Regularization of an abstract ill-posed Cauchy Problem via general quasi-reversibility method

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Abstract. In this paper, we investigate the abstract Cauchy problem for an elliptic equation. This problem is well known as severely ill-posed. The goal of this paper is to present some extensions of the quasi-reversibility method applied to an ill-posed Cauchy problem for elliptic equations. The key point to our analysis is the use of the general modified quasi-reversibility method to construct a family of regularizing operators for the considered problem and we prove the convergence of this method.

M.S.C. 2010: 47A52, 47D60, 41A36, 35R30.
Key words: Ill-posed problems; modified quasi-reversibility regularization; C_0-semigroups; Yosida approximation; Inverse Cauchy Problem.

1 Introduction

Let H be a complex Hilbert space with inner product $(.,.)$ and norm $\|.\|$, and let A be a linear unbounded operator with dense domain $\mathcal{D}(A)$. Assume that A is self-adjoint and positive definite in H, which has a continuous spectrum $\sigma(A) = [\gamma, +\infty[$, $\gamma = \inf(\sigma(A)) > 0$.

We consider the elliptic Cauchy problem of finding a function $u : [0, Z] \longrightarrow H$ such that

$$(1.1) \qquad \begin{cases} \mathfrak{L}u \equiv u_{zz} - Au = 0, & 0 < z < Z \\ u(0) = \varphi, & u_z(0) = 0, \end{cases}$$

where φ is prescribed data in the Hilbert space H.

Such problem aries in a number of applications, such as nondestructive testing techniques [2], geophysics [4], cardiology [10], and other practical industrial processes. There are many various monographs about the historical development of this topic, for more details, we refer the reader to Isakov [18], Lavrent′ev, Romanov and Šišatskiĭ [22], S.I. Kabanikhin and M. Schieck [19], Alexander A. Samarskii, Peter N. Vabishchevich [31], and the recent survey written by Giovanni Alessandrini, Luca Rondi, Edi Rosset, and Sergio Vessella [1].

Unfortunately, the inverse Cauchy problem (1.1), is highly ill-posed i.e., the solution does not depend continuously on the Cauchy data, and thus a small error

Applied Sciences, Vol.18, 2016, pp. 1-17.

in the given data may destroy the numerical solution. This fact was pointed out by Hadamard [15]. Some conditional stability results were given by some papers [1, 2, 18] these results are based on the exact given data. However, in practice, the given data is polluted for a variety of reasons such as measurement error, round-off error in machine representations. Because of these reasons, regularization strategies are necessary in order to compute such a solution in some stable way.

The inverse Cauchy problems associated with the elliptic equations have been studied by using different theoretical and numerical methods, such as, the modified quasi-boundary value method [34], the improved non-local boundary value problem method [35], the boundary element method (BEM) [23, 25], the modified collocation Trefftz method [24], the finite element method (FEM) [8] and the Fourier regularization method [13].

This work is mainly devoted to theoretical aspects of the method of quasi-reversibility to problem (1.1) in the abstract setting, by considering more general self-adjoint operators when A is positive and induces the elliptic case, i.e., has the following properties: for any $\lambda \in (-\infty, 0]$, the resolvent $\mathcal{R}(\lambda; A) = (A - \lambda I)^{-1}$ exists and satisfies the estimates

$$\exists M > 0 : \quad \forall \lambda \geq 0, \ \|(A + \lambda I)^{-1}\| \leq M(1 + \lambda)^{-1}.$$

In the case when A is a linear positive self-adjoint operator with compact inverse, problem (1.1) has been treated by a different methods. We can notably mention the iterative procedure of Kozlov and Maz'ya [20], the nonlocal regularization method [16], the quasi-reversibility method [1, Ch.7, pages 311-314], [5, 6, 7, 26, 27] and recently by the Krylov subspaces method [12].

One method for approaching such problems is the Quasi-reversibility method originally introduced by Lattès and Lions in their pioneering work [21]. The main idea of this method consists in replacing $\mathfrak{L}$ in (1.1) by a sequences of operators $\mathfrak{L}_\alpha \equiv \dfrac{d^2}{dz^2} - f_\alpha(A)$ depending on small parameter $\alpha > 0$, such that the perturbed problem $\mathfrak{L}_\alpha$ is well-posed, and its solution u_α can be taken as a candidate for an approximate solution to the original problem (1.1) in some sense. The perturbation $f_\alpha(A) = (A - \alpha A^2)$ has been exploited for stabilizing a certain class of ill-posed parabolic and elliptic problems. The modified version $f_\alpha(A) = A(I + \alpha A)^{-1}$ also has been used in the parabolic and elliptic case [14]. This paper seeks to make some extensions of this method applied to an abstract ill-posed elliptic problem.

This paper is outlined as follows: In Section 2, we present the notation and the functional setting which will be used in this paper and prepare some material which will be used in our analysis. Section 3 is devoted to the Yosida perturbation method. Finally we give a general perturbation method based on the modified quasi-reversibility method to construct an approximate solution of our problem in section 4.

2 Preliminaries and basic results

We denote by $\mathcal{L}(H)$ the Banach algebra of bounded linear operators acting in H and by $\mathcal{C}(H)$ the set of all closed linear operators densely defined in H. The domain, range and kernel of a linear operator $B \in \mathcal{C}(H)$ are denoted as $\mathcal{D}(B)$, $\mathcal{R}(B)$ and $\mathcal{N}(B)$, and the symbols $\rho(B)$ and $\sigma(B)$ are used for the resolvent set and the spectrum of B, respectively.

Definition 2.1. We denote by $\{H^r\}_{r\in\mathbb{R}}$ the Hilbert scale induced by A according to: $H^0 := H$, $H^r := \mathcal{D}(A^r)$ where $\|u\|_r := \|A^r u\|$ $(r \geq 0)$, $H^{-r} := (H^r)'$, i.e., H^{-r} is the dual space of H^r.

