A study on space forms of semi-transversal lightlike submanifolds of indefinite nearly Kähler manifolds

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Abstract. In this paper, we find conditions on semi-transversal lightlike submanifolds such that the induced connection is metric connection. Further, we prove non-existence of totally umbilical proper semi-transversal lightlike submanifolds of an indefinite nearly Kähler manifold $\bar{M}(c)$ $(c \neq \alpha)$ of constant type α with constant holomorphic sectional curvature c. Also, we establish some new results on mixed geodesic proper semi transversal lightlike submanifolds.

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1 Introduction

A submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be lightlike submanifold if the induced metric g on M is degenerate. Different classes of lightlike submanifolds of indefinite nearly Kähler manifolds have been defined according to the behaviour of distributions on these submanifolds with respect to the action of (1, 1) tensor field $\overline{J}$ in Kähler structure of the ambient manifolds.

The concept of lightlike submanifold has attained various important contributions in semi-Riemannian geometry. It has been successfully employed on different topics of mathematical physics, particularly, relativity theory. Since the study of positive definite metric manifold was not sufficient to understand various problems where metric is indefinite. Therefore, Duggal et al. [8, 9] initiated the study of CR-submanifold with Lorentzian metric and obtained their use in relativity theory. In 1970, a detailed study of an indefinite nearly Kähler manifold was introduced by A. Grey [2]. A new class of lightlike submanifolds, known as $GCR-$lightlike was introduced in [16] which contain $CR-$lightlike and $SCR-$lightlike submanifolds as its sub-cases. In this paper, we study semi-transversal lightlike submanifolds of an indefinite nearly Kähler manifold. The paper is organized as follows:

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In section 2, we describe basics of lightlike submanifolds and indefinite nearly Kähler manifolds. In section 3, we discuss semi-transversal lightlike submanifolds and decomposition of vectors in different distributions. Further, we find conditions on semi-transversal lightlike submanifolds such that the induced connection is metric connection. Some results on mixed geodesic proper semi transversal lightlike submanifolds have been obtained. In section 4, we prove non-existence of totally umbilical proper semi-transversal lightlike submanifolds of an indefinite nearly Kähler manifold $\bar{M}(c)$ $(c \neq \alpha)$ of constant type α with constant holomorphic sectional curvature c. We also obtain some important results on mixed geodesic proper semi transversal lightlike submanifolds.

2 Preliminaries

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $1 \leq q \leq m+n-1$, $m, n \geq 1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ with g as induced metric on M. If g is degenerate on the tangent bundle TM of M, then M is called lightlike submanifold.

For a degenerate metric g on M, $T_xM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but not complementary to each other. Therefore, there exists a subspace $Rad(TM) = T_xM \cap T_xM^{\perp}$, known as Radical subspace. If the mapping $Rad(TM) : M \longrightarrow TM$ such that $x \in M \mapsto Rad(T_xM)$, defines a smooth distribution of rank $r > 0$ on M, then M is said to be be an r-lightlike submanifold and the distribution $Rad(TM)$ is said to be radical distribution on M. The non-degenerate complementary subbundles $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ are known as screen distribution in TM and screen transversal distribution in $TM^{\perp}$ respectively, i.e.,

$$TM = Rad(TM) \perp S(TM) \ \ \& \ \ TM^{\perp} = Rad(TM) \perp S(TM^{\perp}). \tag{2.1}$$

Let $ltr(TM)$(lightlike transversal bundle) and $tr(TM)$(transversal bundle) be complementary but not orthogonal vector bundles to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and TM in $T\bar{M}|_M$ respectively.

Then, the transversal vector bundle $tr(TM)$ is given by[12]

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2.2}$$

From (2.1) and (2.2), we get

$$T\bar{M}|_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.3}$$

Theorem 2.1. *[11] Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)|u)$ consisting of a smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}|_u$, where u is a coordinate neighbourhood of M such that*

$$\bar{g}_{ij}(N_i, \xi_j) = \delta_{ij}, \ \ \bar{g}_{ij}(N_i, N_j) = 0, \tag{2.4}$$

for any i, $j \in \{1, 2, ..., r\}$, where $\{\xi_1, \xi_2, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be a Levi-Civita connection on $\bar{M}$. For $X,\ Y \in \Gamma(Rad(TM))$ and $U \in \Gamma(tr(TM))$, the Gauss-Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.6}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(Rad(TM))$ and $\Gamma(tr(TM))$ respectively, ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(tr(TM))$ which is known as the second fundamental form, A_U is a linear operator on M which is known as the shape operator and $\nabla^{\perp}$ is a linear connection on $tr(TM)$ which is known as the transversal linear connection.
Now, let $L : tr(TM) \longrightarrow ltr(TM)$ and $S : tr(TM) \longrightarrow S(TM^{\perp})$ be projection maps, then (2.5) and (2.6) reduce to

