Dual jet h-normal N-linear connections in time-dependent Hamilton geometry

M. Neagu, A. Oană and V. Balan

Abstract. In this paper we study the local adapted components of the h-normal N-linear connections in the dual jet time-dependent Hamilton geometry. The corresponding local d-torsions and d-curvatures are also analyzed.

M.S.C. 2020: 53B40, 53C60, 53C07.
Key words: dual 1-jet space; d-tensors; h-normal N-linear connections; d-torsions; d-curvatures.

1 Introduction

The 1-jet spaces and their duals are the fundamental ambient spaces used in the study of classical and quantum field theories in the Lagrangian and Hamiltonian approaches (see Olver [8] and Atanasiu, Neagu and Oană [3]). Moreover, following the Cartan covariant Hamiltonian approach of classical Mechanics, the studies of Miron [4] and Atanasiu ([1], [2]) led to the development of the *Hamilton geometry of cotangent bundles* exposed by Miron, Hrimiuc, Shimada and Sabău in the monograph [5]. We underline that, via the Legendre duality of the Hamilton spaces with the Lagrange spaces, the preceding authors showed in [5] that the theory of Hamilton spaces has the same symmetry as the Lagrange geometry, giving thus a geometrical framework for the Hamiltonian theory of Analytical Mechanics.

In this physical and geometrical context, suggested by the cotangent bundle Miron's approach, the papers [6] and [7] are devoted to developing the *time-dependent covariant Hamilton geometry on dual* 1*-jet spaces* which is a natural dual jet extension of the Hamilton geometry on the cotangent bundle from [5]. The geometrical study from [6] and [7] of the usual objects as d-tensors, time-dependent semisprays of momenta, nonlinear connections, N-linear connections, d-torsions and d-curvatures is realized on the *dual* 1*-jet vector bundle* $J^{1*}(\mathbb{R}, M) \equiv \mathbb{R} \times T^*M \to \mathbb{R} \times M$, whose local coordinates are denoted by (t, x^i, p_i^1). Here M^n is a smooth real manifold of dimension

BSG Proceedings 29. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2022, 26-29 August 2021, Mangalia-Romania, pp. 68-73.

n, whose local coordinates are $(x^i)_{i=\overline{1,n}}$. Further, the coordinates p_i^1 are called *momenta*, and the dual 1-jet space $J^{1*}(\mathbb{R},M)$ is called the *time-dependent phase space of momenta.*

The transformations of coordinates $(t,x^i,p_i^1)\longleftrightarrow(\tilde{t},\tilde{x}^i,\tilde{p}_i^1)$, induced from $\mathbb{R}\times M$ on the dual 1-jet space $J^{1*}(\mathbb{R},M)$, have the expressions

$$\left\{\begin{array}{l}\tilde{t}=\tilde{t}\,(t)\\ \tilde{x}^i=\tilde{x}^i\left(x^j\right)\\ \tilde{p}_i^1=\dfrac{\partial x^j}{\partial\tilde{x}^i}\dfrac{d\tilde{t}}{dt}p_j^1,\end{array}\right. \tag{1.1}$$

where $d\tilde{t}/dt\neq 0$ and $\det(\partial\tilde{x}^i/\partial x^j)\neq 0$. Consequently, in the present dual jet geometrical approach, we use a kind of "*relativistic*" time t.

For instance, in the Hamiltonian Miron's approach from [5], the authors use the trivial bundle $\mathbb{R}\times T^*M\to T^*M$, whose local coordinates induced by T^*M are $\left(t,x^i,p_i\right)$. Thus, the changes of coordinates are given by relations

$$\left\{\begin{array}{l}\tilde{t}=t\\ \tilde{x}^i=\tilde{x}^i\left(x^j\right)\\ \tilde{p}_i=\dfrac{\partial x^j}{\partial\tilde{x}^i}p_j,\end{array}\right.$$

pointing out the *absolute* character of the time variable t.