Proposition 2.1. *Let $(H^r)_{r\in\mathbb{R}}$ be the Hilbert scale induced by A. Let $-\infty < r_1 \leq r_2 < \infty$. Then the space H^{r_2} is densely and continuously embedded in H^{r_1}.*

We introduce the Lebesgue space $L^2(0, Z; H^s)$ of strongly measurable functions $(0, Z) \ni z \longrightarrow u(z) \in H^s$ with inner product and norm

$$(u, v)_{L^2(0,Z;H^s)} = \int_0^Z (u, v)_s \, dz, \quad \|u\|^2_{L^2(0,Z;H^s)} = \int_0^Z \|u\|^2_s \, dt,$$

and the Sobolev space

$$W^{m,2}(0, Z; H^s) := \{u \in L^2(0, Z; H^s) : u^{(i)} \in L^2(0, Z; H^s), \ i = 1 \ldots, m\}$$

with the usual norm

$$\|u\|^2_{W^{m,2}(0,Z;H^s)} := \sum_{i=0}^{m} \|u^{(i)}\|^2_{L^2(0,Z;H^s)}.$$

By $\mathcal{C}([0, Z]; H^s)$ we denote the space of continuous functions $[0, Z] \ni z \longrightarrow H^s$ with norm

$$\|u\|_{\infty,s} := \max_{z\in[0,Z]} \|u\|_{H^s}.$$

For any integer $m \in \mathbb{N}^*$, we denote by

$$W_m(0, Z; H^1, H) := \{u : u \in L^2(0, Z; H^1), \ u^{(m)} \in L^2(0, Z; H)\}$$

the completion of $\mathcal{C}([0, Z]; H^1)$ in the norm

$$|||u|||^2_m := \|u\|^2_{L^2(0,Z;H^1)} + \|u^{(m)}\|^2_{L^2(0,Z;H)}.$$

Remark 2.2. Note that if $z \in W_m(0, Z; H^1, H)$ then

$$z^{(i)} \in L^2(0, T; H^{1-i/m}) \cap \mathcal{C}([0, Z]; H^{1-\frac{i+1/2}{m}}), \quad 0 \leq i \leq m - 1.$$

which guarantees that

$$z^{(i)}(0), \ z^{(i)}(Z) \in H^{1-\frac{i+1/2}{m}}, \quad 0 \leq i \leq m - 1.$$

Finally, $V_0^2 := \{u \in W_2(0, Z; H^1, H) : u(0) = u'(0) = 0\}$
(resp. $V_Z^2 := \{u \in W_2(0, Z; H^1, H) : u(Z) = u'(Z) = 0\}$) denotes the closed subspace of $W_2(0, Z; H^1, H)$.

Definition 2.3. By a weak solution to problem (1.1) we mean a function $u(z)$ satisfying the following conditions:

1. $u \in \mathcal{C}([0, Z]; H)$;
2. $\int_0^Z (u, \mathfrak{L}v) = (\varphi, v'(0)), \quad \forall v \in V_Z^2.$

2.1 Spectral theorem and properties

By the spectral theorem, for each positive self-adjoint operator A, there is a unique right continuous family $\{E_\lambda\}_{\lambda\in[0,\infty[} : [0,\infty[\longrightarrow \mathcal{L}(H)$ of orthogonal projection operators such that $A = \int_0^\infty \lambda dE_\lambda$ with

$$\mathcal{D}(A) = \{v \in H : \int_0^\infty \lambda^2 \, d(E_\lambda v, v) < \infty\}.$$

In our case, we have $A = \int_\gamma^\infty \lambda \, dE_\lambda$ because $A \geq \gamma I, \ \gamma > 0$.

Theorem 2.2. *[11, Theorem 6, XII.2.5, p. 1196-1198] Let $\{E_\lambda, \ \lambda \geq \gamma > 0\}$ be the spectral resolution of the identity associate to A and let ϕ be a complex Borel function defined E-almost everywhere on the real axis. Then $f(A)$ is a closed operator with dense domain. Moreover*

(i) $\mathcal{D}(A) := \{h \in H : \int_\gamma^\infty |f(\lambda)|^2 \, d(E_\lambda v, v) < \infty\}$,

(ii) $(\phi(A)h, z) = \int_\gamma^\infty \phi(\lambda) \, d(E_\lambda h, z), \quad h \in \mathcal{D}(\phi(A)), \ z \in H$,

(iii) $\|\phi(A)h\|^2 = \int_\gamma^\infty |\phi(\lambda)|^2 \, d(E_\lambda h, h), \quad h \in \mathcal{D}(\phi(A))$,

(iv) $\phi(A)^* = \overline{\phi}(A)$. *In particular, if ϕ is real Borel function, then $\phi(A)$ is self-adjoint,*

(v) The operator $\phi(A)$ is bounded if and only if $\phi(\lambda)$ is bounded on $\sigma(A) = [\gamma, +\infty[$. In this case, $\|\phi(A)\| = \sup_{\lambda\in[\gamma,+\infty[} |\phi(\lambda)|$.

We denote by $S(z) = e^{-z\sqrt{A}} = \int_\gamma^\infty e^{-z\sqrt{\lambda}} \, dE_\lambda \in \mathcal{L}(H), \ y \geq 0$, the C_0-semigroup generated by $-\sqrt{A}$. Some basic properties of $S(z)$ are listed in the following theorem:

Theorem 2.3. *(see [28], chapter 2, Theorem 6.13, page 74). For this family of operators we have:*

1. $\|S(z)\| \leq 1, \quad \forall z \geq 0$;
2. *the function* $z \longmapsto S(z), \ z > 0$, *is analytic;*
3. *for every real* $r \geq 0$ *and* $z > 0$, *the operator* $S(z) \in \mathcal{L}(H, \mathcal{D}(A^{r/2}))$;
4. *for every integer* $k \geq 0$ *and* $z > 0$, $\|S^{(k)}(z)\| = \|A^{k/2}S(z)\| \leq c(k)z^{-k}$;
5. *for every* $x \in \mathcal{D}(A^{r/2})$, $r \geq 0$ *we have* $S(z)A^{r/2}x = A^{r/2}S(z)x$.

Theorem 2.4. *For $z > 0$, $S(z)$ is self-adjoint and one to one operator with dense range* ($S(z) = S(z)^*, \ \overline{\mathcal{R}(S(z))} = H$).

Proof. Let $\psi_z : [\gamma, +\infty[\to \mathbb{R},\ s \mapsto \psi_z(s) = e^{-z\sqrt{s}}$. Then by virtue of (4) of Theorem 2.3, we can write $(S(z))^* = \overline{\psi_z}(A) = \psi_z(A) = e^{-z\sqrt{A}} = S(z)$.

