Spin 1/2 particle with two mass parameters in external Coulomb field

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Abstract. The generalized wave equation for a spin 1/2 particle with two mass parameters is studied in presence of the external Coulomb field. After separating the variables the problem reduces the system of 8 differential equations of the first order. Taking into account the diagonalization of the space reflection operator, we derive two independent systems of 4 equations, referring to states of opposite parity. In each case, we derive two systems of linked 2-nd order equations, referring to states with different parities. They leads to 4-th order differential equations for separate functions. Their Frobenius type solutions are constructed, which involve power series with 13-terms recurrent relations. Two solutions are appropriate to describe bound states. As quantization rule, we apply the known transcendency condition; in this way we derive two analytical formulas for energy spectra. They are similar to relativistic spectra for ordinary spin 1/2 particle, but being governed by the masses M_1 and M_2 respectively.

M.S.C. 2020: 33E30, 34B30.
Key words: Spin 1/2 particle; two mass parameters; Coulomb field; spatial reflection; separation of the variables; solutions of Frobenius type; bound states; energy spectra.

1 Introduction

In [1]-[4], the wave equation for spin 1/2 particle with two mass parameters was proposed. In absence of external electromagnetic fields, this equation splits into two unlinked Dirac-like equations with different masses, M_1 and M_2. In presence of external fields, there arises a complicated wave equation, in which two bispinor components are joined in a unified system. There were found exact solutions for this equation in presence of the uniform magnetic field.

The system under consideration for two bispinors $\Psi_1(x), \Psi_2(x)$ has the structure

$$\begin{aligned}&\{\gamma^\alpha[i(\partial_\alpha+\Gamma_\alpha)-eA_\alpha]-M_1+b\Lambda_1\Sigma(x)\}\,\Psi_1(x)-a\Lambda_1\Sigma(x)\Psi_2(x)=0\ ,\\&\{\gamma^\alpha[i(\partial_\alpha+\Gamma_\alpha)-eA_\alpha]-M_2-a\Lambda_2\Sigma(x)\}\,\Psi_2(x)+b\Lambda_2\Sigma(x)\Psi_1(x)=0\ ,\end{aligned} \tag{1.1}$$

BSG Proceedings 29. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2022, 26-29 August 2021, Mangalia-Romania, pp. 74-86.

where

$$\gamma^{\alpha}(x) = e^{\alpha}_{(b)}\gamma^{b}, \quad \Sigma(x) = -ieF_{\alpha\beta}\sigma^{\alpha\beta}(x) \, , \tag{1.2}$$
$$\sigma^{\alpha\beta}(x) = \frac{\gamma^{\alpha}(x)\gamma^{\beta}(x) - \gamma^{\beta}(x)\gamma^{\alpha}(x)}{4} \, .$$

In eq. (1.1), the following parameters are used (note that $\gamma \in [0, \pi/2]$):

$$M_1 = \frac{M}{(1+\cos\gamma)/2}, \quad M_2 = \frac{M}{(1-\cos\gamma)/2} \, ,$$
$$a = \frac{1}{2}\frac{1}{M} \left(\, 4 - 3\sqrt{1 + (1/3)\sin^2\gamma} - \cos\gamma \, \right),$$
$$b = \frac{1}{2}\frac{1}{M} \left(\, 4 - 3\sqrt{1 + (1/3)\sin^2\gamma} + \cos\gamma \, \right),$$
$$\Lambda_1 = (1 + \sqrt{1 + (1/3)\sin^2\gamma}\,) \, \frac{\cos\gamma - \sqrt{1 + (1/3)\sin^2\gamma}}{\cos\gamma(1+\cos\gamma)} \, , \tag{1.3}$$
$$\Lambda_2 = (1 + \sqrt{1 + (1/3)\sin^2\gamma}\,) \, \frac{\cos\gamma + \sqrt{1 + (1/3)\sin^2\gamma}}{\cos\gamma(1-\cos\gamma)} \, ,$$

and the parameter M - with dimension of inverse length - is arbitrary.

2 Separating the variables

We further consider this equation in presence of external Coulomb field (assuming the use of spherical tetrads)

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2, \quad x^{\alpha} = (t, r, \theta, \phi) \, , \tag{2.1}$$
$$e^{\alpha}_{(3)} = (0,1,0,0), \quad e^{\alpha}_{(1)} = (0,0,\frac{1}{r},0), \quad e^{\alpha}_{(2)} = (0,0,0,\frac{1}{r\sin\theta}) \, ,$$

allowing for the notations

$$A_t = -\frac{e}{r}, \quad F_{tr} = -\partial_r A_0 = -\frac{e}{r^2}, \quad \Sigma(x) = i\frac{e^2}{r^2}\gamma^0\gamma^3,$$

$$a\Lambda_1 e^2 = \alpha_1, \quad a\Lambda_2 e^2 = \alpha_2, \quad b\Lambda_1 e^2 = \beta_1, \quad b\Lambda_2 e^2 = \beta_2 \, .$$

The system (1.1) takes on the form

$$\left[\gamma^0(i\partial_t - \frac{\alpha}{r}) + i\gamma^3\partial_r + \frac{1}{r}\Sigma_{\theta\phi} - M_1 + i\frac{\beta_1}{r^2}\gamma^0\gamma^3 \right] \Psi_1 - i\frac{\alpha_1}{r^2}\gamma^0\gamma^3\Psi_2 = 0 \, , \tag{2.2}$$
$$\left[\gamma^0(i\partial_t - \frac{\alpha}{r}) + i\gamma^3\partial_r + \frac{1}{r}\Sigma_{\theta\phi} - M_2 - i\frac{\alpha_2}{r^2}\gamma^0\gamma^3 \right] \Psi_2 + i\frac{\beta_2}{r^2}\gamma^0\gamma^3\Psi_1 = 0 \, ,$$

where the angular term $\Sigma_{\theta,\phi}$ is given by

$$\Sigma_{\theta,\phi} = i\gamma^1\partial_\theta + \gamma^2\frac{i\partial_\phi + i\sigma^{12}\cos\theta}{\sin\theta} \, .$$

Below we shall have in mind the following expressions for the involved parameters:

$$
\alpha_1 = -e^2 \frac{1}{3} \frac{(1-\cos\gamma)\left(-\cos\gamma\sqrt{12-3\cos^2\gamma}+\cos^2\gamma+2\right)}{M\cos\gamma\,(1+\cos\gamma)}\,,
$$

$$
\alpha_2 = e^2\,\frac{2}{3}\,\frac{\sin^2\gamma}{M\,\cos\gamma}\,,\quad \beta_1 = -e^2\,\frac{2}{3}\,\frac{\sin^2\gamma}{M\,\cos\gamma} < 0\,, \tag{2.3}
$$

$$
\beta_2 = -\frac{1}{3}\,\frac{e^2\,(1+\cos\gamma)\left(\cos\gamma\sqrt{12-3\cos^2\gamma}+\cos^2\gamma+2\right)}{M\cos\gamma\,(\cos\gamma-1)} > 0\,,
$$

subject to the identities

$$
\alpha_2 = -\beta_1\,,\qquad \alpha_1\beta_2 = -{\beta_1}^2\,. \tag{2.4}
$$

We search for solutions with quantum numbers ϵ, j, m (assuming that the diagonalization of the operators $i\partial_t, \vec{J}^2, J_3$; $D^j_{-m,\sigma}(\phi,\theta,o)$ stands for Wigner functions, for $j = 1/2, 3/2, ...; m = -j, ..., +j$):

$$
\Psi_1(x) = \frac{e^{-i\epsilon t}}{r}\left|\begin{array}{c} f_1(r)D_{-1/2} \\ f_2(r)D_{+1/2} \\ f_3(r)D_{-1/2} \\ f_4(r)D_{+1/2} \end{array}\right|,\quad \Psi_2(x) = \frac{e^{-i\epsilon t}}{r}\left|\begin{array}{c} g_1(r)D_{-1/2} \\ g_2(r)D_{+1/2} \\ g_3(r)D_{-1/2} \\ g_4(r)D_{+1/2} \end{array}\right|. \tag{2.5}
$$

