Non-Abelian monopole problem, the method of geometrical KCC-invariants

N. Krylova, E. Ovsiyuk, V. Balan, V. Red'kov

Abstract. In this paper we apply the geometrical approach based on the Kosambi-Cartan-Chern theory to study the 't-Hooft – Polyakov monopole on the background of the spaces of constant curvature. We consider the two connected problems. The first is the non-Abelian monopole in Bogomol'nyi - Prasad - Sommerfield (BPS) limit on the background of the Euclid, Riemann, and Lobachevsky spaces. The second problem is the doublet of Dirac fermions in the non-Abelian monopole potential. We calculate the first and the second KCC-invariants. The Lagrangian functions related to the relevant system of nonlinear differential equations are found in explicit form. It is shown that the Lagrangian has the arbitrariness up to certain terms, which may be considered as a specific gauge freedom.

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1 Introduction

The quantum-mechanical problems of the particle with spins are usually faced the necessity of solving the complex differential equation systems that can not be solved in terms of hypergeometric functions and require the utilizing of Heun functions or more complicated ones [1]–[21]. Because of that, the search of the analytical solutions for quantum-mechanical problems is straitened or even impossible. It stipulates the demand for new methods of differential system analysis. One of such perspective theory is the Kosambi-Cartan-Chern (KCC) geometrical approach [5]-[6]. It allows to study qualitative behavior of solutions of differential equations from the point of view of the Jacobi stability.

In KCC-approach, one considers a system of second order differential equations of the form

$$\dot{y}^{\,i}(r) + 2Q^i(r,x,y) = 0\,, i \in \overline{1,n}, \tag{1.1}$$

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which corresponds to the Euler – Lagrange equations associated to the second-order extension of some dynamical system with a corresponding Lagrangian L. In (1.1), the $(x^1, \ldots, x^n)$ designates the coordinates, their derivatives in argument r are $y^i = dx^i/dr = \dot{x}^i$, $i \in \overline{1,n}$, and the quantities Q_i are determined by some Lagrangian L in accordance with the formulas

$$Q^i = \frac{1}{4} g^{il} \left(\frac{\partial^2 L}{\partial x^k \partial y^l} y^k - \frac{\partial L}{\partial x^l} + \frac{\partial^2 L}{\partial y^l \partial r} \right), \qquad g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}. \tag{1.2}$$

The first and the second invariants, $\varepsilon^i(r,x,y)$ and $P^i_{\ j}$, are introduced by

$$\begin{aligned} \varepsilon^i &= \tfrac{\partial Q^i}{\partial y^j} y^j - 2Q^i, \\ P^i_{\ j} &= 2\tfrac{\partial Q^i}{\partial x^j} + 2Q^s \tfrac{\partial^2 Q^i}{\partial y^j \partial y^s} - \tfrac{\partial^2 Q^i}{\partial y^j \partial x^s} y^s - \tfrac{\partial Q^i}{\partial y^s} \tfrac{\partial Q^s}{\partial y^j} - \tfrac{\partial^2 Q^i}{\partial y^j \partial r}. \end{aligned} \tag{1.3}$$

The second invariant $P^i_{\ j}$ relates to the Jacobi stability of the differential system. There is an analogy between the equations of geodesic deviation expressed in terms of the Riemannian curvature and the equations of geodesic deviation expressed in terms of the second KCC-invariant:

$$\frac{D^2 \xi^i}{Ds^2} = R^i_{kjl} \frac{dx^k}{ds} \frac{dx^l}{ds} \xi^j = -K^i_j \xi^j, \qquad \frac{D^2 \xi^i}{Dr^2} = P^i_j \xi^j. \tag{1.4}$$

It is known that a pencil of geodesic curves from some point r_0 converges (or diverges) if the real parts of all the eigenvalues of the invariant $P^i{}_j$ are negative (or positive) ones.

Early [11, 13, 14, 12] we applied the KCC-theory to study the particle with a spin and additional electro-magnetic characteristics in the external Coulomb field [11, 13, 14] or on the background of the Schwarzschild spacetime [12].

In this paper, we consider the quantum-mechanical problem of 't-Hooft – Polyakov monopole [8, 7] on the background of the spaces of constant curvature. In the first part, we geometrize the non-Abelian monopole problem in the Bogomol'nyi – Prasad – Sommerfield (BPS) limit [19, 7], calculate and analyze the behavior of the relevant KCC-invariants. In the second part, we apply KCC-approach to study the monopole-caused physical phenomena which arise when an isotopic doublet of Dirac fermions is placed into background of the non-Abelian monopole potential [20, 17].

2 Geometrization of the non-Abelian monopole problem, KCC-invariants

We start with the system of radial Yang – Mills equations in the Bogomol'nyi – Prasad – Sommerfield limit that can be presented as follows [1]

$$\begin{aligned} &\frac{d^2\Phi}{dr^2} + \frac{4}{r}\frac{d\Phi}{dr} - 2e\Phi(2 + er^2K)K - \frac{\dot{\Sigma}}{\Sigma}(\frac{d\Phi}{dr} + \frac{\Phi}{r}) = 0, \\ &\frac{d^2 f}{dr^2} + \frac{4}{r}\frac{df}{dr} - 2e\Phi(2 + er^2K)K - \frac{\dot{\Sigma}}{\Sigma}(\frac{df}{dr} + \frac{f}{r}) = 0, \\ &\frac{d^2K}{dr^2} + \frac{4}{r}\frac{dK}{dr} - e(3 + er^2K)K^2 + \frac{\dot{\Sigma}}{\Sigma}(\frac{dK}{dr} + \frac{2K}{r}) + e\frac{(f^2 - \Phi^2)(1 + er^2K)}{\Sigma^2} = 0, \end{aligned} \tag{2.1}$$

where the quantity Σ determines the metric of the geometrical model:

$$dS^2 = dt^2 + \frac{1}{\Sigma^2(r)}[dx_1^2 + dx_2^2 + dx_3^2], \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