Let $h \in \mathcal{N}(S(z_0))$, $z_0 > 0$, then $S(z_0)h = 0$, which implies that $S(z)S(z_0)h = S(z+z_0)h = 0$, $z \geq 0$. Using analyticity, one obtains that $S(z)h = 0$, $z \geq 0$. Strong continuity at 0 now gives $h = 0$. This shows that $\mathcal{N}(S(z_0)) = 0$. Thanks to $\overline{\mathcal{R}(S(z_0))} = \mathcal{N}(S(z_0))^{\perp} = \{0\}^{\perp} = H$, we conclude that $\mathcal{R}(S(z_0))$ is dense in H. □

3 The Yosida perturbation method

In this section we use quasi-reversibility method, where the main idea consists in replacing the operator A by the Yosida approximation $A_\alpha = A(I+\alpha A)^{-1}$. Then let u_α be the solution of the perturbed problem

$$\begin{cases} u''_\alpha(z) - A_\alpha u_\alpha(z) = 0, & z \in [0, Z], \\ u_\alpha(0) = \varphi^\delta, \\ u'_\alpha(0) = 0, \end{cases} \tag{3.1}$$

where the operator A is replaced by

$$A_\alpha = A(I+\alpha A)^{-1}. \tag{3.2}$$

We show that

$$\sup_{0<z<Z} \|u_\alpha(z) - u(z)\| \to 0, \quad as\ \alpha \to 0, \tag{3.3}$$

$$\|u_\alpha(Z) - u(Z)\| \to 0, \quad as\ \alpha \to 0. \tag{3.4}$$

We show that the problem (3.1) is well posed, i.e., its solution

$$u^\delta_\alpha(z) = \cosh(z\sqrt{A_\alpha})\varphi^\delta, \tag{3.5}$$

is dependent continuously on the data φ^δ. Moreover, it is an approximation of the exact solution $u(z)$.

Lemma 3.1. *The problem* (1.1) *has a unique solution if and only if* $\varphi \in \{\varphi \in H : \|\varphi\|_1^2 = \int_\gamma^{+\infty} e^{2Z\sqrt{\lambda}}\, d\|E_\lambda \varphi\|^2 < +\infty\}$*, and its unique solution represented by*

$$u\,(z) = \cosh(z\sqrt{A})\varphi. \tag{3.6}$$

We will derive a bound on the difference between the solutions of the problem (1.1) and (3.1). However, before doing that, we need to assume that $\|u(Z)\|$ is bounded, i.e., $\|u(Z)\| \leq E$, where $E > 0$ is a constant.

The relation between any two regularized solutions of (3.1) is given by the following lemma.

Lemma 3.2. *Suppose we have two regularized solutions u_α^1 and u_α^2 defined by* (3.5) *with φ_1^δ and φ_2^δ, satisfying $\|\varphi_1^\delta - \varphi_2^\delta\| \le \delta$. If we choose $\sqrt{\alpha} = Z/\ln(2E/\delta)$. then we get the error bound*

$$\|u_\alpha^1(z) - u_\alpha^2(z)\| \le (2E)^{z/Z}\delta^{1-z/Z} \tag{3.7}$$

Proof. From (3.5) we have

$$\begin{aligned}\|u_\alpha^1(z) - u_\alpha^2(z)\|^2 &= \|\cosh(z\sqrt{A_\alpha})\varphi_1^\delta - \cosh(z\sqrt{A_\alpha})\varphi_2^\delta\|^2 \\ &\le \|\varphi_1^\delta - \varphi_2^\delta\|^2 \cosh^2(z/\sqrt{\alpha}) \\ &\le \delta^2 e^{2z/\sqrt{\alpha}}.\end{aligned}$$

The choice of parameter $\sqrt{\alpha} = Z/\ln(2E/\delta)$ leads to $\|u_\alpha^1(z) - u_\alpha^2(z)\| \le (2E)^{z/Z}\delta^{1-z/Z}$. □

From lemma (3.2) we see that the solution defined by (3.5) depends continuously on the data φ^δ.

Lemma 3.3. *Let u and u_α be the solutions of problem* (1.1) *and* (3.1) *with the same exact data φ. Suppose that $\|u(Z)\| \le E$. Then we have*

$$\|u(z) - u_\alpha(z)\| \le C_E(z)\alpha, \tag{3.8}$$

where $C_E(z) = \Big(\dfrac{3}{(Z-z)e}\Big)^3 Ez/2.$

Proof. From (3.6) the assumption $\|u(Z)\| \le E$ is equivalent to

$$\|u(Z)\|^2 = \int_\gamma^{+\infty} \cosh^2(Z\sqrt{\lambda})d\|E_\lambda\varphi\|^2 \le E^2. \tag{3.9}$$

Consequently,

$$\begin{aligned}\|u(z) - u_\alpha(z)\|^2 &= \int_\gamma^{+\infty} H_\alpha^2(z,\lambda)\cosh^2(Z\sqrt{\lambda})d\|E_\lambda\varphi\|^2 \\ &\le \Big(\sup_{\lambda\ge\gamma} e^{-(Z-z)\sqrt{\lambda}}F_\alpha(z,\lambda)\Big)^2 \int_\gamma^{+\infty} \cosh^2(Z\sqrt{\lambda})d\|E_\lambda\varphi\|^2,\end{aligned}$$

then, using the inequality $1 - e^{-r} \le r$ $(r \ge 0)$, we have

$$e^{-(Z-z)\sqrt{\lambda}}F_\alpha(z,\lambda) \le \frac{\alpha z}{2}\lambda^{\frac{3}{2}}e^{-(Z-z)\sqrt{\lambda}}.$$

Then the function $J_\alpha(\lambda) = \lambda^{\frac{3}{2}}e^{-(Z-z)\sqrt{\lambda}}$ satisfies the properties

$$J_\alpha(0) = 0,\ J_\alpha(+\infty) = 0 \quad \text{and } J_\alpha'(\lambda) = \frac{3}{2}\sqrt{\lambda}e^{-(Z-z)\sqrt{\lambda}}(3 - (Z-z)\sqrt{\lambda}),$$

$$J_\alpha'(\lambda) = 0 \Rightarrow \lambda_* = \Big(\frac{3}{Z-z}\Big)^3.$$

The function J_α attains its maximum at $\lambda_* = \Big(\frac{3}{Z-z}\Big)^3$ and $\sup_{\lambda\ge\gamma} J_\alpha(\lambda) = J_\alpha(\lambda_*) = \Big(\frac{3}{(Z-z)e}\Big)^3$, so we have $\|u(z) - u_\alpha(z)\| \le C_E(z)\alpha$. □