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.7}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.8}$$

In particular,

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X,N), \tag{2.9}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X,W), \tag{2.10}$$

where $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM)^{\perp})$. The equations (2.5), (2.7) are known as Gauss equations and, (2.8), (2.9),(2.10) are known as Weingarten equations respectively, for the lightlike submanifold M of $\bar{M}$[3].
Now, let P be the projection of TM on $S(TM)$, by using metric connection $\bar{\nabla}$ and (2.5), (2.6), (2.7), (2.9), (2.10), we get the following equations:

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.11}$$

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y, h^l(X,\xi) + g(Y, \nabla_X \xi) = 0, \tag{2.12}$$

$$\bar{g}(D^s(X,N),W) = g(N, A_W X), \tag{2.13}$$

$$\bar{g}(A_{N'} X, N) + \bar{g}(A_N X, N^{'}) = 0, \tag{2.14}$$

for $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM)^{\perp})$ and $N, N^{'} \in \Gamma(ltr(TM))$.
In particular, the induced connection ∇ and transversal connection ∇^t_X are not metric connections. For $X, Y, Z \in \Gamma(TM)$ and $U, U^{'} \in \Gamma(tr(TM))$, following formulae represent induced connection and transversal connection respectively

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y) \tag{2.15}$$

$$(\nabla^t_X \bar{g})(U, U') = -\{\bar{g}(A_U X, U') + \bar{g}(A_{U'} X, U)\}. \tag{2.16}$$

For any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, we have the following equations on the basis of projection P:

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{2.17}$$

$$\nabla_X \xi = A^*_\xi X + \nabla^{*t}_X(\xi), \tag{2.18}$$

where $\{h^*(X, PY), \nabla^{*t}_X(\xi)\}$ and $\{\nabla^*_X PY, A^*_\xi X\}$ belong to $\Gamma(Rad(TM))$ and $\Gamma(S(TM)))$ respectively, ∇ and ∇^{*t} are linear connections on $(S(TM))$ and $Rad(TM)$ respectively, h^* and A^* are $\Gamma(Rad(TM))$-valued and $\Gamma(S(TM))$-valued bilinear functions and are known as second fundamental forms of distributions $S(TM)$ and $Rad(TM)$ respectively.
Then, by using (2.17), (2.18), (2.7),(2.9) and (2.10), we have the following results:

$$\bar{g}(h^l(X, PY), \xi) = g(A^*_\xi X, PY), \tag{2.19}$$

$$\bar{g}(h^l(X, PY), \xi) = g(A^*_\xi X, PY), \tag{2.20}$$

$$\bar{g}(h^*(X, PY), N) = g(A_N X, PY), \tag{2.21}$$

$$\bar{g}(h^*(X, \xi), \xi) = 0, \;\; A^*_\xi \xi = 0. \tag{2.22}$$

We have the following relation between the curvature tensor $\bar{R}$ and R [10]

$$\begin{aligned}\bar{R}(X,Y)Z =& R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z)\\ &- (\nabla_Y h^l)(X,Z) + (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) + D^l(X, h^s(Y,Z))\\ &+ D^l(Y, h^s(X,Z)) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)).\end{aligned} \tag{2.23}$$

Definition 2.1. [10] Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold with almost complex structure $\bar{J}$ of type $(1,1)$ tensor, Hermitian metric $\bar{g}$ and Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$, then for all $X, Y \in \Gamma(T\bar{M})$ we have:

$$\bar{J}^2 = -1, \;\; \bar{g}(\bar{J}X, \; \bar{J}Y) = \bar{g}(X, Y),$$

$$(\bar{\nabla}_X \bar{J})Y = \bar{\nabla}_X \bar{J}Y - \bar{J}\bar{\nabla}_X Y. \tag{2.24}$$

Definition 2.2. [2] An indefinite nearly Kähler manifold is an almost Hermitian manifold which satisfies the equation:

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}_Y \bar{J})X = 0 \tag{2.25}$$

for all $X, Y \in \Gamma(TM)$.

