda + 2\lambda' t)\alpha_1 g_{jk} + \beta_1 p_j p_k], \\ RicPP^{jk} - \rho G^{jk}_{(2)} = \lambda(\lambda + 2\lambda' t)^2 [(\lambda + 2\lambda' t)\alpha_2 g^{jk} + 2\lambda\beta_2 g^{0j} g^{0k}], \\ RicQP_j{}^k - \rho G3^k_j = (\lambda + 2\lambda' t)^2 [\lambda(\lambda + 2\lambda' t)\alpha_3 \delta^k_j + \beta_3 p_j g^{0k}], \end{cases}$$

where $\alpha_1,\ \alpha_2,\ \alpha_3,\ \beta_1,\ \beta_2,\ \beta_3$ are rational functions depending on $a_1,\ a_3,\ \lambda$, their derivatives of the first two orders, and ρ. We do not present here the explicit expressions of the functions, since they are quite long.

Using lemma 2.1, and taking into account that $\lambda \neq 0,\ \lambda + 2\lambda' t \neq 0$, we obtain that $\alpha_1,\ \alpha_2,\ \alpha_3,\ \beta_1,\ \beta_2,\ \beta_3$ must vanish.

Solving the equations $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$ with respect to ρ we get the same value of ρ, which is quite long and we shall not write here.

Next, from $\beta_1 = 0$, $\beta_2 = 0$, and $\beta_3 = 0$, we obtain another three values for ρ, which we denote respectively by ρ_1, ρ_2, and ρ_3. This values must coincide with ρ.

When we impose the conditions $\rho_2 - \rho = 0$, $\rho_3 - \rho = 0$, we obtain two equations:

$$(a_1^2 + a_1^2 a_3^2 - 4a_1 a_1' t - 4a_1 a_1' a_3^2 t + 4a_1^2 a_3 a_3' t + 4a_1'^2 t^2 + 4a_1'^2 a_3^2 t^2 - \\ -8a_1 a_1' a_3 a_3' t^2 + 4a_1^2 a_3'^2 t^2)(An + B)/N_1 = 0 \tag{3.1}$$

$$(a_1^3 a_3 - 2a_1^2 a_1' a_3 t + 2a_1^3 a_3' t + 2a_1 a_3 ct + 2a_1 a_3^3 ct - 4a_1' a_3 ct^2 - 4a_1' a_3^3 ct^2 - \\ -4a_1 a_3' ct^2 + 4a_1 a_3^2 a_3' ct^2)(An + B)/N_2 = 0 \tag{3.2}$$

where the expressions of $A,\ B,\ N_1,\ N_2$ are quite long, depending on $a_1,\ a_3,\ \lambda$, and their derivatives.

Let us study the first parenthesis from (3.1) and (3.2), namely

$$E = a_1^2 + a_1^2 a_3^2 - 4a_1 a_1' t - 4a_1 a_1' a_3^2 t + 4a_1^2 a_3 a_3' t +$$

$$+4a_1'^2 t^2 + 4a_1'^2 a_3^2 t^2 - 8a_1 a_1' a_3 a_3' t^2 + 4a_1^2 a_3'^2 t^2,$$

$$F = a_1^3 a_3 - 2a_1^2 a_1' a_3 t + 2a_1^3 a_3' t + 2a_1 a_3 ct + 2a_1 a_3^3 ct -$$
$$-4a_1' a_3 ct^2 - 4a_1' a_3^3 ct^2 - 4a_1 a_3' ct^2 + 4a_1 a_3^2 a_3' ct^2,$$

The sign of E may be studied thinking it as a second degree function of the variable a_3'. The associated equation has the discriminant $\Delta = -(a_1^2 t^2 (a_1 - 2a_1' t)^2) < 0, \forall t > 0$ and the coefficient of $a_3'^2$, $4a_1^2 t^2 > 0, \forall t > 0$. Thus, $E > 0$ for every $t > 0$. If $t = 0$, the expression becomes $a_1^2(1 + a_3^2) > 0$. Hence we obtained that E is always positive.

Taking into account of the values of a_3' and a_1' from (2.3) and then multiplying by $\frac{a_1+2b_1 t}{a_3+2b_3 t} > 0$, $F = 0$ becomes an equation of the second order with respect to a_1^2

$$(a_1^2)^2 - 4a_1^2(1 - a_3^2)ct + 4c^2 t^2 (1 + a_3^2)^2 = 0, \tag{3.3}$$

with the discriminant $\Delta = -64a_3^2 c^2 t^2 < 0,\ \forall t > 0$. Thus $F > 0, \forall t > 0$ and if $t = 0$, $F = a_1^4 > 0$.

Since E and F are always positive, the relations (3.1) and (3.2) are fulfilled if and only if $An + B = 0$. The obtained equations does not depend on the dimension n of the base manifold, so we get that both A and B must vanish.

From the conditions $A = 0$ and $B = 0$ we get two quite long expression of λ'' and λ''', respectively.

By doing some computations with RICCI, we prove that the differences $\rho_1 - \rho$, $\rho_2 - \rho$ and $\rho_3 - \rho$ vanish when we replace the obtained values for λ'' and λ'''. Hence all the expressions obtained for the constant ρ coincide.