Theorem 3.4. *Let u the solution of problem* (1.1) *with exact data φ and u_α^δ is given by* (3.5) *with measured data φ^δ. Suppose that $\|u(Z)\| \le E$, and the measured function φ^δ satisfies $\|\varphi - \varphi^\delta\| \le \delta$ and if we choose $\sqrt{\alpha} = Z/\ln(2E/\delta)$. Then we have*

$$\|u(z) - u_\alpha^\delta(z)\| \le (2E)^{z/Z}\delta^{Z-z} + \frac{C_E(z)}{\ln^2(2E/\delta)}, \tag{3.10}$$

where $C_E(z) = \Big(\dfrac{3}{(Z-z)e}\Big)^3 Ez/2.$

Proof. Let u_α be the solution defined by (3.5) with exact data. Then the theorem is straightforward by using the triangle inequality $\|u - u_\alpha^\delta\| \le \|u - u_\alpha\| + \|u_\alpha - u_\alpha^\delta\|$ and the two previous lemmas □

Theorem (3.4) does not give any information about the continuous dependence of the solution of (1.1)-(3.9) at $z = Z$ on the data, as the condition (3.9) is too weak. We show several error estimates according to conditions that we impose on the final data $u(Z)$.

Theorem 3.5. *Let u and u_α be the solutions of problem* (1.1) *and* (3.1) *with the same exact data φ. Suppose that*
(i) $u(Z) \in \mathcal{D}(A^{\frac{3}{2}})$ *Then we have*

$$\|u(Z) - u_\alpha(Z)\| \le \sqrt{2c_Z(\alpha,1)}E_3. \tag{3.11}$$

(ii) $u(Z) \in \mathcal{D}(A)$ *Then we have*

$$\|u(Z) - u_\alpha(Z)\| \le \sqrt{2c_Z(\alpha,\frac{2}{3})}E_2. \tag{3.12}$$

(iii) $u(Z) \in \mathcal{D}(A^{\frac{1}{2}})$ *Then we have*

$$\|u(Z) - u_\alpha(Z)\| \le \sqrt{2c_Z(\alpha,\frac{1}{3})}E_1. \tag{3.13}$$

Proof. **(i)** Using the assumption $u(Z) \in \mathcal{D}(A^{\frac{3}{2}})$ who is equivalent to $\int_\gamma^{+\infty} \lambda^3 \cosh^2(Z\sqrt{\lambda}) d\|E_\lambda\varphi\|^2 \le E_3^2$, the difference $(u(Z) - u_\alpha(Z))$ can be estimated as follows

$$\begin{aligned}
\|u(Z) - u_\alpha(Z)\|^2 &= \int_\gamma^{+\infty} \left(\cosh\left(z\sqrt{\lambda}\right) - \cosh\left(z\sqrt{\lambda_\alpha}\right)\right)^2 d\|E_\lambda\varphi\|^2 \\
&\le \int_\gamma^{\lambda^*} F_\alpha^2(Z,\lambda)\cosh^2(Z\sqrt{\lambda})d\|E_\lambda\varphi\|^2 + \int_{\lambda^*}^{\infty} F_\alpha^2(Z,\lambda)\cosh^2(Z\sqrt{\lambda})d\|E_\lambda\varphi\|^2 \qquad (3.14)\\
&= I_1 + I_2,
\end{aligned}$$

where $F_\alpha(z,\lambda) = 1 - e^{\frac{-\alpha z \lambda^{\frac{3}{2}}}{2}}$. The function I_2 can be estimated as follows:

$$I_2 \le c_Z(\alpha,\theta)\|A^{\frac{3\theta}{2}}u(Z)\|^2, \tag{3.15}$$

where $c_Z(\alpha,\theta) = (\alpha Z/2)^{2\theta}$.
Indeed, let $m = (\alpha Z\lambda^{\frac{3}{2}})/2 \geq 1 \Rightarrow \lambda \geq (2/\alpha Z)^{\frac{2}{3}} = \lambda^*$, by virtue of $(1 - e^{-m} \leq m^\theta,\ m \geq 1,\ \theta > 0)$, then I_2 can be estimated as follows:

$$
\begin{aligned}
I_2 &\leq \int_{\lambda^*}^{\infty} \Big(\frac{\alpha Z\lambda^{\frac{3}{2}}}{2}\Big)^{2\theta} \cosh^2(Z\sqrt{\lambda}) d\|E_\lambda\varphi\|^2 \\
&\leq \Big(\frac{\alpha Z}{2}\Big)^{2\theta} \int_{\gamma}^{\infty} \lambda^{3\theta} \cosh^2(Z\sqrt{\lambda}) d\|E_\lambda\varphi\|^2 = c_Z(\alpha,\theta)\|A^{\frac{3\theta}{2}}u(Z)\|^2. \qquad (3.16)
\end{aligned}
$$

Let $\mathcal{N}_\alpha(\lambda) = F_\alpha(Z,\lambda)/(\alpha Z\lambda^{\frac{3}{2}}/2)$, then the function I_1 can be estimated as follows:

$$
I_1 \leq c_Z(\alpha,1)\Big(\sup_{\gamma\leq\lambda\leq\lambda^*} \mathcal{N}_\alpha(\lambda)\Big)^2 \|A^{\frac{3}{2}}u(Z)\|^2,
$$

we now set $f(s) = \dfrac{1-e^{-s}}{s}$, where

$$
0 < \alpha Z\gamma^{\frac{3}{2}}/2 \leq s = \alpha Z\lambda^{\frac{3}{2}}/2 \leq \alpha Z(\lambda^*)^{\frac{3}{2}}/2 = 1,\ \textit{for all}\ \lambda \in [\gamma,\lambda^*],
$$

then $f(s) \geq 0$, for all $s \in [0,1]$, and we have

$$
f'(s) = \frac{(1+s)e^{-s} - 1}{s^2} = \frac{M(s)}{z^2}.
$$

The function M satisfies the properties

$$
M(0) = 0, \qquad M'(s) = -se^{-s} \leq 0 \Rightarrow M \downarrow,
$$

this implies that

$$
M(s) \leq 0,\ \textit{for all}\ s \in [0,1],
$$

and

$$
f'(s) \leq 0 \Rightarrow f \downarrow \Rightarrow f(s) \leq f(\widetilde{s}),\ \textit{for all}\ s \geq \widetilde{s} = \alpha Z\gamma^{\frac{3}{2}}/2,
$$

it follows that

$$
\sup_{\widetilde{s}\leq s\leq 1} f(s) = f(\widetilde{s}) = \frac{1-e^{\widetilde{s}}}{\widetilde{s}}, \qquad (3.17)
$$

but,

$$
\lim_{\alpha\to 0} \widetilde{s} = 0 \Rightarrow 0 < \lim_{\widetilde{s}\to 0} f(\widetilde{s}) = 1 \Rightarrow \sup_{\widetilde{s}\leq s\leq 1} f(s) \leq 1. \qquad (3.18)
$$