The curvature tensor field $\bar{R}$ of a nearly Kähler manifold $\bar{M}(c)$ with constant holomorphic curvature c is given by [2]

$$\bar{R}(X,Y,Z,W) = \frac{c}{4}\{\bar{g}(X,W)\bar{g}(Y,Z) - \bar{g}(X,Z)\bar{g}(Y,W) + \bar{g}(X,\bar{J}W)\bar{g}(Y,\bar{J}Z) - \bar{g}(X,\bar{J}Z)\bar{g}(Y,\bar{J}W)$$

(2.26)

$$-2\bar{g}(X,\bar{J}Y)\bar{g}(Z,\bar{J}W)\} + \frac{1}{4}\{\bar{g}\big((\bar{\nabla}_X\bar{J})(W),(\bar{\nabla}_Y\bar{J})(Z)\big) - \bar{g}\big((\bar{\nabla}_X\bar{J})Z,(\bar{\nabla}_Y\bar{J})W\big) - 2\bar{g}\big((\bar{\nabla}_X\bar{J})Y,(\bar{\nabla}_Z\bar{J})W\big)\}$$

and the sectional curvature is given by
(2.27)

$$\bar{R}(X,Y,X,Y) = \frac{c}{4}\{\bar{g}(X,Y)^2 - \bar{g}(X,X)\bar{g}(Y,Y) - 3\bar{g}(X,\bar{J}Y)^2\} - \frac{3}{4}\|(\bar{\nabla}_X\bar{J})(Y)\|.$$

A nearly Kähler manifold is said to be of constant type α(for detail see [2]), if there exist a real valued C^∞ function α on $\bar{M}$ such that

$$\|(\bar{\nabla}_X\bar{J})(Y)\| = \alpha\{g(X,X)^2g(Y,Y)^2 - g(X,Y)^2 - g(X,\bar{J}Y)^2\}. \tag{2.28}$$

3 Semi-transversal lightlike submanifolds

Definition 3.1. [18] Let $\bar{M}$ be an indefinite nearly Kähler manifold, then its lightlike submanifold M is said to be semi-transversal lightlike submanifold, if it satisfies the following conditions:

(i) $Rad(TM)$ is transversal with respect to $\bar{J}$.

(ii) There exists a real non-null distribution $D \subset S(TM)$ such that
$S(TM) = D \oplus D^\perp$, $\bar{J}D^\perp \subset S(TM^\perp)$, $\bar{J}(D) = D$,
where $D^\perp$ is orthogonal complementary to D in $S(TM)$.
Thus, we decompose TM as $TM = D \perp D^\perp$, where $D' = D^\perp \perp Rad(TM)$.

M is said to be proper semi-transversal lightlike submanifold if both D and $D^\perp$ are non-zero. The decomposition of $T\bar{M}$ is as follows:

$$T\bar{M} = D \oplus D' \oplus \bar{J}D' \perp \mu,$$

where μ is orthogonal part of $\bar{J}D^\perp$ in $S(TM^\perp)$.
Now, we state the existence theorem of semi-transversal lightlike submanifolds of an indefinite nearly Kähler manifold.

Let Q, P_1, P_2 and P be the projection morphisms from TM to D, $Rad(TM)$, $D^\perp$ and $S(TM)$ respectively. Then for any $Y \in TM$, we can decompose it in the below form:

$$Y = QY + P_1Y + P_2Y, \tag{3.1}$$

On applying $\bar{J}$ to (3.1), we obtain

$$\bar{J}Y = TY + wP_1Y + wP_2Y. \tag{3.2}$$

(3.2) can be rewrite as

$$\bar{J}(Y) = TY + wY, \tag{3.3}$$

where T and w are projection morphisms of $\bar{J}(Y)$ on TM and $tr(TM)$ respectively.