Using Dirac matrices in spinor basis, we derive 8 radial equations

$$
\begin{aligned}
&\left(\epsilon+\frac{\alpha}{r}\right) f_3 - i\frac{d}{dr}f_3 - i\frac{\nu}{r}f_4 - M_1 f_1 + \frac{i\beta_1}{r^2}f_1 - \frac{i\alpha_1}{r^2}g_1 = 0\,,\\
&\left(\epsilon+\frac{\alpha}{r}\right) f_4 + i\frac{d}{dr}f_4 + i\frac{\nu}{r}f_3 - M_1 f_2 - \frac{i\beta_1}{r^2}f_2 + \frac{i\alpha_1}{r^2}g_2 = 0\,,\\
&\left(\epsilon+\frac{\alpha}{r}\right) f_1 + i\frac{d}{dr}f_1 + i\frac{\nu}{r}f_2 - M_1 \delta f_2 - \frac{i\beta_1}{r^2}f_3 + \frac{i\alpha_1}{r^2}g_3 = 0\,,\\
&\left(\epsilon+\frac{\alpha}{r}\right) f_2 - i\frac{d}{dr}f_2 - i\frac{\nu}{r}f_1 - M_1 f_4 + \frac{i\beta_1}{r^2}f_4 - \frac{i\alpha_1}{r^2}g_4 = 0\,;
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
&\left(\epsilon+\frac{\alpha}{r}\right) g_3 - i\frac{d}{dr}g_3 - i\frac{\nu}{r}g_4 - M_2 g_1 - \frac{i\alpha_2}{r^2}g_1 + \frac{i\beta_2}{r^2}f_1 = 0\,,\\
&\left(\epsilon+\frac{\alpha}{r}\right) g_4 + i\frac{d}{dr}g_4 + i\frac{\nu}{r}g_3 - M_2 g_2 + \frac{i\alpha_2}{r^2}g_2 - \frac{i\beta_2}{r^2}f_2 = 0\,,\\
&\left(\epsilon+\frac{\alpha}{r}\right) g_1 + i\frac{d}{dr}g_1 + i\frac{\nu}{r}g_2 - M_2 g_3 + \frac{i\alpha_2}{r^2}g_3 - \frac{i\beta_2}{r^2}\delta f_2 = 0\,,\\
&\left(\epsilon+\frac{\alpha}{r}\right) g_2 - i\frac{d}{dr}g_2 - i\frac{\nu}{r}g_1 - M_2 g_4 - \frac{i\alpha_2}{r^2}g_4 + \frac{i\beta_2}{r^2}f_4 = 0\,,
\end{aligned} \tag{2.7}
$$

where $\nu = j + 1/2$. The system (2.6)–(2.7) is consistent with the constraints (which follow from the diagonalization of the spacial reflection operator)

$$
f_3 = \delta f_2\,,\quad f_4 = \delta f_1\,,\quad \delta = \pm 1,\quad g_3 = \delta g_2\,,\quad g_4 = \delta g_1\,,\quad \delta = \pm 1\,. \tag{2.8}
$$

In this way we derive

$$\begin{aligned}
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,\delta f_2 - i\frac{d}{dr}\delta f_2 - i\frac{\nu}{r}\delta f_1 - M_1 f_1 + \frac{i\beta_1}{r^2}f_1 - \frac{i\alpha_1}{r^2}g_1 = 0\,,\\
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,\delta f_1 + i\frac{d}{dr}\delta f_1 + i\frac{\nu}{r}\delta f_2 - M_1 f_2 - \frac{i\beta_1}{r^2}f_2 + \frac{i\alpha_1}{r^2}g_2 = 0\,,\\
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,f_1 + i\frac{d}{dr}f_1 + i\frac{\nu}{r}f_2 - M_1\delta f_2 - \frac{i\beta_1}{r^2}\delta f_2 + \frac{i\alpha_1}{r^2}\delta g_2 = 0\,,\\
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,f_2 - i\frac{d}{dr}f_2 - i\frac{\nu}{r}f_1 - M_1\delta f_1 + \frac{i\beta_1}{r^2}\delta f_1 - \frac{i\alpha_1}{r^2}\delta g_1 = 0\,;\\
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,\delta g_2 - i\frac{d}{dr}\delta g_2 - i\frac{\nu}{r}\delta g_1 - M_2 g_1 - \frac{i\alpha_2}{r^2}g_1 + \frac{i\beta_2}{r^2}f_1 = 0\,,\\
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,\delta g_1 + i\frac{d}{dr}\delta g_1 + i\frac{\nu}{r}\delta g_2 - M_2 g_2 + \frac{i\alpha_2}{r^2}g_2 - \frac{i\beta_2}{r^2}f_2 = 0\,,\\
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,g_1 + i\frac{d}{dr}g_1 + i\frac{\nu}{r}g_2 - M_2\delta g_2 + \frac{i\alpha_2}{r^2}\delta g_2 - \frac{i\beta_2}{r^2}\delta f_2 = 0\,,\\
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,g_2 - i\frac{d}{dr}g_2 - i\frac{\nu}{r}g_1 - M_2\delta g_1 - \frac{i\alpha_2}{r^2}\delta g_1 + \frac{i\beta_2}{r^2}\delta f_1 = 0;
\end{aligned}$$

where only 4 equations are different:

$$\begin{aligned}
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,f_1 + i\frac{d}{dr}f_1 + i\frac{\nu}{r}f_2 - M_1\delta f_2 - \frac{i\beta_1}{r^2}\delta f_2 + \frac{i\alpha_1}{r^2}\delta g_2 = 0\,,\\
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,f_2 - i\frac{d}{dr}f_2 - i\frac{\nu}{r}f_1 - M_1\delta f_1 + \frac{i\beta_1}{r^2}\delta f_1 - \frac{i\alpha_1}{r^2}\delta g_1 = 0\,,\\
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,g_1 + i\frac{d}{dr}g_1 + i\frac{\nu}{r}g_2 - M_2\delta g_2 + \frac{i\alpha_2}{r^2}\delta g_2 - \frac{i\beta_2}{r^2}\delta f_2 = 0\,,\\
&\Big(\epsilon+\frac{\alpha}{r}\Big)\,g_2 - i\frac{d}{dr}g_2 - i\frac{\nu}{r}g_1 - M_2\delta g_1 - \frac{i\alpha_2}{r^2}\delta g_1 + \frac{i\beta_2}{r^2}\delta f_1 = 0\,.
\end{aligned} \tag{2.9}$$