A geometrization of Hamiltonians $H:\mathbb{R}\times T^*M\to\mathbb{R}$ on the dual 1-jet vector bundle $J^{1*}(\mathbb{R},M)\equiv\mathbb{R}\times T^*M\to\mathbb{R}\times M$ was initiated in the papers [6] and [7] and it is in progress in this paper and other future articles.

2 h-Normal N-linear connections

Generally speaking, a linear connection on the dual 1-jet space $J^{1*}(\mathbb{R},M)$ is characterized by 27 adapted local components. To work with these components is very complicated through of the big numbers of coefficients. For this reason, in the paper [7], were introduced and studied the N-linear connections. Because the number of components which characterize an N-linear connection on $E^*=J^{1*}(\mathbb{R},M)$ is still a big one (9 local components), we are constrained to study only a particular class of N-linear connections on E^*, which must be characterized by a reduced number of components. In this direction, let us consider a semi-Riemannian metric $h_{11}(t)$ on the time manifold $\mathbb{R}$, together with its Christoffel symbol

$$\chi_{11}^1=\frac{h^{11}}{2}\frac{dh_{11}}{dt}.$$

Let $\mathbb{J}$ be the h-normalization d-tensor field on E^*, locally expressed by $\mathbb{J}=J_{(1)1j}^{(i)}\delta p_i^1\otimes dt\otimes dx^j$, where $J_{(1)1j}^{(i)}=h_{11}\delta_j^i$. In this context, we introduce the following geometrical concept:

Definition 2.1. An N-linear connection $D\Gamma(N)$ on E^*, whose local components

$$D\Gamma\,(N)=\left(A^1_{11},A^i_{j1},-A^{(1)(j)}_{(i)(1)1},H^1_{1k},H^i_{jk},-H^{(1)(j)}_{(i)(1)k},\right.$$
$$\left.C^{1(k)}_{1(1)},C^{i(k)}_{j(1)},-C^{(1)(j)(k)}_{(i)(1)(1)}\right),$$

satisfy the relations $A^1_{11}=\chi^1_{11},\ H^1_{1i}=0,\ C^{1(i)}_{1(1)}=0$ and $D\mathbb{J}=0$, is called an *h-normal N-linear connection* on the dual 1-jet space E^*.

Theorem 2.1. *The local adapted components of an h-normal N-linear connection $D\Gamma(N)$ verify the following identities:*

$$\begin{gathered}A^1_{11}=\chi^1_{11},\quad H^1_{1i}=0,\quad C^{1(i)}_{1(1)}=0,\\ A^{(1)(j)}_{(i)(1)1}=A^j_{i1}-\delta^j_i\chi^1_{11},\quad H^{(1)(j)}_{(i)(1)k}=H^j_{ik},\\ C^{(1)(j)(k)}_{(i)(1)(1)}=C^{j(k)}_{i(1)}.\end{gathered}\tag{2.1}$$

Proof. It is obvious that the first three relations immediately come from the definition of an h-normal N-linear connection. To prove the other three relations, we emphasize that, taking into account the definition of the local $\mathbb{R}$-horizontal ("$_{/1}$"), M-horizontal ("$_{|s}$") and vertical ("$|^{(s)}_{(1)}$") covariant derivatives produced by $D\Gamma(N)$, the condition $D\mathbb{J}=0$ is equivalent to

$$J^{(i)}_{(1)1j/1}=0,\qquad J^{(i)}_{(1)1j|s}=0,\qquad J^{(i)}_{(1)1j}|^{(s)}_{(1)}=0.$$