Put $wP_1 = w_1$ and $wP_2 = w_2$, we get

$$\bar{J}Y = TY + w_1 Y + w_2 Y, \tag{3.4}$$

where $TY \in \Gamma(D)$, $w_1 Y \in \Gamma(ltr(TM))$ and $w_2 Y \in \Gamma(\bar{J}(D^{\perp}))$. Similarly, for any $V \in \Gamma(tr(TM))$,

$$\bar{J}V = BV + CV, \tag{3.5}$$

where BV and CV are the sections of TM and $tr(TM)$, respectively. Differentiating (3.4) with respect to X and using (2.7), (2.8), (2.9), (2.10) and (3.5), we obtain

$$(\nabla_X T)Y + (\nabla_Y T)X = A_{w_1 Y}X + A_{w_2 Y}X + A_{w_1 X}Y + A_{w_2 Y}X + 2Bh(X,Y), \tag{3.6}$$

$$\begin{aligned} D^s(X, wP_1 Y) + D^s(Y, wP_1 X) = {} & -\nabla^s_X wP_2 Y - \nabla^s_Y wP_2 X + wP_2 \nabla_X Y + wP_2 \nabla_Y X \\ & - h^s(X, TY) - h^s(TX, Y) + 2Ch^s(X,Y), \end{aligned} \tag{3.7}$$

$$\begin{aligned} D^l(X, wP_2 Y) + D^l(Y, wP_2 X) = {} & -\nabla^l_X wP_1 Y - \nabla^l_Y wP_1 X + wP_1 \nabla_X Y + wP_1 \nabla_Y X \\ & - h^l(X, TY) - h^l(TX, Y), \end{aligned} \tag{3.8}$$

for any $X, Y \in \Gamma(TM)$.
Using (2.7), (2.8) in (2.25), we have the following lemma:

Lemma 3.1. *[18] Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$. Then, for any $X, Y \in \Gamma(TM)$, we have*

$$(\nabla_X T)Y + (\nabla_Y T)X = A_{wX}Y + A_{wY}X + 2Bh(X,Y) \tag{3.9}$$

$$(\nabla^t_X w)Y + (\nabla^t_Y w)X = 2Ch(X,Y) - h(X,TY) - h(TX,Y), \tag{3.10}$$

where

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (\nabla^t_X w)Y = \nabla^t_X wY - w\nabla_X Y. \tag{3.11}$$

Lemma 3.2. *[13] If $\bar{M}$ is a nearly Kähler manifold then, for any $X, Y \in \Gamma(T\bar{M})$,*

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}_{\bar{J}X}\bar{J})\bar{J}Y = 0, N(X,Y) = -4\bar{J}\big((\bar{\nabla}_X \bar{J})(Y)\big), \tag{3.12}$$

where $N(X,Y)$ is the Nijenhuis tensor given by

$$N(X,Y) = [\bar{J}X, \bar{J}Y] - \bar{J}[X, \bar{J}Y] - \bar{J}[\bar{J}X, Y] - [X,Y]. \tag{3.13}$$

Theorem 3.3. *[18] Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$. Then for any $X, Y \in \Gamma(D)$,*

(i) if D is integrable, then

$$h(X, \bar{J}Y) = h(\bar{J}X, Y), \tag{3.14}$$

(ii) if D defines totally geodesic foliation in M, then

$$h(X, \bar{J}Y) = h(\bar{J}X, Y) = \bar{J}h(X, Y). \tag{3.15}$$

Theorem 3.4. *Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$. Then the induced connection is a metric connection if and only if, for any $X \in \Gamma(TM)$, $Y \in \Gamma(Rad(TM))$, the following conditions hold:*

(i) $T(A_{\bar{J}Y}X - \nabla_{\bar{J}Y}X) = 0$

(ii) $B(D^s(X, \bar{J}Y) - h^s(\bar{J}Y, X)) = 0$

(iii) $\nabla_{\bar{J}Y}TX - A_{wX}\bar{J}Y \in \Gamma(Rad(TM))$

Proof. Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$. Then, for any $X \in \Gamma(TM)$, $Y \in \Gamma(Rad(TM))$, using the fact that $\bar{J}$ is almost complex structure of $\bar{M}$, we obtain

$$\bar{\nabla}_X Y = -\nabla_X \bar{J}^2 Y.$$

Now, using (2.24) and (2.25) in above equation, we get

$$\bar{\nabla}_X Y = -\bar{J}\{\nabla_X \bar{J}Y + \bar{\nabla}_{\bar{J}Y}X\} + \bar{\nabla}_{\bar{J}Y}\bar{J}X. \tag{3.16}$$