It is convenient to use the new combinations of functions

$$f = (f_2+f_1), \quad F = i(f_2-f_1)\,; \quad g = (g_2+g_1), \quad G = i(g_2-g_1)\,,$$

which lead to

$$\begin{aligned}
&\Big(\frac{d}{dr} - \frac{\nu}{r} + \delta\frac{\beta_1}{r^2}\Big)\,F - \Big(\epsilon+\frac{\alpha}{r} - \delta M_1\Big)\,f - \delta\frac{\alpha_1}{r^2}G = 0\,,\\
&\Big(\frac{d}{dr} + \frac{\nu}{r} - \delta\frac{\beta_1}{r^2}\Big)\,f + \Big(\epsilon+\frac{\alpha}{r} + \delta M_1\Big)\,F + \delta\frac{\alpha_1}{r^2}g = 0\,,\\
&\Big(\frac{d}{dr} - \frac{\nu}{r} - \delta\frac{\alpha_2}{r^2}\Big)\,G - \Big(\epsilon+\frac{\alpha}{r} - \delta M_2\Big)\,g + \delta\frac{\beta_2}{r^2}F = 0\,,\\
&\Big(\frac{d}{dr} + \frac{\nu}{r} + \delta\frac{\alpha_2}{r^2}\Big)\,g + \Big(\epsilon+\frac{\alpha}{r} + \delta M_2\Big)\,G - \delta\frac{\beta_2}{r^2}f = 0\,.
\end{aligned} \tag{2.10}$$

We shall separately study the cases of different parities

$$\begin{aligned}
\delta = +1, \quad &\Big(\frac{d}{dr} - \frac{\nu}{r} + \frac{\beta_1}{r^2}\Big)\,F - \Big(\epsilon+\frac{\alpha}{r} - M_1\Big)\,f - \frac{\alpha_1}{r^2}G = 0\,,\\
&\Big(\frac{d}{dr} + \frac{\nu}{r} - \frac{\beta_1}{r^2}\Big)\,f + \Big(\epsilon+\frac{\alpha}{r} + M_1\Big)\,F + \frac{\alpha_1}{r^2}g = 0\,,\\
&\Big(\frac{d}{dr} - \frac{\nu}{r} - \frac{\alpha_2}{r^2}\Big)\,G - \Big(\epsilon+\frac{\alpha}{r} - M_2\Big)\,g + \frac{\beta_2}{r^2}F = 0\,,\\
&\Big(\frac{d}{dr} + \frac{\nu}{r} + \frac{\alpha_2}{r^2}\Big)\,g + \Big(\epsilon+\frac{\alpha}{r} + M_2\Big)\,G - \frac{\beta_2}{r^2}f = 0\,;
\end{aligned} \tag{2.11}$$

$$
(2.12)\qquad
\begin{aligned}
\delta = -1, \quad & \Big(\frac{d}{dr} - \frac{\nu}{r} - \frac{\beta_1}{r^2} \Big) F - \Big(\epsilon + \frac{\alpha}{r} + M_1 \Big) f + \frac{\alpha_1}{r^2} G = 0 \,, \\
& \Big(\frac{d}{dr} + \frac{\nu}{r} + \frac{\beta_1}{r^2} \Big) f + \Big(\epsilon + \frac{\alpha}{r} - M_1 \Big) F - \frac{\alpha_1}{r^2} g = 0 \,, \\
& \Big(\frac{d}{dr} - \frac{\nu}{r} + \frac{\alpha_2}{r^2} \Big) G - \Big(\epsilon + \frac{\alpha}{r} + M_2 \Big) g - \frac{\beta_2}{r^2} F = 0 \,, \\
& \Big(\frac{d}{dr} + \frac{\nu}{r} - \frac{\alpha_2}{r^2} \Big) g + \Big(\epsilon + \frac{\alpha}{r} - M_2 \Big) G + \frac{\beta_2}{r^2} f = 0 \,.
\end{aligned}
$$

The systems for states with different parities mutually relate by symmetry:

$$
(2.13)\qquad M_1, M_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \quad \Longrightarrow \quad -M_1, -M_2, -\alpha_1, -\alpha_2, -\beta_1, -\beta_2 \,.
$$

It should be noted that, far from the origin $r = 0$ and neglecting the terms r^{-2}, we obtain much more simple equations:

$$
(2.14)\qquad
\begin{aligned}
\delta = +1, \quad & \Big(\frac{d}{dr} - \frac{\nu}{r} \Big) F - \Big(\epsilon + \frac{\alpha}{r} - M_1 \Big) f = 0 \,, \\
& \Big(\frac{d}{dr} + \frac{\nu}{r} \Big) f + \Big(\epsilon + \frac{\alpha}{r} + M_1 \Big) F = 0 \,, \\
& \Big(\frac{d}{dr} - \frac{\nu}{r} \Big) G - \Big(\epsilon + \frac{\alpha}{r} - M_2 \Big) g = 0 \,, \\
& \Big(\frac{d}{dr} + \frac{\nu}{r} \Big) g + \Big(\epsilon + \frac{\alpha}{r} + M_2 \Big) G = 0 \,;
\end{aligned}
$$

$$
(2.15)\qquad
\begin{aligned}
\delta = -1, \quad & \Big(\frac{d}{dr} - \frac{\nu}{r} \Big) F - \Big(\epsilon + \frac{\alpha}{r} + M_1 \Big) f = 0 \,, \\
& \Big(\frac{d}{dr} + \frac{\nu}{r} \Big) f + \Big(\epsilon + \frac{\alpha}{r} - M_1 \Big) F = 0 \,, \\
& \Big(\frac{d}{dr} - \frac{\nu}{r} \Big) G - \Big(\epsilon + \frac{\alpha}{r} + M_2 \Big) g = 0 \,, \\
& \Big(\frac{d}{dr} + \frac{\nu}{r} \Big) g + \Big(\epsilon + \frac{\alpha}{r} - M_2 \Big) G = 0 \,.
\end{aligned}
$$

These subsystems coincide with those for ordinary Dirac particles with masses M_1 and M_2. This means that far from the origin we observe two ordinary particles with fixed masses. Also, this asymptotic behavior assumes that from analyzing the exact equations in the whole region of the variable $r \in (0, +\infty)$, we may expect the same energy spectra as for ordinary particles with masses M_1 and M_2.

3 Solving the radial equations

It suffices to follow in detail only the case of parity $\delta = +1$. With the help of the first two equations in (2.11), we can exclude the variables $G(r)$ and $g(r)$; in this way we derive the system of 2-nd order for the functions f and F. Since their explicit form is rather complicated, we write down only their general structure (we temporarily use the notation $M_1 - M_2 = M$):

$$\left[\frac{d^2}{dr^2} + \left(\frac{a_1}{r} + \frac{a_2}{r^2} \right) \frac{d}{dr} + b + \frac{b_1}{r} + ... \frac{b_4}{r^4} \right] f + \left[M \frac{d}{dr} + \frac{C_1}{r} + \frac{C_2}{r^2} + \frac{C_3}{r^3} \right] F = 0 \, , \tag{3.1}$$

$$\left[\frac{d^2}{dr^2} + \left(\frac{A_1}{r} + \frac{A_2}{r^2} \right) \frac{d}{dr} + B + \frac{B_1}{r} + ... \frac{B_4}{r^4} \right] F + \left[M \frac{d}{dr} f + \frac{c_1}{r} + \frac{c_2}{r^2} + \frac{c_3}{r^3} \right] f = 0 \, , \tag{3.2}$$

while the remaining functions $g(r)$ and $G(r)$ can be expressed in terms of $f(r)$ and $F(r)$.