We now return to I_1 and use (3.18) to write

$$
\begin{aligned}
I_1 &\leq c_Z(\alpha,1)\Big(\sup_{\gamma\leq\lambda\leq\lambda^*} \mathcal{N}_\alpha(\lambda)\Big)^2 \|A^{\frac{3}{2}}u(Z)\varphi\|^2 \\
&= c_Z(\alpha,1)\Big(\sup_{\widetilde{s}\leq s\leq 1} f(s)\Big)^2 \|A^{\frac{3}{2}}u(Z)\|^2 \\
&\leq c_Z(\alpha,1)\|A^{\frac{3}{2}}u(Z)\|^2, \qquad (3.19)
\end{aligned}
$$

and by virtue of inequality (3.16) with $\theta = 1$ and the inequality (3.19) we obtain the desired estimates. The claims (ii) and (iii) follow with the same manner. □

4 The general perturbation method

Quasi-reversibility is a regularization technique for ill-posed problems that is designed to generate approximate solutions to the problem in question. The central idea of quasi-reversibility is to solve the original problem backward, after first replacing A by $f(A)$, whose spectrum is bounded above. By following the idea in [3] and by using a modified quasi-reversibility method(M.Q.R.M) we construct an approximate solution of the considered problem. In Theorem 4.2, we will demonstrate that we obtain Hölder continuous dependence for the CONTROL PROBLEM when f satisfies Condition $(\mathcal{A})$.

Definition 4.1. Let $f : [0, \infty) \to \mathbb{R}^+$ be a Borel function, and assume that there exists $\omega \in \mathbb{R}^+$ such that $f(\lambda) \leq \omega^2$ for all $\lambda \in [0, \infty)$.

We consider the general approximate problem

$$\begin{cases} v''(z) = f(A)v(z), \quad z \in [0, Z], \\ v(0) = \varphi, \\ v'(0) = 0. \end{cases} \tag{4.1}$$

In this case, the problem 4.1 is well-posed, and the solution is given by

$$v(z) = \cosh\left(z\sqrt{f(A)}\right)\varphi = \frac{1}{2}\int_0^\infty \left(e^{z\sqrt{f(\lambda)}} + e^{-z\sqrt{f(\lambda)}}\right)dE_\lambda\varphi, \tag{4.2}$$

where $\left\{e^{z\sqrt{f(A)}}\right\}_{z\geq 0}$ is a strongly continuous semigroup of bounded operators.

In order to establish continuous dependence on modeling, in addition we assume that f satisfy the following condition

The Condition $(\mathcal{A})$. There exist positive constants β, δ, with $0 < \beta < 1$, for which $\mathcal{D}(A^{(1+\delta)/2}) \subseteq \mathcal{D}(\sqrt{f(A)})$, and

$$\left\|\left(-\sqrt{A} + \sqrt{f(A)}\right)\psi\right\| \leq \beta\|A^{(1+\delta)/2}\psi\|, \tag{4.3}$$

for all $\psi \in \mathcal{D}(A^{(1+\delta)/2})$, we use implicitly the fact that $\mathcal{D}(A^{(1+\delta)/2}) \subseteq \mathcal{D}(\sqrt{A})$, which follows immediately from the Spectral Theorem.

Next, we note that for $\psi \in \mathcal{D}(\sqrt{f(A)})$, $\left(\sqrt{f(A)}\psi, \psi\right) \leq \omega(\psi, \psi)$, so that $\sqrt{f(A)}$ is the generator of a strongly continuous semigroup $\left\{e^{z\sqrt{f(A)}}\right\}_{z\geq 0}$ of bounded operators, with $\|e^{z\sqrt{f(A)}}\| \leq e^{\omega z}$. If we set $g(\lambda) = -\sqrt{\lambda} + \sqrt{f(\lambda)}$, for $\lambda \in [0, \infty)$, then $g(A)$ is a self-adjoint operator, with domain

$$\mathcal{D}(g(A)) = \left\{\psi \in H \mid \int_0^\infty |g(\lambda)|^2 d(E(\lambda)\psi, \psi) < \infty\right\}.$$

It follows from properties of the functional calculus (cf. [30]) that $-\sqrt{A}+\sqrt{f(A)} \subseteq g(A)$, in the sense of unbounded operators; that is, $\mathcal{D}(-\sqrt{A} + \sqrt{f(A)}) = \mathcal{D}(\sqrt{A}) \cap$

$\mathcal{D}(\sqrt{f(A)}) \subseteq \mathcal{D}(g(A))$, and $g(A)\psi = (-\sqrt{A}+\sqrt{f(A)})\psi$ for all $\psi \in \mathcal{D}(-\sqrt{A}+\sqrt{f(A)})$. Since $g(A)$ is self-adjoint, and $(g(A)\psi,\psi) \leq \omega(\psi,\psi)$ for all $\psi \in \mathcal{D}(g(A))$, it follows that $g(A)$ is also the generator of a strongly continuous semigroup $\{e^{zg(A)}\}_{z\geq 0}$ of bounded operators, with $\|e^{zg(A)}\| \leq e^{\omega z}$. Before stating our main result, we shall need the following:

Lemma 4.1. *For all $z \geq 0$,*

$$e^{zg(A)} = e^{-z\sqrt{A}}e^{z\sqrt{f(A)}}. \tag{4.4}$$

Proof. First, note that $\mathcal{D}(\sqrt{A}) \cap \mathcal{D}(\sqrt{f(A)})$ is a core for $g(A)$, indeed, set

$$e_n = \{\lambda \in [0,\infty)/\ |g(\lambda)| \leq n\}$$

and let $E_n = E(e_n)$. Then if $\lambda \in e_n$, $\lambda \leq (n+\omega)^2$, so that e_n is a bounded Borel set, and hence E_n is a bounded projection on H. Now, if $x \in \mathcal{D}(g(A))$, then $E(e_n)x \in \mathcal{D}(\sqrt{A}) \cap \mathcal{D}(\sqrt{f(A)})$, and $E(e_n)x \to x$. In addition, $g(A)E(e_n)x = E(e_n)g(A)x \to g(A)x$, and so $\mathcal{D}(\sqrt{A}) \cap \mathcal{D}(\sqrt{f(A)})$ is a core for $g(A)$. Thus $g(A)$ is essentially self-adjoint on $\mathcal{D}(\sqrt{A}) \cap \mathcal{D}(\sqrt{f(A)})$. Since the bounded operators $e^{-z\sqrt{A}}$ and $e^{z\sqrt{f(A)}}$ commute, (4.4) is now a consequence of the version of the Trotter Product Formula given in ([29], VIII.31). (Note: A shorter proof follows immediately from Property (c) of the functional calculus for unbounded self-adjoint operators given in ([11], XII.2.7, Corollary 7), and the fact that both sides of (4.4) are bounded operators.)