Using (2.7) and (2.8) in (3.16), we obtain

$$\begin{aligned} \nabla_X Y + h(X, Y) = &- \bar{J}\big(- A_{\bar{J}Y}X + \nabla^l_X \bar{J}Y + D^s(X, \bar{J}Y) + \nabla_{\bar{J}Y}X + h(\bar{J}Y, X)\big) - \\ &\nabla_{\bar{J}Y}TX + h(\bar{J}Y, TX) - A_{wX}\bar{J}Y + \nabla^{\perp}_{\bar{J}Y}wX. \end{aligned} \tag{3.17}$$

Now, by equating the tangential part for any $Y \in \Gamma(Rad(TM))$ in (3.17), we obtain

$$\begin{aligned} \nabla_X Y = &T(A_{\bar{J}Y}X) - \bar{J}(\nabla^l_X \bar{J}Y) - B(D^s(X, \bar{J}Y)) - T(\nabla_{\bar{J}Y}X) - B(h^l(\bar{J}Y, X)) - B(h^s(\bar{J}Y, X)) + \\ &\nabla_{\bar{J}Y}TX - A_{wX}\bar{J}Y. \end{aligned} \tag{3.18}$$

Thus from (3.18), $\nabla_X Y \in \Gamma(Rad(TM))$, if and only if $T(A_{\bar{J}Y}X - \nabla_{\bar{J}Y}X) = 0$, $B(D^s(X, \bar{J}Y) - h^s(\bar{J}Y, X)) = 0$ and $\nabla_{\bar{J}Y}TX - A_{wX}\bar{J}Y \in \Gamma(Rad(TM))$.
Which follow the assertions.

□

Definition 3.2. A semi-transversal lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$ is said to be D geodesic (respectively mixed geodesic) if its second fundamental form h satisfies $h(X, Y) = 0$ for any $X, Y \in \Gamma(D)$ (respectively, $X \in \Gamma(D)$ and $Y \in \Gamma(D')$)

Theorem 3.5. *Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$. If D defines a totally geodesic foliation in $\bar{M}$, then M is D geodesic.*

Proof. Let D defines a totally geodesic foliation in $\bar{M}$.
This implies, for any $X, Y \in \Gamma(D)$, $\bar{\nabla}_X Y \in \Gamma(D)$. Then using (2.7) and (2.8) for any $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, we get

$$\bar{g}(h^l(X,Y),\xi) = \bar{g}(\bar{\nabla}_X Y,\xi) = 0,$$
$$\bar{g}(h^s(X,Y),W) = \bar{g}(\bar{\nabla}_X Y,W) = 0.$$

Therefore $h^l(X,Y) = h^s(X,Y) = 0$, that is, $h(X,Y) = 0$.
Hence by definition 3.1., we get M is D geodesic. □

Theorem 3.6. *Let M be a mixed geodesic proper semi-transversal lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}(c)$ of constant type α with constant holomorphic sectional curvature c. If the distribution D defines totally geodesic foliation in $\bar{M}$, then it is necessary that $c = \alpha$.*

Proof. For any $X, Y \in \Gamma(\bar{M})$, using (2.24), we obtain,

$$(\bar{\nabla}_X \bar{J})(\bar{J}Z) = \bar{\nabla}_X \bar{J}^2 Z - \bar{J}\bar{\nabla}_X \bar{J}Z.$$

Since $\bar{J}$ is almost complex structure, above equation can be written as,

$$(\bar{\nabla}_X \bar{J})(\bar{J}Z) = -\bar{J}^2\bar{\nabla}_X \bar{J}^2 Z - \bar{J}\bar{\nabla}_X \bar{J}Z,$$

which implies,

$$(\bar{\nabla}_X \bar{J})(\bar{J}Z) = -\bar{J}(\bar{\nabla}_X \bar{J})(Z). \tag{3.19}$$

For any $X \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$, using (2.26), (3.12) and (3.19), we obtain

$$\bar{g}(\bar{R}(X,\bar{J}X)Z,\bar{J}Z) = -\frac{c}{2}g(X,X)g(Z,Z) + \frac{1}{2}\|(\bar{\nabla}_X \bar{J})(Z)\|^2 \tag{3.20}$$