Let us explain the method of deriving 4-th order equations from equations (3.1)–(3.2). First, in order to simplify the structure of equations, we introduce special multipliers at F and f:

$$F = \Phi \bar{F}, \quad \left(M \frac{d}{dr} + \frac{C_1}{r} + \frac{C_2}{r^2} + \frac{C_3}{r^3} \right) \Phi \bar{F} = \Phi M \frac{d}{dr} \bar{F}, \quad \Phi = x^{\frac{-C_1}{M}} e^{\frac{C_2}{Mx}} e^{\frac{C_3}{2Mx^2}} ,$$

$$f = \varphi \bar{f}, \quad \left(M \frac{d}{dr} + \frac{c_1}{r} + \frac{c_2}{r^2} + \frac{c_3}{r^3} \right) \varphi \bar{f} = \varphi M \frac{d}{dr} \bar{f}, \quad \varphi = x^{\frac{-c_1}{M}} e^{\frac{c_2}{Mx}} e^{\frac{c_3}{2Mx^2}} \, .$$

Correspondingly, the equations (3.1)–(3.2) take on the form

$$\frac{1}{\Phi M} \left[\frac{d^2}{dr^2} + \left(\frac{a_1}{r} + \frac{a_2}{r^2} \right) \frac{d}{dr} + b + \frac{b_1}{r} + ... \frac{b_4}{r^4} \right] \varphi \bar{f} + \frac{d}{dr} \bar{F} = 0 \, , \tag{3.3}$$

$$\frac{1}{\varphi M} \left[\frac{d^2}{dr^2} + \left(\frac{A_1}{r} + \frac{A_2}{r^2} \right) \frac{d}{dr} + B + \frac{B_1}{r} + ... \frac{B_4}{r^4} \right] \Phi \bar{F} + \frac{d}{dr} \bar{f} = 0 \, . \tag{3.4}$$

With the use of the temporary notations $\bar{f}(r) = f_1(r)$, $\bar{F}(r) = -f_2(r)$, we can transform the equations to a more symmetric form,

$$\left(K_2(x) \frac{d^2}{dx^2} + K_1(x) \frac{d}{dx} + K_0(x) \right) f_1 = \frac{df_2}{dx} \, , \qquad \left(L_2(x) \frac{d^2}{dx^2} + L_1(x) \frac{d}{dx} + L_0(x) \right) f_2 = \frac{df_1}{dx} \, . \tag{3.5}$$

Let us exclude the variable f_2:

$$f_2(x) = \int \left(K_2(x) \frac{d^2}{dx^2} + K_1(x) \frac{d}{dx} + K_0(x) \right) f_1 \, ,$$

$$\left(L_2 \frac{d}{dx} + L_1 \right) \left(K_2 \frac{d^2}{dx^2} + K_1 \frac{d}{dx} + K_0 \right) f_1 + L_0 \int dx \left(K_2 \frac{d^2}{dx^2} + K_1 \frac{d}{dx} + K_0 \right) f_1 = 0.$$

The second equation should be divided by $L_0(x)$ and differentiated, which provides a 4-th order equation for $f_1(x)$:

$$\left\{\frac{d}{dx}\left(\frac{L_2}{L_0}\frac{d}{dx}+\frac{L_1}{L_0}\right)\left(K_2\frac{d^2}{dx^2}+K_1\frac{d}{dx}+K_0\right)+\left(K_2\frac{d^2}{dx^2}+K_1\frac{d}{dx}+K_0\right)\right\}f_1(x)=0.$$

Similarly, we derive the 4-th order equation for f_2:

$$\left\{\frac{d}{dx}\left(\frac{K_2}{K_0}\frac{d}{dx}+\frac{K_1}{K_0}\right)\left(L_2\frac{d^2}{dx^2}+L_1\frac{d}{dx}+L_0\right)+\left(L_2\frac{d^2}{dx^2}+L_1\frac{d}{dx}+L_0\right)\right\}f_2(x)=0.$$

The general structure of the 4-th order equation for $F(r)$ is:

$$\begin{aligned}
&\tfrac{d^4F}{dr^4}+\left(\tfrac{m_1}{r}+\tfrac{m_2}{r^2}+\tfrac{m_3\,r^5+m_4\,r^4+m_5\,r^3+m_6\,r^2+m_7\,r+m_8}{P}\right)\tfrac{d^3F}{dr^3}\\
&+\left(n_0+\tfrac{n_1}{r}+\tfrac{n_2}{r^2}+\tfrac{n_3}{r^3}+\tfrac{n_4}{r^4}+\tfrac{n_5\,r^5+n_6\,r^4+n_7\,r^3+n_8\,r^2+n_9\,r+n_{10}}{P}\right)\tfrac{d^2F}{dr^2}\\
&+\left(\tfrac{p_1}{r}+\tfrac{p_2}{r^2}+\tfrac{p_3}{r^3}+\tfrac{p_4}{r^4}+\tfrac{p_5}{r^5}+\tfrac{p_6\,r^5+p_7\,r^4+p_8\,r^3+p_9\,r^2+p_{10}\,r+p_{11}}{P}\right)\tfrac{dF}{dr}\\
&+\Big(q_0+\tfrac{q_1}{r}+\tfrac{q_2}{r^2}+\tfrac{q_3}{r^3}+\tfrac{q_4}{r^4}+\tfrac{q_5}{r^5}+\tfrac{q_6}{r^6}\\
&+\tfrac{q_7\,r^5+q_8\,r^4+q_9\,r^3+q_{10}\,r^2+q_{11}\,r+q_{12}}{P}\Big)F=0\,,
\end{aligned}$$

where

$$\begin{aligned}
P= \;&(M_1-M_2)^2\,(\epsilon+M_2)\,(\epsilon-M_1)\,r^6-(M_1-M_2)^2\,\alpha\,(-M_2-2\,\epsilon+M_1)\,r^5\\
&+\left[\left(4\,\nu^2-6\,\nu+2+\alpha^2\right){M_1}^2+\left(\left(-6-8\,\nu^2+14\,\nu\right)\epsilon-2\,M_2\,\left(-1+\nu+\alpha^2\right)\right)M_1\right.\\
&\left.+4\,(-1+\nu)^2\,\epsilon^2+2M_2\,(-1+\nu)\,\epsilon+{M_2}^2\alpha^2\right]r^4\\
&-8[\beta_1M_1-\epsilon\beta_1+(-\tfrac{1}{2}+\nu)\alpha](-1+\nu)(-\epsilon+M_1)\,r^3\\
&+\left[(-\alpha-2\,\epsilon\,\beta_1+2\,\nu\,\alpha+\beta_1\,M_1+M_2\,\beta_1)^2\right.\\
&+2(-\alpha-2\epsilon\beta_1+2\nu\alpha+\beta_1M_1+M_2\beta_1)(M_1-M_2)\beta_1+2\alpha\beta_1(M_1-M_2)(2+2\nu)\\
&\left.-6\,(M_1-M_2)\,\alpha\,\beta_1-4\,(-2\,\epsilon+2\,\nu\,\epsilon-\nu\,M_1+2\,M_1-M_2\,\nu)\,\alpha\,\beta_1\right]r^2\\
&-8\,\beta_1\,[(-\tfrac{1}{2}+\nu)\alpha+\beta_1\,(-\epsilon+M_1)]\alpha\,r+4\,\alpha^2{\beta_1}^2\,.
\end{aligned}$$

All the remaining coefficients in the above equation are complicated, and for this reason, are omitted.