Once again using the Spectral Theorem, we note that for each $n = 1, 2, ...$, $\{e^{z\sqrt{A}}E_n\}_{z\geq 0}$ is a strongly continuous semigroup of bounded operators on H. In fact, a consequence of the Spectral Theorem is that $\{e^{\alpha\sqrt{A}}E_n\}_{\alpha\in\mathbb{C}}$ is an entire group of bounded operators on H, in the sense of [9], as are $\{e^{\alpha\sqrt{f(A)}}E_n\}_{\alpha\in\mathbb{C}}$ and $\{e^{\alpha g(A)}E_n\}_{\alpha\in\mathbb{C}}$. Moreover, observe that (4.4) holds for all complex values of the parameter, when applied to E_n:

$$e^{\alpha g(A)}E_n = e^{-\alpha\sqrt{A}}e^{\alpha\sqrt{f(A)}}E_n, \quad \alpha \in \mathbb{C}, n = 1, 2, ...$$

Indeed, from (4.4), we have equality of these two entire functions for all values of $z \geq 0$.

We now prove the following:

Theorem 4.2. *Let A be a positive self-adjoint operator acting on H, let f satisfy Condition $(\mathcal{A})$, and assume that there exists a constant γ independent of β and ω, such that $(g(A)\psi,\psi) \leq \gamma(\psi,\psi)$, for all $\psi \in H$. If $u(t)$ and $v(t)$ are solutions of (1.1) and (4.1), respectively, and $\|u(Z)\| \leq \widetilde{M}$, then there exist constants C and M, independent of β, such that for $0 \leq z < Z$,*

$$\|u(z) - v(z)\| \leq C\beta^{1-z/Z}M^{z/Z}.$$

Proof. Let $\varphi_n = E(e_n) = E_n\varphi$, and for $h \in H$ we define

$$\phi_n(\alpha) = \Big(e^{\alpha^2}\Big[\cosh(\alpha\sqrt{A}) - \cosh(\alpha\sqrt{f(A)})\Big]\varphi_n, h\Big), \tag{4.5}$$

Our aim is to show that $\phi_n(\alpha)$ is bounded in the strip $0 \le \mathcal{R}\alpha \le Z$, so that we might apply the Three Lines Theorem (cf. [30], p. 33). We set $\alpha = z + i\eta$, where $0 \le z \le Z$, and $\eta \in \mathbb{R}$. Notice that because A and $\sqrt{f(A)}$ are self-adjoint,

$$\|e^{i\eta\sqrt{A}}\| = \|e^{i\eta\sqrt{f(A)}}\| = 1.$$

Then

$$|\phi_n(\alpha)| \le \frac{1}{2}\, e^{(z^2-\eta^2)}\Big(\|\mathcal{B}_1\varphi_n\| + \|\mathcal{B}_2\varphi_n\|\Big)\|h\|, \tag{4.6}$$

where

$$\mathcal{B}_1 = e^{(z+i\eta)\sqrt{A}} - e^{(z+i\eta)\sqrt{f(A)}}, \text{ and } \mathcal{B}_2 = e^{-(z+i\eta)\sqrt{A}} - e^{-(z+i\eta)\sqrt{f(A)}}.$$

First we have

$$\begin{aligned} \|\mathcal{B}_1\varphi_n\| &= \|e^{(z+i\eta)\sqrt{A}}\varphi_n - e^{(z+i\eta)\sqrt{A}}e^{(z+i\eta)g(A)}\varphi_n\| \\ &\le \Big(\|e^{(z+i\eta)\sqrt{A}}\varphi_n - e^{(z+i\eta)\sqrt{A}}e^{i\eta g(A)}\varphi_n\| \\ &+ \|e^{(z+i\eta)\sqrt{A}}e^{i\eta g(A)}\varphi_n - e^{(z+i\eta)\sqrt{A}}e^{(z+i\eta)g(A)}\varphi_n\|\Big), \end{aligned} \tag{4.7}$$

then

$$\|\mathcal{B}_1\varphi_n\| \le \|\mathcal{I}_1\varphi_n\| + \|\mathcal{I}_2\varphi_n\|, \tag{4.8}$$

where

$$\|\mathcal{I}_1\varphi_n\| = \|e^{(z+i\eta)\sqrt{A}}\varphi_n - e^{(z+i\eta)\sqrt{A}}e^{i\eta g(A)}\varphi_n\| \le \|(I - e^{i\eta g(A)})e^{z\sqrt{A}}\varphi_n\|.$$

We have repeatedly used (4.4) for complex values of the parameter, and by standard properties of semigroups, if $\psi \in \mathcal{D}(g(A))$ and $\eta \in \mathbb{R}$, then

$$I - e^{i\eta g(A)}\psi = -i\int_0^\eta e^{isg(A)}g(A)\psi ds, \tag{4.9}$$

so that

$$\|(I - e^{i\eta g(A)})\psi\| \le |\eta|\|g(A)\psi\|.$$

Since $e^{z\sqrt{A}}\varphi_n \in \mathcal{D}(\sqrt{A}) \cap \mathcal{D}(\sqrt{f(A)}) \subseteq \mathcal{D}(g(A))$ for all $z \ge 0$, and $e^{z\sqrt{A}}\varphi_n \in \mathcal{D}(A^{(1+\delta)/2})$, we have from Condition $(\mathcal{A})$ and the above inequality that

$$\|\mathcal{I}_1\varphi_n\| = \|(I - e^{i\eta g(A)})e^{z\sqrt{A}}\varphi_n\| \le \beta|\eta|\|A^{(1+\delta)/2}e^{z\sqrt{A}}\varphi_n\|. \tag{4.10}$$