Furthermore, for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, assuming M is mixed geodesic and using (2.23), we get

$$\bar{g}(\bar{R}(X,\bar{J}X)Z,\bar{J}Z) = \bar{g}\big((\nabla_X h^s)(\bar{J}X,Z) - (\nabla_{\bar{J}X} h^s)(X,Z),\bar{J}Z\big), \tag{3.21}$$

where

$$(\nabla_X h^s)(\bar{J}X,Z) = -h^s(\nabla_X \bar{J}X,Z) - h^s(\bar{J}X,\nabla_X Z) \tag{3.22}$$

and

$$(\nabla_{\bar{J}X} h^s)(X,Z) = -h^s(\nabla_{\bar{J}X}\bar{J}X,Z) - h^s(X,\nabla_{\bar{J}X} Z). \tag{3.23}$$

From (3.22) and (3.23), we obtain

$$(\nabla_X h^s)(\bar{J}X,Z) - (\nabla_{\bar{J}X} h^s)(X,Z) = h^s([\bar{J}X,X],Z) - h^s(\bar{J}X,\nabla_X Z) + h^s(X,\nabla_{\bar{J}X} Z). \tag{3.24}$$

For any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$, using the fact that D defines a totally geodesic foliation in $\bar{M}$, we obtain

$$g(T\nabla_X Z, Y) = -g(\nabla_X Z, TY) = -g(\bar{\nabla}_X Z, TY) = g(Z, \bar{\nabla}_X TY) = 0.$$

Since D is non-degenerate, from above equation we obtain, $\nabla_X Z \in \Gamma(D^{'})$. Thus (3.24) reduces to

$$(\nabla_X h^s)(\bar{J}X, Z) - (\nabla_{\bar{J}X} h^s)(X, Z) = 0.$$

From (3.20) and (3.21), we have

$$cg(X, X)g(Z, Z) = \|\bar{\nabla}_X \bar{J})(Z)\|^2. \tag{3.25}$$

Since $\bar{M}$ is of constant type α, from (2.28) and (3.25), we obtain

$$cg(X, X)g(Z, Z) = \alpha\{g(X, X)g(Z, Z) - g(X, Z)^2 - g(X, \bar{J}Z)^2\}.$$

Therefore, for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, we get $(c - \alpha)g(X, X)g(Z, Z) = 0$. Since D and $D^{\perp}$ are non-degenerate distributions, hence $c = \alpha$. □

4 Totally umbilical semi-transversal lightlike submanifolds

Definition 4.1. A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M such that, for $X, Y \in \Gamma(TM)$

$$h(X, Y) = H\bar{g}(X, Y). \tag{4.1}$$

From (2.9) and (2.10), it is clear that M is totally umbilical, if and only if, on each coordinate neighborhood u there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$ such that

$$h^l(X, Y) = H^l\bar{g}(X, Y), \quad h^s(X, Y) = H^s\bar{g}(X, Y), D^l(X, W) = 0 \tag{4.2}$$

for $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$. M is called totally geodesic if $H = 0$, that is, $h(X, Y) = 0$.

Theorem 4.1. *Let M be a totally umbilical proper semi-transversal lightlike submanifold of an indefinite nearly Kähler $\bar{M}$. If D defines a totally geodesic foliation in M, then the induced connection ∇ is a metric connection. Moreover, $h^s = 0$.*

Proof. Let $X, Y \in \Gamma(D)$. From (3.7) and (3.8), we obtain

$$h^l(X, \bar{J}Y) + h^l(\bar{J}X, Y) = 0.$$

$$h^s(X, TY) + h^s(TX, Y) = 2Ch^s(X, Y),$$

Since D defines a totally geodesic foliation in M, using Theorem (3.2) in above equation, we obtain

$$h^s(X, \bar{J}Y) = Ch^s(X, Y). \tag{4.3}$$

$$h^l(X, \bar{J}Y) = 0. \tag{4.4}$$

Since M is totally umbilical semi-transversal lightlike submanifold, (4.3) reduces to

$$H^s g(X, \bar{J}Y) = CH^s g(X, Y).$$

By putting $X = \bar{J}Y$ in above equation, we obtain

$$H^s g(\bar{J}Y, \bar{J}Y) = CH^s g(\bar{J}Y, Y) = 0.$$

Since D is non-degenerate, from above equation, we get $H^s = 0$. Using $H^s = 0$ in (4.2), we obtain $h^s = 0$.
Similarly, from (4.4) we obtain, $H^l g(X, \bar{J}Y) = 0$. By putting $X = \bar{J}Y$ in $H^l g(X, \bar{J}Y) = 0$ we obtain,

$$H^l g(\bar{J}Y, \bar{J}Y) = 0.$$

Since D is non-degenerate, from above equation, we get $H^l = 0$. Using $H^l = 0$ in (4.2), we obtain $h^l = 0$. Hence from (2.15), the induced connection ∇ is a metric connection. □