First we apply the substitution $F(r)=e^{Kr}\bar{F}(r)$, which gives

$$\begin{aligned}
&\tfrac{d^4\bar{F}}{dr^4}+\left[4\,K+\tfrac{m_1}{r}+\tfrac{m_2}{r^2}+\tfrac{m_3\,r^5+m_4\,r^4+m_5\,r^3+m_6\,r^2+m_7\,r+m_8}{P}\right]\tfrac{d^3\bar{F}}{dr^3}\\
&+\Big\{n_0+6\,K^2+\tfrac{3\,m_1\,K+n_1}{r}+\tfrac{n_2+3\,m_2\,K}{r^2}+\tfrac{n_3}{r^3}+\tfrac{n_4}{r^4}\\
&+\tfrac{1}{P}\left[(n_5+3\,m_3\,K)\,r^5+(3\,m_4\,K+n_6)\,r^4+(n_7+3\,m_5\,K)\,r^3\right.\\
&\left.+(n_8+3\,m_6\,K)\,r^2+(3\,m_7\,K+n_9)\,r+3\,m_8\,K+n_{10}\right]\Big\}\tfrac{d^2\bar{F}}{dr^2}\\
&+\Big\{2\,n_0\,K+4\,K^3+\tfrac{p_1+3\,m_1\,K^2+2\,n_1\,K}{r}+\tfrac{p_2+3\,m_2\,K^2+2\,n_2\,K}{r^2}+\tfrac{p_3+2\,n_3\,K}{r^3}\\
&+\tfrac{p_4+2\,n_4\,K}{r^4}+\tfrac{p_5}{r^5}+\tfrac{1}{P}\left[\left(3\,m_3\,K^2+2\,n_5\,K+p_6\right)r^5+\left(3\,m_4\,K^2+p_7+2\,n_6\,K\right)r^4\right.\\
&+\left(p_8+3\,m_5\,K^2+2\,n_7\,K\right)r^3+\left(2\,n_8\,K+p_9+3\,m_6\,K^2\right)r^2\\
&\left.+\left(2\,n_9\,K+3\,m_7\,K^2+p_{10}\right)r+2\,n_{10}\,K+p_{11}+3\,m_8\,K^2\right]\Big\}\tfrac{d\bar{F}}{dr}\\
&+\Big\{(n_0\,K^2+K^4+q_0)+\tfrac{q_1+p_1\,K+m_1\,K^3+n_1\,K^2}{r}+\tfrac{q_2+p_2\,K+m_2\,K^3+n_2\,K^2}{r^2}\\
&+\tfrac{q_3+p_3\,K+n_3\,K^2}{r^3}+\tfrac{q_4+p_4\,K+n_4\,K^2}{r^4}+\tfrac{q_5+p_5\,K}{r^5}+\tfrac{q_6}{r^6}
\end{aligned}$$

$$
+\frac{1}{P}\Big[\big(q_7+p_6\,K+n_5\,K^2+m_3\,K^3\big)\,r^5+\big(q_8+p_7\,K+m_4\,K^3+n_6\,K^2\big)\,r^4
+\big(p_8\,K+q_9+n_7\,K^2+m_5\,K^3\big)\,r^3+\big(n_8\,K^2+p_9\,K+m_6\,K^3+q_{10}\big)\,r^2
+\big(n_9\,K^2+q_{11}+p_{10}\,K+m_7\,K^3\big)\,r+m_8\,K^3+p_{11}\,K+n_{10}\,K^2+q_{12}\Big]\Big\}\,\bar F=0\,.
$$

We impose the following restriction on K: $K^4+n_0\,K^2+q_0=0$, and allow for

$$
n_0=-{M_1}^2-{M_2}^2+2\,\epsilon^2\,,\quad q_0=\big(\epsilon^2-M_1^2\big)\big(\epsilon^2-M_2^2\big)\,,
$$

which leads to

$$
\big(K^2-{M_1}^2+\epsilon^2\big)\big(K^2-{M_2}^2+\epsilon^2\big)=0\,;
$$

therefore four different variants for K are possible:

$$
K_1=\pm\sqrt{{M_1}^2-\epsilon^2}<0\,,\quad K_2=\pm\sqrt{{M_2}^2-\epsilon^2}<0\,. \tag{3.6}
$$

We shall further follow only the case of negative K (since such solutions may correspond to bound states).

Now, let us apply the substitution $\bar F=r^H e^{L/r}\tilde F$; we shall omit the explicit form of the resulting equation. The coefficient of $\tilde F$ requires the vanishing of the term

$$
\frac{L^2\,\big(L^2-m_2\,L+n_4\big)}{r^8}=0\quad\Longrightarrow\quad m_2=0\,,\quad n_4=-4\beta_1^2\,;
$$

so we have four roots:

$$
L=+2\beta_1,\quad -2\beta_1,\quad 0,\quad 0. \tag{3.7}
$$

Thus, there exist four variants:

$$
\begin{aligned}
&I. && L=2\beta_1<0\,,\quad H=\nu\ >0;\\
&II. && L=-2\beta_1>0\,,\quad H=1-\nu\le 0;\\
&III. && L=0\,,\quad H=+\sqrt{\nu^2-\alpha^2}>0;\\
&IV. && L=0\,,\quad H=-\sqrt{\nu^2-\alpha^2}<0,
\end{aligned} \tag{3.8}
$$

and we note that only the variants I and III are appropriate for bound states.

The 4-th order equation in the case I, after multiplication with r^6P, gives (we write down only the general structure)

$$
\begin{aligned}
&\big(P_{12}\,r^{12}+P_{11}\,r^{11}+P_{10}\,r^{10}+P_9\,r^9+P_8\,r^8+P_7\,r^7+P_6\,r^6\big)\,\tfrac{d^4\tilde F}{dr^4}\\
&+(Q_{12}r^{12}+Q_{11}r^{11}+Q_{10}r^{10}+Q_9r^9+Q_8r^8+Q_7r^7+Q_6r^6+Q_5r^5+Q_4r^4\big)\,\tfrac{d^3\tilde F}{dr^3}\\
&+\big(M_{12}r^{12}+M_{11}r^{11}+M_{10}r^{10}+M_9r^9+M_8r^8+M_7r^7\\
&+M_6r^6+M_5r^5+M_4r^4+M_3r^3+M_2r^2\big)\,\tfrac{d^2\tilde F}{dr^2}\\
&+\big(N_{12}\,r^{12}+N_{11}\,r^{11}+N_{10}\,r^{10}+N_9\,r^9+N_8\,r^8+N_7\,r^7\\
&+N_6r^6+N_5r^5+N_4r^4+N_3r^3+N_2r^2+N_1r+N_0\big)\,\tfrac{d\tilde F}{dr}\\
&+\big(L_{11}\,r^{11}+L_{10}\,r^{10}+L_9\,r^9+L_8\,r^8+L_7\,r^7+L_6\,r^6\\
&+L_5\,r^5+L_4\,r^4+L_3\,r^3+L_2\,r^2+L_1\,r+L_0\big)\,\tilde F=0\,.
\end{aligned}
$$