Consequently, the condition $D\mathbb{J}=0$ provides the local identities

$$h_{11}A^{(1)(i)}_{(j)(1)1}=h_{11}A^i_{j1}-\delta^i_j\left(\frac{dh_{11}}{dt}-h_{11}\chi^1_{11}\right)=h_{11}A^i_{j1}-\delta^i_j\frac{1}{2}\frac{dh_{11}}{dt},$$
$$h_{11}H^{(1)(i)}_{(j)(1)k}=h_{11}H^i_{jk},\quad h_{11}C^{(1)(i)(k)}_{(j)(1)(1)}=h_{11}C^{i(k)}_{j(1)}.$$

Contracting now the above relations by h^{11}, we obtain the last required identities from (2.1). □

Remark 2.2. The above Theorem shows that an h-normal N-linear connection on E^* is an N-linear connection determined only by *four* effective components (instead of nine in the general case):

$$D\Gamma(N)=\left(\chi^1_{11},\ A^i_{j1},\ H^i_{jk},\ C^{i(k)}_{j(1)}\right).$$

The other five components either vanish or are provided by the relations (2.1).

Example 2.3. The Berwald N_0-linear connection associated with the pair of metrics $(h_{11}(t),\varphi_{ij}(x))$ is an h-normal N_0-linear connection on E^*, whose four effective components are $B\Gamma\,(N_0)=\left(\chi^1_{11}(t),\ 0,\ \gamma^i_{jk}(x),\ 0\right)$, where $\gamma^i_{jk}(x)$ are the Christoffel symbols of the metric $\varphi_{ij}(x)$. For more details, see the papers [6] and [7].

3 Torsion and curvature d-tensors

The study of the adapted components of the torsion and curvature tensors of an arbitrary N-linear connection $D\Gamma(N)$ on E^* was done in [7]. In that context, it turned out that the torsion tensor $\mathbf{T}$ is determined by *ten* effective local adapted d-tensors, while the curvature tensor $\mathbf{R}$ is determined by *fifteen* local adapted d-tensors. In what follows, we study the adapted components of the torsion and curvature tensors for an h-normal N-linear connection $D\Gamma(N)$.

Theorem 3.1. *The torsion tensor* $\mathbf{T}$ *of an h-normal N-linear connection* $D\Gamma(N)$ *is determined by* eight *effective local adapted d-tensors (instead of ten in the general case):*

(3.1)

	$h_{\mathbb{R}}$	h_M	v
$h_{\mathbb{R}}h_{\mathbb{R}}$	0	0	0
$h_M h_{\mathbb{R}}$	0	T^r_{1j}	$\underset{(r)1j}{\overset{(1)}{R}}$
$vh_{\mathbb{R}}$	0	0	$P^{(1)\ (j)}_{(r)1(1)}$
$h_M h_M$	0	T^r_{ij}	$R^{(1)}_{(r)ij}$
vh_M	0	$P^{r(j)}_{i(1)}$	$P^{(1)\ (j)}_{(r)i(1)}$
vv	0	0	$S^{(1)(i)(j)}_{(r)(1)(1)}$

where

$$T^r_{1j} = -A^r_{j1}, \quad T^r_{ij} = H^r_{ij} - H^r_{ji}, \quad P^{r(j)}_{i(1)} = C^{r(j)}_{i(1)},$$

$$P^{(1)\ (j)}_{(r)1(1)} = \frac{\partial \underset{1}{N}{}^{(1)}_{(r)1}}{\partial p^1_j} + A^j_{r1} - \delta^j_r \chi^1_{11}, \quad P^{(1)\ (j)}_{(r)i(1)} = \frac{\partial \underset{2}{N}{}^{(1)}_{(r)i}}{\partial p^1_j} + H^j_{ri},$$

$$R^{(1)}_{(r)1j} = \frac{\delta \underset{1}{N}{}^{(1)}_{(r)1}}{\delta x^j} - \frac{\delta \underset{2}{N}{}^{(1)}_{(r)j}}{\delta t},$$

$$R^{(1)}_{(r)ij} = \frac{\delta \underset{2}{N}{}^{(1)}_{(r)i}}{\delta x^j} - \frac{\delta \underset{2}{N}{}^{(1)}_{(r)j}}{\delta x^i}, \quad S^{(1)(i)(j)}_{(r)(1)(1)} = -\left(C^{i(j)}_{r(1)} - C^{j(i)}_{r(1)}\right).$$