Similarly,

$$(I - e^{zg(A)})e^{z\sqrt{A}}\varphi_n = -\int_0^z e^{sg(A)}g(A)e^{z\sqrt{A}}\varphi_n ds, \tag{4.11}$$

so that

$$\|\mathcal{I}_2\varphi_n\| = \|(I - e^{zg(A)})e^{z\sqrt{A}}\varphi_n\| \le \beta z e^{\gamma z}\|A^{(1+\delta)/2}e^{z\sqrt{A}}\varphi_n\|, \tag{4.12}$$

using (4.10) and (4.12) the inequality (4.8) becomes

$$\|\mathcal{B}_1\varphi_n\| \le \beta(|\eta| + ze^{\gamma z})\|A^{(1+\delta)/2}e^{z\sqrt{A}}\varphi_n\| \le \beta(1+Te^{\gamma T})\|A^{(1+\delta)/2}e^{z\sqrt{A}}\varphi_n\|. \tag{4.13}$$

On the other hand, we have

$$\begin{aligned}
\|\mathcal{B}_2\varphi_n\| &= \|e^{-(z+i\eta)\sqrt{A}}\varphi_n - e^{-(z+i\eta)\sqrt{A}}e^{-(z+i\eta)g(A)}\varphi_n\| \\
&\le \Big(\|e^{-(z+i\eta)\sqrt{A}}\varphi_n - e^{-(z+i\eta)\sqrt{A}}e^{-i\eta g(A)}\varphi_n\| \\
&+ \|e^{-(z+i\eta)\sqrt{A}}e^{-i\eta g(A)}\varphi_n - e^{(z+i\eta)\sqrt{A}}e^{-(z+i\eta)g(A)}\varphi_n\|\Big) \\
&= \|\mathcal{J}_1\varphi_n\| + \|\mathcal{J}_2\varphi_n\|,
\end{aligned}$$

where

$$\begin{aligned}
\|\mathcal{J}_1\varphi_n\| &= \|e^{-(z+i\eta)\sqrt{A}}\varphi_n - e^{-(z+i\eta)\sqrt{A}}e^{-i\eta g(A)}\varphi_n\| \\
&\le \|(I - e^{-i\eta g(A)})e^{z\sqrt{A}}\varphi_n\|.
\end{aligned}$$

If $\psi \in \mathcal{D}(g(A))$, and $\eta \in \mathbb{R}$, then

$$I - e^{-i\eta g(A)}\psi = i\int_0^\eta e^{-isg(A)}g(A)\psi ds,$$

so that

$$\|\mathcal{J}_1\varphi_n\| = \|(I - e^{-i\eta g(A)})\varphi_n\| \le \beta|\eta|\|A^{(1+\delta)/2}\varphi_n\|. \tag{4.14}$$

Similarly,

$$(I - e^{-zg(A)})\varphi_n = \int_0^z e^{-sg(A)}g(A)\varphi_n ds,$$

so that

$$\|\mathcal{J}_2\varphi_n\| = \|(I - e^{-zg(A)})\varphi_n\| \le \beta z\|A^{(1+\delta)/2}\varphi_n\|. \tag{4.15}$$

Thus

$$\|\mathcal{B}_2\varphi_n\| \le \beta(|\eta| + z)\|A^{(1+\delta)/2}\varphi_n\| \le \beta(1+Z)\|A^{(1+\delta)/2}\varphi_n\|, \tag{4.16}$$

according to (4.13) and (4.16) the inequality (4.6) becomes

$$\begin{aligned}
|\phi_n(\alpha)| &\le \frac{1}{2}e^{(z^2-\eta^2)}\Big((|\eta|+z)\|A^{(1+\delta)/2}\varphi_n\| + \beta(|\eta| + ze^{\gamma z})\|A^{(1+\delta)/2}e^{z\sqrt{A}}\varphi_n\|\Big)\|h\| \\
&\le \frac{1}{2}e^{Z^2}\beta\Big((1+Z)\|A^{(1+\delta)/2}\varphi_n\| + (1+Ze^{\gamma Z})\|A^{(1+\delta)/2}e^{z\sqrt{A}}\varphi_n\|\Big)\|h\| \\
&\le \frac{1}{2}e^{Z^2}\beta\max\Big(1+Z, 1+Ze^{\gamma Z}\Big)\|A^{(1+\delta)/2}e^{z\sqrt{A}}\varphi_n\|\|h\|,
\end{aligned}$$

and so, using strong continuity of the semigroup $\left(e^{z\sqrt{A}}E_n\right)$, it follows that ϕ_n is bounded in the strip $0 \le z \le Z$. By the Three Lines Theorem,

$$|\phi_n(z)| \le M(0)^{1-z/Z}M(Z)^{z/Z}, \qquad \text{for } 0 \le z \le Z, \tag{4.17}$$

where

$$M(z) = \max_{\alpha = z + i\eta,\, \eta\in\mathbb{R}} |\phi_n(\alpha)|. \tag{4.18}$$

Since $M(0) \leq \frac{1}{2}\beta\|A^{(1+\delta)/2}\varphi_n\|\|h\|$, we have

$$|\phi_n(z)| \leq \left(\frac{1}{2}\beta\|A^{(1+\delta)/2}\varphi_n\|\|h\|\right)^{1-z/Z} M(Z)^{z/Z}, \qquad \text{for } 0 \leq z \leq Z.$$

Also, returning to (4.6) and using (4.10), (4.12), (4.14) and (4.15) we have

$$\begin{aligned} |\phi_n(Z+i\eta)| &\leq \Big(\|(I - e^{i\eta g(A)})e^{Z\sqrt{A}}\varphi_n\| + \|(I - e^{Zg(A)})e^{Z\sqrt{A}}\varphi_n\| \\ &+ \|(I - e^{-i\eta g(A)})\varphi_n\| + \|(I - e^{-Zg(A)})\varphi_n\|\Big)\|h\| \\ &\leq 2\|e^{Z\sqrt{A}}\varphi_n\|\|h\| + (1 + e^{\gamma Z})\|e^{Z\sqrt{A}}\varphi_n\|\|h\| + 4\|\varphi_n\|\|h\| \\ &\leq (7 + e^{\gamma Z})\|e^{Z\sqrt{A}}\varphi_n\|\|h\|, \end{aligned}$$