The solutions may be constructed as power series $\tilde{F} = \sum_{l=0}^{\infty} d_l r^l$, and hence we derive

$$P_{12}\sum_{k=12}^{\infty}(k-8)(k-9)(k-10)(k-11)d_{k-8}r^k + P_{11}\sum_{k=11}^{\infty}(k-7)(k-8)(k-9)(k-10)d_{k-7}r^k$$
$$+P_{10}\sum_{k=10}^{\infty}(k-6)(k-7)(k-8)(k-9)d_{k-6}r^k + P_9\sum_{k=9}^{\infty}(k-5)(k-6)(k-7)(k-8)d_{k-5}r^k$$
$$+P_8\sum_{k=8}^{\infty}(k-4)(k-5)(k-6)(k-7)d_{k-4}r^k + P_7\sum_{k=7}^{\infty}(k-3)(k-4)(k-5)(k-6)d_{k-3}r^k$$
$$+P_6\sum_{k=6}^{\infty}(k-2)(k-3)(k-4)(k-5)d_{k-2}r^k + Q_{12}\sum_{k=12}^{\infty}(k-9)(k-10)(k-11)d_{k-9}r^k$$
$$+Q_{11}\sum_{k=11}^{\infty}(k-8)(k-9)(k-10)d_{k-8}r^k + Q_{10}\sum_{k=10}^{\infty}(k-7)(k-8)(k-9)d_{k-7}r^k$$
$$+Q_9\sum_{k=9}^{\infty}(k-6)(k-7)(k-8)d_{k-6}r^k + Q_8\sum_{k=8}^{\infty}(k-5)(k-6)(k-7)d_{k-5}r^k$$
$$+Q_7\sum_{k=7}^{\infty}(k-4)(k-5)(k-6)d_{k-4}r^k + Q_6\sum_{k=6}^{\infty}(k-3)(k-4)(k-5)d_{k-3}r^k$$
$$+Q_5\sum_{k=5}^{\infty}(k-2)(k-3)(k-4)d_{k-2}r^k + Q_4\sum_{k=4}^{\infty}(k-1)(k-2)(k-3)d_{k-1}r^k$$
$$+M_{12}\sum_{k=12}^{\infty}(k-10)(k-11)d_{k-10}r^k + M_{11}\sum_{k=11}^{\infty}(k-9)(k-10)d_{k-9}r^k$$
$$+M_{10}\sum_{k=10}^{\infty}(k-8)(k-9)d_{k-8}r^k + M_9\sum_{k=9}^{\infty}(k-7)(k-8)d_{k-7}r^k$$
$$+M_8\sum_{k=8}^{\infty}(k-6)(k-7)d_{k-6}r^k + M_7\sum_{k=7}^{\infty}(k-5)(k-6)d_{k-5}r^k$$
$$+M_6\sum_{k=6}^{\infty}(k-4)(k-5)d_{k-4}r^k + M_5\sum_{k=5}^{\infty}(k-3)(k-4)d_{k-3}r^k$$
$$+M_4\sum_{k=4}^{\infty}(k-2)(k-3)d_{k-2}r^k + M_3\sum_{k=3}^{\infty}(k-1)(k-2)d_{k-1}r^k$$
$$+M_2\sum_{k=2}^{\infty}k(k-1)d_k r^k + N_{12}\sum_{k=12}^{\infty}(k-11)d_{k-11}r^k + N_{11}\sum_{k=11}^{\infty}(k-10)d_{k-10}r^k$$
$$+N_{10}\sum_{k=10}^{\infty}(k-9)d_{k-9}r^k + N_9\sum_{k=9}^{\infty}(k-8)d_{k-8}r^k + N_8\sum_{k=8}^{\infty}(k-7)d_{k-7}r^k$$
$$+N_7\sum_{k=7}^{\infty}(k-6)d_{k-6}r^k + N_6\sum_{k=6}^{\infty}(k-5)d_{k-5}r^k + N_5\sum_{k=5}^{\infty}(k-4)d_{k-4}r^k$$
$$+N_4\sum_{k=4}^{\infty}(k-3)d_{k-3}r^k + N_3\sum_{k=3}^{\infty}(k-2)d_{k-2}r^k + N_2\sum_{k=2}^{\infty}(k-1)d_{k-1}r^k$$
$$+N_1\sum_{k=1}^{\infty}kd_k r^k + N_0\sum_{k=0}^{\infty}(k+1)d_{k+1}r^k + L_{11}\sum_{k=11}^{\infty}d_{k-11}r^k + L_{10}\sum_{k=10}^{\infty}d_{k-10}r^k$$
$$+L_9\sum_{k=9}^{\infty}d_{k-9}r^k + L_8\sum_{k=8}^{\infty}d_{k-8}r^k + L_7\sum_{k=7}^{\infty}d_{k-7}r^k + L_6\sum_{k=6}^{\infty}d_{k-6}r^k + L_5\sum_{k=5}^{\infty}d_{k-5}r^k$$
$$+L_4\sum_{k=4}^{\infty}d_{k-4}r^k + L_3\sum_{k=3}^{\infty}d_{k-3}r^k + L_2\sum_{k=2}^{\infty}d_{k-2}r^k + L_1\sum_{k=1}^{\infty}d_{k-1}r^k + L_0\sum_{k=0}^{\infty}d_k r^k = 0.$$

The recurrent formulas start as follow

$$k=0\,, \quad N_0\,d_1 + L_0\,d_0 = 0\,, \quad k=1\,, \quad 2\,N_0\,d_2 + (N_1+L_0)\,d_1 + L_1\,d_0 = 0\,, \ ...,$$

and in this way we get the 13-terms recurrent relations with the general structure

$$Q_{k-11}d_{k-11} + Q_{k-10}d_{k-10} + ... + Q_k d_k + Q_{k+1}d_{k+1} = 0\,. \tag{3.9}$$

The constraint which determines transcendental Frobenius reads

$$Q_{k-10} = 0 \quad \Longrightarrow \quad L_{11} + N_{12}\,(k-11) = 0\,, \qquad k-11 = n > 0 \tag{3.10}$$

and has the explicit form

$$\begin{aligned} &-4\,(M_1-\epsilon)\,(M_2+\epsilon)\,(M_1-M_2)^2\,\big\{(k-10+H)\,K^3 + \alpha\,\epsilon\,K^2 \\ &+\big[\epsilon^2(H-10+k) + \big(\frac{9}{2} - \frac{1}{2}\,k - \frac{1}{2}\,H\big){M_1}^2 - \frac{1}{2}\,(H-11+k)\,{M_2}^2\big]K \\ &-\frac{1}{2}\,\alpha\,\epsilon\,\big({M_1}^2 + {M_2}^2 - 2\,\epsilon^2\big)\,\big\} = 0\,. \end{aligned} \tag{3.11}$$