Proof. Particularizing the general local expressions of the torsion tensor of an N-linear connection from [7] (which generally have ten d-components) for an h-normal N-linear connection $D\Gamma(N)$ on E^*, we deduce that the adapted components T^1_{1j} and $P^{1(k)}_{1(1)}$ are zero, while the other eight are given by the formulas from Theorem. □

Remark 3.1. All torsion d-tensors of the Berwald h-normal N_0-linear connection $B\Gamma(N_0)$ (associated with the semi-Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$) are zero, except $R^{(1)}_{(r)ij} = -\mathfrak{R}^s_{rij} p^1_s$, where $\mathfrak{R}^s_{rij}(x)$ are the local curvature tensors of the metric $\varphi_{ij}(x)$.

Theorem 3.2. *The curvature tensor* $\mathbf{R}$ *of an* h*-normal* N*-linear connection* $D\Gamma(N)$ *is characterized by* five *effective local d-tensors (instead of fifteen in the general case):*

(3.2)

	$h_{\mathbb{R}}$	h_M	v
$h_{\mathbb{R}}h_{\mathbb{R}}$	0	0	0
$h_M h_{\mathbb{R}}$	0	R^l_{i1k}	$-R^{(1)(i)}_{(l)(1)1k} = -R^i_{l1k}$
$wh_{\mathbb{R}}$	0	$P^{l\ \ (k)}_{i1(1)}$	$-P^{(1)(i)\ (k)}_{(l)(1)1(1)} = -P^{i\ (k)}_{l1(1)}$
$h_M h_M$	0	R^l_{ijk}	$-R^{(1)(i)}_{(l)(1)jk} = -R^i_{ljk}$
wh_M	0	$P^{l\ (k)}_{ij(1)}$	$-P^{(1)(i)\ (k)}_{(l)(1)j(1)} = -P^{i\ (k)}_{lj(1)}$
ww	0	$S^{l(j)(k)}_{i(1)(1)}$	$-S^{(1)(i)(j)(k)}_{(l)(1)(1)(1)} = -S^{i(j)(k)}_{l(1)(1)}$

where

$$R^l_{i1k} = \frac{\delta A^l_{i1}}{\delta x^k} - \frac{\delta H^l_{ik}}{\delta t} + A^r_{i1}H^l_{rk} - H^r_{ik}A^l_{r1} + C^{l(r)}_{i(1)}R^{(1)}_{(r)1k},$$

$$P^{l\ (k)}_{i1(1)} = \frac{\partial A^l_{i1}}{\partial p^1_k} - C^{l(k)}_{i(1)/1} + C^{l(r)}_{i(1)}P^{(1)\ (k)}_{(r)1(1)},$$

$$R^l_{ijk} = \frac{\delta H^l_{ij}}{\delta x^k} - \frac{\delta H^l_{ik}}{\delta x^j} + H^r_{ij}H^l_{rk} - H^r_{ik}H^l_{rj} + C^{l(r)}_{i(1)}R^{(1)}_{(r)jk},$$

$$P^{l\ (k)}_{ij(1)} = \frac{\partial H^l_{ij}}{\partial p^1_k} - C^{l(k)}_{i(1)|j} + C^{l(r)}_{i(1)}P^{(1)\ (k)}_{(r)j(1)},$$

$$S^{l(j)(k)}_{i(1)(1)} = \frac{\partial C^{l(j)}_{i(1)}}{\partial p^1_k} - \frac{\partial C^{l(k)}_{i(1)}}{\partial p^1_j} + C^{r(j)}_{i(1)}C^{l(k)}_{r(1)} - C^{r(k)}_{i(1)}C^{l(j)}_{r(1)}.$$