Lemma 4.2. *Let M be a totally umbilical semi-transversal lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$. Then, for any $X \in \Gamma(D)$, $\nabla_X X \in \Gamma(D)$.*

Proof. Let M be a totally umbilical semi-transversal lightlike submanifold of an indefinite nearly Kähler manifold. Now $\nabla_X X \in \Gamma(D)$, if and only if, $g(\nabla_X X, \bar{J}\xi) = 0$ and $g(\nabla_X X, \bar{J}W) = 0$, for any $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(\bar{J}(D^\perp))$.
For any $X \in \Gamma(D)$ and $\xi \in \Gamma(Rad(TM))$, we have

$$\begin{aligned} g(\nabla_X X, \bar{J}\xi) &= -\bar{g}(\nabla_X \bar{J}X, \xi) \\ &= -\bar{g}(\nabla_X \bar{J}X + h^l(X, \bar{J}X) + h^s(X, \bar{J}X), \xi) \\ &= -\bar{g}(h^l(X, \bar{J}(X)), \xi) \\ &= -\bar{g}(H^l, \xi) g(X, \bar{J}X) = 0. \end{aligned}$$

Similarly, for any $X \in \Gamma(D)$ and $W \in \Gamma(\bar{J}(D^\perp))$, we have

$$\begin{aligned} g(\nabla_X X, \bar{J}W) &= -\bar{g}(\nabla_X \bar{J}X, W) \\ &= -\bar{g}(\nabla_X \bar{J}X + h^l(X, \bar{J}X) + h^s(X, \bar{J}X), W) \\ &= -\bar{g}(h^s(X, \bar{J}X), W) \\ &= -\bar{g}(H^s, W) g(X, \bar{J}X) = 0. \end{aligned}$$

This implies $\nabla_X X \in \Gamma(D)$.

□

Theorem 4.3. *Let M be a locally umbilical proper semi-transversal lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}$. Then one of the following holds:*

(i) M is totally geodesic, if D defines a totally geodesic foliation in M.

(ii) $h^s = 0$ or $dim(\bar{J}(D^{\perp})) = 1$, if D does not define a totally geodesic foliation in M.

Proof. Let D defines a totally geodesic foliation in M. Then from Theorem (4.1), we obtain $h^l = h^s = 0$. Hence M is totally geodesic.
On the other hand, suppose D does not define totally geodesic foliation in M, using (2.7), (2.8), (2.9), (2.10), (3.3), (3.5) in $(\nabla_Z \bar{J})W + (\nabla_W \bar{J})Z = 0$, we get

$$-A_{\bar{J}W}Z+\nabla^s_X(\bar{J}Z)+D^l(X,\bar{J}Z)-A_{\bar{J}Z}W+\nabla^s_Z(\bar{J}X)+D^l(Z,\bar{J}X) = \bar{J}(\nabla_Z W+\nabla_W Z+h(Z,W)+h(W,Z))$$

$$\begin{aligned} -A_{\bar{J}W}Z + \nabla^s_X(\bar{J}Z) + D^l(X,\bar{J}Z) - A_{\bar{J}Z}W + \nabla^s_Z(\bar{J}X) + D^l(Z,\bar{J}X) = T(\nabla_Z W + \nabla_W Z) + \\ w(\nabla_Z W + \nabla_W Z) + B(h(Z,W) + h(Z,W)) + C(h(Z,W) + h(Z,W)) \end{aligned} \tag{4.5}$$