Let

$$K = -\sqrt{M_1^2 - \epsilon^2}\,, \quad H = \nu = 1, 2, 3... \;; \tag{3.12}$$

then eq. (3.11) takes on the form

$$-2\,(M_1 - \epsilon)\,(M_2 + \epsilon)\,(M_1 - M_2)^2$$
$$\times \Big\{ (-2\,k + 20 - 2\,\nu)\left({M_1}^2 - \epsilon^2\right)^{3/2} + \left[(-2\,k + 20 - 2\,\nu)\,\epsilon^2 + (k - 9 + \nu)\,{M_1}^2\right.$$
$$\left. + (k - 11 + \nu)\,{M_2}^2\right]\sqrt{{M_1}^2 - \epsilon^2} + \alpha\,\epsilon\,\left(M_1^2 - M_2^2\right)\Big\} = 0\,,$$

whence we get the real valued roots ϵ:

$$\epsilon = +M_1\,,\; -M_2\,, \quad \epsilon = \pm\frac{M_1}{\sqrt{1 + \alpha^2/(k - 11 + \nu)^2}}\,,$$

while (quasi) physical is the root

$$\epsilon = +\frac{M_1}{\sqrt{1 + \alpha^2/(k - 11 + \nu)^2}} > 0\,, \quad k \geq 12\,. \tag{3.13}$$

Let

$$K = -\sqrt{M_2^2 - \epsilon^2}, \quad H = \nu = 1, 2, 3, \dots\,. \tag{3.14}$$

In this case we have the roots

$$\epsilon = M_1\,,\; -M_2\,, \quad \epsilon = \pm\frac{M_2}{\sqrt{1 + \alpha^2/(k - 9 + \nu)^2}}\,, \tag{3.15}$$

and (quasi) physical is the root

$$\epsilon = +\frac{M_2}{\sqrt{1 + \alpha^2/(k - 9 + \nu)^2}}, \quad k \geq 12\,. \tag{3.16}$$

It should be noticed that both these spectra cannot be considered as relativistic spectra for a spin 1/2 particle in the Coulomb field, because they do not contain the specific combination $\sqrt{\nu^2 - \alpha^2}$; besides, these formulas do not agree with the study of the problem at infinity (see (2.14)).

For this reason, we further study the equation for the variant III; after multiplying it by $r^5 P$, we obtain the following equation (we write down again only its general structure):

$$\left(P_{11}\,r^{11} + P_{10}\,r^{10} + P_9\,r^9 + P_8\,r^8 + P_7\,r^7 + P_6\,r^6 + P_5\,r^5\right)\tfrac{d^4\tilde{F}}{dr^4}$$
$$+ \left(Q_{11}r^{11} + Q_{10}r^{10} + Q_9r^9 + Q_8r^8 + Q_7r^7 + Q_6r^6 + Q_5r^5 + Q_4r^4 + Q_3\,r^3\right)\tfrac{d^3\tilde{F}}{dr^3}$$
$$+ \left(M_{11}r^{11} + M_{10}r^{10} + M_9r^9 + M_8r^8 + M_7r^7 + M_6r^6\right.$$
$$\left.+ M_5r^5 + M_4r^4 + M_3r^3 + M_2r^2 + M_1r\right)\tfrac{d^2\tilde{F}}{dr^2}$$
$$+ \left(N_{11}r^{11} + N_{10}r^{10} + N_9r^9 + N_8r^8 + N_7r^7 + N_6r^6\right.$$
$$\left.+ N_5r^5 + N_4r^4 + N_3r^3 + N_2r^2 + N_1r + N_0\right)\tfrac{d\tilde{F}}{dr}$$
$$+ \left(L_{10}\,r^{10} + L_9\,r^9 + L_8\,r^8 + L_7\,r^7 + L_6\,r^6\right.$$
$$\left.+ L_5\,r^5 + L_4\,r^4 + L_3\,r^3 + L_2\,r^2 + L_1\,r + L_0\right)\tilde{F} = 0\,.$$

Its solutions are constructed as power series, so we obtain small

$$
\begin{gathered}
P_{11}\sum_{k=11}^{\infty}(k-7)(k-8)(k-9)(k-10)d_{k-7}r^k+P_{10}\sum_{k=10}^{\infty}(k-6)(k-7)(k-8)(k-9)d_{k-6}r^k\\
+P_9\sum_{k=9}^{\infty}(k-5)(k-6)(k-7)(k-8)d_{k-5}r^k+P_8\sum_{k=8}^{\infty}(k-4)(k-5)(k-6)(k-7)d_{k-4}r^k\\
+P_7\sum_{k=7}^{\infty}(k-3)(k-4)(k-5)(k-6)d_{k-3}r^k+P_6\sum_{k=6}^{\infty}(k-2)(k-3)(k-4)(k-5)d_{k-2}r^k\\
+P_5\sum_{k=5}^{\infty}(k-1)(k-2)(k-3)(k-4)d_{k-1}r^k+Q_{11}\sum_{k=11}^{\infty}(k-8)(k-9)(k-10)d_{k-8}r^k\\
+Q_{10}\sum_{k=10}^{\infty}(k-7)(k-8)(k-9)d_{k-7}r^k+Q_9\sum_{k=9}^{\infty}(k-6)(k-7)(k-8)d_{k-6}r^k\\
+Q_8\sum_{k=8}^{\infty}(k-5)(k-6)(k-7)d_{k-5}r^k+Q_7\sum_{k=7}^{\infty}(k-4)(k-5)(k-6)d_{k-4}r^k\\
+Q_6\sum_{k=6}^{\infty}(k-3)(k-4)(k-5)d_{k-3}r^k+Q_5\sum_{k=5}^{\infty}(k-2)(k-3)(k-4)d_{k-2}r^k\\
+Q_4\sum_{k=4}^{\infty}(k-1)(k-2)(k-3)d_{k-1}r^k+Q_3\sum_{k=3}^{\infty}k(k-1)(k-2)d_kr^k\\
+M_{11}\sum_{k=11}^{\infty}(k-9)(k-10)d_{k-9}r^k+M_{10}\sum_{k=10}^{\infty}(k-8)(k-9)d_{k-8}r^k\\
+M_9\sum_{k=9}^{\infty}(k-7)(k-8)d_{k-7}r^k+M_8\sum_{k=8}^{\infty}(k-6)(k-7)d_{k-6}r^k\\
+M_7\sum_{k=7}^{\infty}(k-5)(k-6)d_{k-5}r^k+M_6\sum_{k=6}^{\infty}(k-4)(k-5)d_{k-4}r^k\\
+M_5\sum_{k=5}^{\infty}(k-3)(k-4)d_{k-3}r^k+M_4\sum_{k=4}^{\infty}(k-2)(k-3)d_{k-2}r^k\\
+M_3\sum_{k=3}^{\infty}(k-1)(k-2)d_{k-1}r^k+M_2\sum_{k=2}^{\infty}k(k-1)d_kr^k+M_1\sum_{k=1}^{\infty}(k+1)kd_{k+1}r^k\\
+N_{11}\sum_{k=11}^{\infty}(k-10)d_{k-10}r^k+N_{10}\sum_{k=10}^{\infty}(k-9)d_{k-9}r^k+N_9\sum_{k=9}^{\infty}(k-8)d_{k-8}r^k\\
+N_8\sum_{k=8}^{\infty}(k-7)d_{k-7}r^k+N_7\sum_{k=7}^{\infty}(k-6)d_{k-6}r^k+N_6\sum_{k=6}^{\infty}(k-5)d_{k-5}r^k\\
+N_5\sum_{k=5}^{\infty}(k-4)d_{k-4}r^k+N_4\sum_{k=4}^{\infty}(k-3)d_{k-3}r^k+N_3\sum_{k=3}^{\infty}(k-2)d_{k-2}r^k\\
+N_2\sum_{k=2}^{\infty}(k-1)d_{k-1}r^k+N_1\sum_{k=1}^{\infty}kd_kr^k+N_0\sum_{k=0}^{\infty}(k+1)d_{k+1}r^k\\
+L_{10}\sum_{k=10}^{\infty}d_{k-10}r^k+L_9\sum_{k=9}^{\infty}d_{k-9}r^k+L_8\sum_{k=8}^{\infty}d_{k-8}r^k+L_7\sum_{k=7}^{\infty}d_{k-7}r^k\\
+L_6\sum_{k=6}^{\infty}d_{k-6}r^k+L_5\sum_{k=5}^{\infty}d_{k-5}r^k+L_4\sum_{k=4}^{\infty}d_{k-4}r^k+L_3\sum_{k=3}^{\infty}d_{k-3}r^k\\
+L_2\sum_{k=2}^{\infty}d_{k-2}r^k+L_1\sum_{k=1}^{\infty}d_{k-1}r^k+L_0\sum_{k=0}^{\infty}d_kr^k=0\,.
\end{gathered}
$$