Next we have to find the conditions under which the derivative of λ'' is equal to λ''':

$$(\lambda'')' - \lambda''' = 0.$$

Computing the above difference, we obtain that its numerator decomposes into three factors, the vanishing condition for third one, reducing to the expression (3.3), after replacing the values of a_1' and a_3' given by (2.3) and multiplying by the denominator $(a_1+2b_1t)>0$.

Since we have proved that the obtained expression is always positive, we have to study only the next two cases, obtained from the vanishing conditions for the other two factors of the numerator of the difference $(\lambda'')'-\lambda'''$:

$$I)\ a_1^2a_1'\lambda+2a_1c\lambda+2a_1a_3^2c\lambda+a_1^3\lambda'-2a_1'c\lambda t-2a_1'a_3^2c\lambda t+4a_1a_3a_3'c\lambda t+$$
$$+2a_1c\lambda' t+2a_1a_3^2c\lambda' t=0,$$
$$II)\ a_1^2t(a_1^4-4a_1^2ct+4a_1^2a_3^2ct+4c^2t^2+8a_3^2c^2t^2+4a_3^4c^2t^2){\lambda'}^2+a_1^2(a_1^4-4a_1^2ct+$$
$$+4a_1^2a_3^2ct+4c^2t^2+8a_3^2c^2t^2+4a_3^4c^2t^2)\lambda'\lambda+(a_1^5a_1'+2a_1^4a_3^2c-a_1^4{a_1'}^2t-4a_1^3a_1'ct-$$
$$-4a_1^3a_1'a_3^2ct+4a_1^4a_3a_3'ct+4a_1^2{a_1'}^2ct^2+4a_1^2{a_1'}^2a_3^2ct^2-8a_1^3a_1'a_3a_3'ct^2+$$
$$+4a_1a_1'c^2t^2+8a_1a_1'a_3^2c^2t^2+4a_1a_1'a_3^4c^2t^2-8a_1^2a_3a_3'c^2t^2-8a_1^2a_3^3a_3'c^2t^2-$$
$$-4{a_1'}^2c^2t^3-8{a_1'}^2a_3^2c^2t^3-4{a_1'}^2a_3^4c^2t^3+16a_1a_1'a_3a_3'c^2t^3+16a_1a_1'a_3^3a_3'c^2t^3-$$
$$-16a_1^2a_3^2{a_3'}^2c^2t^3)\lambda^2=0.$$

In the case *I*, we may obtain the following expression of λ'

$$\lambda'=-\lambda\frac{a_1(a_1a_1'+2c(1+a_3^2))-2ct(a_1'+2a_1'a_3^2-4a_1a_3a_3')}{a_1[a_1^2+2ct(1+a_3^2)]}.$$

Replacing this expression of λ' in the first value obtained for ρ, we get

$$\lambda=\frac{2a_1c(n+1)}{\rho[a_1^2+2ct(1+a_3^2)]}. \tag{3.4}$$

Now we may state:

Theorem 3.2. *Let (M,g) be a smooth n-dimensional Riemannian manifold. If (G,J) is a general natural Kählerian structure on the cotangent bundle T^*M and the parameter λ is expressed by* (3.4), *where ρ is a nonzero real constant, then (T^*M,G,J) is a Kähler-Einstein manifold, i.e. $Ric=\rho G$.*

Remark 3.1. Taking into account of a theorem from [5], the expression (3.4) of λ implies that (T^*M,G,J) is a Kählerian manifold of constant holomorphic sectional curvature $k=\frac{2\rho}{n+1}$.

Example 3.1. The Kähler-Einstein structure on T^*M, from the paper [21] by Oproiu and Poroşniuc, may be obtained from the theorem 3.2, as a particular case. If in the expression (3.4) we impose the condition $a_3=0$, we get the same expression of λ obtained in [21], in the case of the natural structure of diagonal lifted type on the cotangent bundle T^*M of a Riemannian manifold (M,g).

In the case *II*, we obtain a homogeneous equation of second order in λ' and λ which may be solved with respect to $\frac{\lambda'}{\lambda}$. Then we obtain two expressions for λ'

$$\lambda'=\lambda(\pm\frac{1}{2t}+\frac{a_1^3-2a_1^2a_1't-2a_1ct-2a_1a_3^2ct+4a_1'ct^2+4a_1'a_3^2ct^2-8a_1a_3a_3'ct^2}{2a_1t\sqrt{a_1^4-4a_1^2ct+4a_1^2a_3^2ct+4c^2t^2+8a_3^2c^2t^2+4a_3^4c^2t^2}}).$$

When we replace this expression of λ' and its derivative λ'' in the first value of ρ, we obtain that in this case λ is defined on the set $T_0M\subset TM$ of the nonzero cotangent vectors to M, and it is given by

$$\lambda = \frac{n(a_1^2 + 2ct + 2a_3^2ct \pm \sqrt{a_1^4 - 4a_1^2ct + 4a_1^2a_3^2ct + 4c^2t^2 + 8a_3^2c^2t^2 + 4a_3^4c^2t^2})}{4a_1\rho t}. \tag{3.5}$$

Now we may formulate the next theorem:

Theorem 3.3. *Let (G, J) be a general natural Kählerian structure on the cotangent bundle T^*M of a smooth n-dimensional Riemannian manifold. If the parameter λ is expressed by* (3.5), *where ρ is a nonzero real constant, then (G, J) is a Kähler-Einstein structure on the bundle T_0^*M, of nonzero cotangent vectors to M, i.e. $Ric = \rho G$.*

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