we have used the fact that $\|e^{Zg(A)}\| \leq e^{\gamma Z}$. Thus

$$|\phi_n(z)| \leq \Big(\beta\|A^{(1+\delta)/2}\varphi_n\|\Big)^{1-z/Z} \Big(C_1\|e^{Z\sqrt{A}}\varphi_n\|\Big)^{z/Z} \|h\|. \tag{4.19}$$

for a suitable constant C_1 that is independent of β

We now assume that $\|\cosh(Z\sqrt{A})\varphi\| \leq \widetilde{M}$ (which serves to stabilize the problem), from which it follows that $\|A^{(1+\delta)/2}\varphi\| \leq \widetilde{M}$ for a possibly different value of $\widetilde{M}$. If we let $n \to \infty$ in (4.19), we obtain

$$|\phi(z)| \leq C\beta^{1-z/Z}M^{z/Z}\|h\|,$$

where C and M are computable constants which are independent of β, and

$$\phi(z) = e^{z^2}\Big(\Big[\cosh(z\sqrt{A}) - \cosh(z\sqrt{f(A)})\Big]\varphi, h\Big).$$

Taking the supremum over all $h \in H$, with $\|h\| \leq 1$, we obtain

$$e^{z^2}\|u(z) - v(z)\| \leq C\beta^{1-z/Z}M^{z/Z},$$

and the proof is complete. □

4.1 Example

let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$, with smooth boundary $\partial\mathcal{D}$, and we consider the following ill-posed problem

$$\begin{cases} u_{zz} + \Delta u = 0, \quad \textit{in } \mathcal{D}\times[0,Z), \\ u(x,0) = \varphi(x), \quad \textit{in } \mathcal{D} \\ u'(x,0) = 0, \quad \textit{in } \mathcal{D} \\ u = 0, \quad \textit{in } \partial\mathcal{D}\times[0,Z), \end{cases} \tag{E}$$

where $\varphi(x)$ is a prescribed function and $A = -\Delta$.

For $\epsilon > 0$, we consider the approximate well-posed problem

$$\begin{cases} v_{zz} + \Delta v + \epsilon \Delta v_{zz} = 0, \quad in\ \mathcal{D} \times [0, Z), \\ v(x, 0) = \varphi(x), \quad in\ \mathcal{D} \\ v'(x, 0) = 0, \quad in\ \mathcal{D} \\ v = 0, \quad in\ \partial\mathcal{D} \times [0, Z). \end{cases} \tag{E_ϵ}$$

Let $f(\lambda) = \lambda(1+\epsilon\lambda)^{-1}$, for $\epsilon > 0$. Then f is a bounded Borel function, and clearly satisfies the Condition $(\mathcal{A})$, with $\omega = \frac{1}{\epsilon}$, $\beta = \epsilon$, and $\delta = 1$, since

$$\|(\sqrt{(I+\epsilon A)^{-1}} - I)\sqrt{A}\psi\| \leq \beta \|A^{\frac{3}{2}}\psi\|,$$

for all $\psi \in \mathcal{D}(A^{\frac{3}{2}})$. Moreover, $g(A) = -\sqrt{A} + \sqrt{A(I+\epsilon A)^{-1}}$, generates a semigroup of contractions, so that $\gamma \leq 0$. In both cases, Theorem (4.2) yields the result

$$\|u(z) - v(z)\| \leq C\beta^{1-z/Z} M^{z/Z}.$$

Definition 4.2. A family $\{R_\beta(z),\ \beta > 0,\ z \in [0, Z]\} \subset L(H)$ is called a family of regularizing operators for the problem (1.1) if for each solution $u(z)$, $0 \leq z \leq Z$ of (1.1) with initial element φ, and for any $\delta > 0$, there exists $\beta(\delta) > 0$, such that

$$\beta(\delta) \longrightarrow 0,\ \delta \longrightarrow 0, \tag{R_1}$$

$$\|R_{\beta(\delta)}(z)\varphi_\delta - u(z)\| \longrightarrow 0,\ \delta \longrightarrow 0, \tag{R_2}$$

for each $z \in [0, Z]$ provided that φ_δ satisfies $\|\varphi_\delta - \varphi\| \leq \delta$.

Define

$$R_{\beta(\delta)}(z) = \cosh\left(z\sqrt{f(A)}\right),\ z \geq 0,\ \beta > 0. \tag{4.20}$$

It is clear that $R_{\beta(\delta)}(z) \in L(H)$. In the following we will show that $R_{\beta(\delta)}(z)$ is a family of regularizing operators for (1.1).

Theorem 4.3. *Assuming that $\varphi \in \mathcal{C}_1(A)$, then (R_2) holds.*

Proof. At first, we have

$$\|R_{\beta(\delta)}(z)\varphi_\delta - u(z)\| \leq \|R_{\beta(\delta)}(z)(\varphi_\delta - \varphi)\| + \|R_{\beta(\delta)}(z)\varphi - u(z)\| = \Delta_1(z) + \Delta_2(z). \tag{4.21}$$

By choosing $\omega \leq Z/\ln(M'/\delta)$, we have

$$\begin{aligned} \Delta_1(z) = \|R_{\beta(\delta)}(z)(\varphi_\delta - \varphi)\| &\leq e^{\omega z}\delta \\ &\leq (M'/\delta)^{z/Z}\delta \\ &= (M')^{z/Z}\delta^{1-z/Z} \longrightarrow 0,\ \text{as } \delta \longrightarrow 0 \end{aligned} \tag{4.22}$$

and

$$\Delta_2(z) = \|R_{\beta(\delta)}(z)\varphi - u(z)\|. \tag{4.23}$$

Now, by virtue of theorem 4.2 we have

$$\Delta_2(z) = \|R_{\beta(\delta)}(z)\varphi - u(z)\| \le C\beta^{1-z/Z}M^{z/Z} \longrightarrow 0, \text{ as } \delta \longrightarrow 0, \tag{4.24}$$

uniformly in z. Combining (4.22) and (4.24) we obtain

$$\sup_{0\le z\le Z} \|R_{\beta(\delta)}(z)\varphi_\delta - u(z)\| \longrightarrow 0, \text{ as } \delta \longrightarrow 0. \tag{4.25}$$

This shows that $R_{\beta(\delta)}(z)$ is a family of regularizing operators for (1.1). □

Acknowledgements. The authors would like to thank to the editors for their cooperation and support. The work described in this paper was supported by the MESRS of Algeria (CNEPRU Project B01120090003).

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