Proof. The general formulas which express the local curvature d-tensors of an arbitrary N-linear connection from [7], applied to the particular case of an h-normal N-linear connection $D\Gamma(N)$, imply the above formulas and the relations from the Table (3.2). □

Remark 3.2. For the Berwald h-normal N_0-linear connection $B\Gamma(N_0)$ (associated with the pair of metrics $(h_{11}(t), \varphi_{ij}(x))$), all curvature d-tensors are zero, except $R^{(1)(l)}_{(i)(1)jk} = R^l_{ijk} = \mathfrak{R}^l_{ijk}$.

4 A physical application

As an example, we shall investigate in a forecoming paper, the *dual jet time-dependent Hamiltonian of electrodynamics*,

$$H = \frac{1}{4mc}h_{11}(t)\varphi^{ij}(x)p^1_i p^1_j - \frac{e}{m^2c}A^{(i)}_{(1)}(x)p^1_i + \frac{e^2}{m^3c}F(t,x) - \mathsf{P}(t,x), \tag{4.1}$$

where $A^{(i)}_{(1)}(x)$ is a d-tensor on $J^{1*}(\mathbb{R}, M)$ having the physical meaning of a *potential d-tensor of an electromagnetic field*, $\mathsf{P}(t,x)$ is a *potential function,* and m, c and e are well-known physical constants: *mass of the test body*, *speed of light* and *electric* *charge.* Here, we have $F(t,x) = h^{11}(t)\varphi_{ij}(x)A^{(i)}_{(1)}(x)A^{(j)}_{(1)}(x)$. The geometrization associated with this time-dependent Hamiltonian will consist of a canonical nonlinear connection N, a Cartan canonical N-linear connection $C\Gamma(N)$ (which is an h-normal linear connection) together with its adapted d-torsions and d-curvatures. All these geometrical objects are provided only by the time-dependent Hamiltonian (4.1).

References

[1] Gh. Atanasiu, *The invariant expression of Hamilton geometry*, Tensor N.S., 47, 3 (1988), 225-234.

[2] Gh. Atanasiu, F.C. Klepp, *Nonlinear connections in cotangent bundle*, Publ. Math. Debrecen 39, 1-2 (1991), 107-111.

[3] Gh. Atanasiu, M. Neagu, A. Oană, *The Geometry of Jet Multi-Time Lagrange and Hamilton Spaces. Applications in Theoretical Physics*, Fair Partners, Bucharest, 2013.

[4] R. Miron, *Hamilton Geometry*, An. Şt. "Al. I. Cuza" Univ., Iaşi, Romania, 35 (1989), 33-67.

[5] R. Miron, D. Hrimiuc, H. Shimada, S.V. Sabău, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[6] M. Neagu, A. Oană, *Dual jet geometrical objects of momenta in the time-dependent Hamilton geometry*, "Vasile Alecsandri" University of Bacău, Faculty of Sciences, Scientific Studies and Research. Series Mathematics and Informatics, 30, 2 (2020), 153-164.

[7] A. Oană, M. Neagu, *On dual jet N-linear connections in the time-dependent Hamilton geometry*, Annals of the University of Craiova - Mathematics and Computer Science Series, Romania, 48, 1 (2021), 98-111.

[8] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

Authors' addresses:

Mircea Neagu, Alexandru Oană
Transilvania University of Braşov,
Department of Mathematics and Informatics,
Blvd. Iuliu Maniu 50, 500091 Braşov, Romania.
E-mail: mircea.neagu@unitbv.ro , alexandru.oana@unitbv.ro

Vladimir Balan
University Politehnica of Bucharest, Faculty of Applied Sciences,
Department of Mathematics - Informatics,
313 Splaiul Independentei, Bucharest 060042, Romania.
E-mail: vladimir.balan@upb.ro