We obtain a 12-terms recurrent relation with the structure

$$Q_{k-10}d_{k-10}+Q_{k-9}d_{k-9}+....+Q_kd_k+Q_{k+1}d_{k+1}=0\,.$$

The constraint which determines transcendental Frobenius is

$$Q_{k-10}=0\quad\Longrightarrow\quad L_{10}+N_{11}\,(k-10)=0\,,\qquad k-10=n=1,2,...>0\,,$$

and explicitly reads

$$
\begin{aligned}
&-4\,(M_1-\epsilon)\,(M_2+\epsilon)\,(M_1-M_2)^2\left\{(k-9+H)\,K^3+\alpha\,\epsilon\,K^2\right.\\
&+\Big[(k-9+H)\,\epsilon^2+\big(4-\frac{1}{2}k-\frac{1}{2}H\big)M_1{}^2-\frac{1}{2}\,(k-10+H)\,M_2{}^2\Big]K\\
&\left.-\frac{1}{2}\,\alpha\,\epsilon\,\left(M_1{}^2+M_2{}^2-2\,\epsilon^2\right)\right\}=0\,.
\end{aligned}
\tag{3.17}
$$

Let

$$K=-\sqrt{M_1^2-\epsilon^2}\,,\quad H=\sqrt{\nu^2-\alpha^2};$$

then eq. (3.17) takes the form

$$-4(M_1-\epsilon)(M_2+\epsilon)(M_1-M_2)^2\{-(k-9+\sqrt{\nu^2-\alpha^2})({M_1}^2-\epsilon^2)^{3/2}$$
$$+({M_1}^2-\epsilon^2)\alpha\,\epsilon-\big[(k-9+\sqrt{\nu^2-\alpha^2})\epsilon^2+\big(4-\frac{1}{2}\,k-\frac{1}{2}\sqrt{\nu^2-\alpha^2}\big){M_1}^2$$
$$-\frac{1}{2}\,{M_2}^2(k-10+\sqrt{\nu^2-\alpha^2})\big]\sqrt{{M_1}^2-\epsilon^2}-\frac{1}{2}\,\alpha\,\epsilon\,\left({M_1}^2+{M_2}^2-2\,\epsilon^2\right)\}=0\,,$$

whence we get the roots

$$\epsilon=M_1\,,\ -M_2\,,\quad \epsilon=\pm\frac{M_1}{\sqrt{1+\alpha^2/\left(k-10+\sqrt{\nu^2-\alpha^2}\right)^2}}\,;$$

we note that the physical root is

$$\epsilon=\frac{M_1}{\sqrt{1+\alpha^2/\left(k-10+\sqrt{\nu^2-\alpha^2}\right)^2}},\quad k-10=n=1,2,3,... \tag{3.18}$$

Let

$$K=-\sqrt{M_2^2-\epsilon^2},\quad H=\sqrt{\nu^2-\alpha^2}; \tag{3.19}$$

then the roots are

$$\epsilon=M_1\,,\ -M_2\,,\quad \epsilon=\pm\frac{M_2}{\sqrt{1+\alpha^2/\left(k-8+\sqrt{\nu^2-\alpha^2}\right)^2}}\,,$$

and the physical root is

$$\epsilon=\frac{M_2}{\sqrt{1+\alpha^2/\left(k-8+\sqrt{\nu^2-\alpha^2}\right)^2}},\quad k-10=n=1,2,3,... \tag{3.20}$$

Both spectra (3.18) and (3.20) are typical for a spin 1/2 particle in the Coulomb field: they contain the combination $\sqrt{\alpha^2-\nu^2}$; besides, they agree with the results for studying the equation at infinity (see (2.14)).

The energy spectra for states with opposite parity may be found with the help of the formal changes: $M_1,M_2,\alpha_1,\alpha_2,\beta_1,\beta_2\implies -M_1,-M_2,-\alpha_1,-\alpha_2,-\beta_1,-\beta_2$.

Correspondingly, we have here four possible variants

$$\begin{aligned} \delta=-1,\quad & I. && L=-2\beta_1>0\,, && H=\nu>0\,;\\ & II. && L=+2\beta_1<0\,, && H=1-\nu\le 0\,;\\ & III. && L=0\,, && H=\sqrt{\nu^2-\alpha^2}>0\,;\\ & IV. && L=0\,, && H=-\sqrt{\nu^2-\alpha^2}<0\,, \end{aligned} \tag{3.21}$$

and only the variant III leads to physical spectra, which coincide.

Let us write down the general structure of the solutions

$$\Psi^{(1)}_{\epsilon jm\delta} = \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{c} (f+iF)D_{-1/2} \\ (f-iF)D_{+1/2} \\ \delta(f-iF)D_{-1/2} \\ \delta(f+iF)D_{+1/2} \end{array} \right| , \ \Psi^{(2)}_{\epsilon jm\delta} = \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{c} (g+iG)D_{-1/2} \\ (g-iG)D_{+1/2} \\ \delta(g-iG)D_{-1/2} \\ \delta(g+iG)D_{+1/2} \end{array} \right| .$$

In nonrelativistic approximation, these formulas give (recall that $k-10 = 1, 2, 3, ...$):

$$\epsilon = +\frac{M_1}{\sqrt{1+\alpha^2/(k-10+\sqrt{\nu^2-\alpha^2})^2}} \quad \Longrightarrow \quad \epsilon = M_1 - \frac{M_1\alpha^2}{2(k-10+\nu)^2} ,$$

$$\epsilon = \frac{M_2}{\sqrt{1+\alpha^2/\left(k-8+\sqrt{\nu^2-\alpha^2}\right)^2}} \quad \Longrightarrow \quad \epsilon = M_2 - \frac{M_2\alpha^2}{2(k-8+\nu)^2} ,$$

which lead to the nonrelativistic spectra

$$E_1 = -\frac{M_1\alpha^2}{2(k-10+\nu)^2}, \quad E_2 = -\frac{M_2\alpha^2}{2(k-8+\nu)^2} .$$

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