Now by equating tangential part in (4.5), we obtain

$$-A_{\bar{J}W}Z - A_{\bar{J}Z}W = T\nabla_Z W + T\nabla_W Z + B(h(Z,W) + h(Z,W)),$$

for any $Z, W \in \Gamma(D^{\perp})$.
Taking inner product of above equation with Z and using (2.11) and (3.5), we get

$$\bar{g}(h^s(Z,Z), \bar{J}W) = \bar{g}(h^s(Z,W), \bar{J}Z). \tag{4.6}$$

$$\bar{g}(h^s(Z,Z), \bar{J}W) = \bar{g}(h^s(Z,W), \bar{J}Z). \tag{4.7}$$

Since M is totally umbilical, (4.7) reduces to

$$\bar{g}(H^s, \bar{J}W)g(Z,Z) = \bar{g}(H^s, \bar{J}Z)g(Z,W). \tag{4.8}$$

Now by interchanging role of Z and W, we get

$$\bar{g}(H^s, \bar{J}Z)g(W,W) = \bar{g}(H^s, \bar{J}W)g(Z,W). \tag{4.9}$$

Thus from (4.8) and (4.9), we obtain

$$\bar{g}(H^s, \bar{J}Z) = \frac{g(Z,W)^2}{g(W,W)g(Z,Z)}\bar{g}(H^s, \bar{J}Z), \tag{4.10}$$

which implies,

$$\bar{g}(H^s, \bar{J}Z)\Big(1 - \frac{g(Z,W)^2}{g(W,W)g(Z,Z)}\Big) = 0. \tag{4.11}$$

Using Lemma 4.1. in (3.7), for any $X \in \Gamma(D)$, we obtain

$$h^s(X, \bar{J}X) = Ch^s(X,X).$$

Since M is totally umbilic, we get $g(X,X)CH^s = 0$. Then, the non-degeneracy of D implies that $CH^s = 0$, that is, $H^s \in \bar{J}D^{\perp}$.
Since $D^{\perp}$ is also non degenerate, that is, we can choose two non-zero vector fields Z and W in (4.11), we obtain that either $H^s = 0$ or Z and W are linearly dependent.
□

Theorem 4.4. *There does not exist any totally umbilical proper semi-transversal lightlike submanifold of an indefinite nearly Kähler manifold $\bar{M}(c)$ of constant type α with constant holomorphic sectional curvature c, such that $c \neq \alpha$.*

Proof. Let M be a totally umbilical semi-transversal lightlike submanifold of $\bar{M}(c)$ such that $c \neq \alpha$. Then, for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$, equating (3.20) and (3.21) we obtain,

$$-\frac{c}{2}g(X,X)g(Y,Y) + \frac{1}{2}\|(\bar{\nabla}_X \bar{J})(Y)\|^2 = \bar{g}\big((\nabla_X h^s)(\bar{J}X,Y) - (\nabla_{\bar{J}X} h^s)(X,Y), \bar{J}Y\big) \tag{4.12}$$

Using conditions of totally umbilical submanifold M in (4.2), we obtain

$$\big(\nabla_X h^s)(\bar{J}X,Y) = -g(\nabla_X \bar{J}X,Y)H^s - g(\bar{J}X,\nabla_X Y)H^s. \tag{4.13}$$

Since, for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$, $\bar{g}(\bar{J}X,Y) = 0$, differentiating $\bar{g}(\bar{J}X,Y) = 0$ with respect to X, we get

$$g(\nabla_X \bar{J}X,Y) = -g(\bar{J}X,\nabla_X Y). \tag{4.14}$$

From (4.13) and (4.14), we obtain

$$(\nabla_X h^s)(\bar{J}X,Y) = 0. \tag{4.15}$$

Similarly, we obtain

$$(\nabla_{\bar{J}X} h^s)(X,Y) = 0. \tag{4.16}$$

Using (4.15) and (4.16) in (4.12), we obtain

$$\frac{c}{2}g(X,X)g(Y,Y) = \frac{1}{2}\|(\bar{\nabla}_X \bar{J})(Y)\|^2. \tag{4.17}$$

Using (2.28) in (4.17), we obtain

$$cg(X,X)g(Y,Y) = \alpha\{g(X,X)g(Y,Y) - g(X,Y)^2 - g(X,\bar{J}Y)^2\}.$$

Therefore, for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$, $(c-\alpha)g(X,X)g(Y,Y) = 0$. Since D and $D^{\perp}$ are non-degenerate which implies $(c-\alpha) = 0$, that is, $c = \alpha$. This is a contradiction to our assumption. □

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