The quantity $\Sigma(r) = 1$ corresponds to Euclidean space, $\Sigma(r) = 1 + r^2/4$ corresponds to Riemannian space, and the $\Sigma(r) = 1 - r^2/4$ is associated with Lobachevsky space; the dot above Σ designates the derivative d/dr. Applying the following notations

$$x^i = \{\Phi(r),\ f(r),\ K(r)\}, \qquad y^i = \frac{d}{dr}x^i(r) = \{\dot{\Phi},\ \dot{f},\ \dot{K}\}$$

and comparing equations (2.1) with (1.1), one finds the relevant expressions for the quantities Q^i:

$$\begin{gathered} Q^1 = -eK\Phi\left(eKr^2+2\right) - \frac{\dot{\Sigma}}{2\Sigma}\Big(\frac{\Phi}{r} + \dot{\Phi}\Big) + \frac{2\dot{\Phi}}{r}, \\ Q^2 = -efK\left(eKr^2+2\right) - \frac{\dot{\Sigma}}{2\Sigma}\Big(\dot{f} + \frac{f}{r}\Big) + \frac{2\dot{f}}{r}, \\ Q^3 = \frac{e\left(f^2-\Phi^2\right)\left(eKr^2+1\right)}{2\Sigma^2} - \frac{1}{2}eK^2\left(eKr^2+3\right) + \frac{\dot{\Sigma}}{2\Sigma}\Big(\dot{K} + \frac{2K}{r}\Big) + \frac{2\dot{K}}{r}. \end{gathered} \tag{2.2}$$

By direct calculation, we get the components of the first KCC-invariant:

$$\begin{gathered} \varepsilon^1 = 2eK\Phi\left(eKr^2+2\right) + \frac{1}{2}\dot{\Phi}\Big(\frac{\dot{\Sigma}}{\Sigma} - \frac{4}{r}\Big) + \frac{\Phi}{r}\frac{\dot{\Sigma}}{\Sigma}, \\ \varepsilon^2 = 2eKf\left(eKr^2+2\right) + \frac{1}{2}\dot{f}\Big(\frac{\dot{\Sigma}}{\Sigma} - \frac{4}{r}\Big) + \frac{f}{r}\frac{\dot{\Sigma}}{\Sigma}, \\ \varepsilon^3 = \frac{e\left(\Phi^2-f^2\right)\left(eKr^2+1\right)}{\Sigma^2} + eK^2\left(eKr^2+3\right) - \frac{\dot{\Sigma}}{2r\Sigma}\Big(4K + r\dot{K}\Big) - \frac{2}{r}\dot{K}, \end{gathered} \tag{2.3}$$

and the components of the second invariant (the entries of the matrix are arranged on columns)

$$P^i_{\ j} = \begin{vmatrix} \frac{\ddot{\Sigma}}{2\Sigma} - \frac{3\left(\dot{\Sigma}\right)^2}{4\Sigma^2} + \frac{\dot{\Sigma}}{r\Sigma} - \frac{2\left(eKr^2+1\right)^2}{r^2} & 0 & -4e\Phi\left(eKr^2+1\right) \\ 0 & \frac{\ddot{\Sigma}}{2\Sigma} - \frac{3\left(\dot{\Sigma}\right)^2}{4\Sigma^2} + \frac{\dot{\Sigma}}{r\Sigma} - \frac{2\left(eKr^2+1\right)^2}{r^2} & -4ef\left(eKr^2+1\right) \\ -\frac{2e\Phi\left(eKr^2+1\right)}{\Sigma^2} & \frac{2ef\left(eKr^2+1\right)}{\Sigma^2} & -\frac{2e\Phi\left(eKr^2+1\right)}{\Sigma^2}\,\frac{e^2r^2(f^2-\Phi^2)}{\Sigma^2} + \frac{\dot{\Sigma}^2}{4\Sigma^2} - 3eK\left(eKr^2+2\right) - \frac{2}{r^2} - \frac{\ddot{\Sigma}}{2\Sigma} \end{vmatrix}. \tag{2.4}$$

The eigenvalues Λ_i $(i = 1, 2, 3)$ of the second invariant are given by the formulas

$$\Lambda_1 = -\frac{2\left(eKr^2+1\right)^2}{r^2} + \frac{\dot{\Sigma}}{r\Sigma} + \frac{\ddot{\Sigma}}{2\Sigma} - \frac{3\dot{\Sigma}^2}{4\Sigma^2}, \tag{2.5}$$

$$\Lambda_{2,3} = -\frac{1}{4\Sigma^2}\Big(2e^2r^2\left(\Phi^2-f^2\right) + \dot{\Sigma}^2\Big) - \frac{1}{2r^2}\left(5eKr^2\left(eKr^2+2\right)+4\right) + \frac{\dot{\Sigma}}{2r\Sigma}$$

$$\pm\frac{1}{2r\Sigma^2}\Big[\Big(e^2r^3\Big(f^2-\Phi^2\Big)+r\dot{\Sigma}^2\Big)^2+r^2\Sigma^2\Big(2e^2\Big(\Phi^2-f^2\Big)\Big(17eKr^2\Big(eKr^2+2\Big)+16\Big)+\ddot{\Sigma}^2\Big)$$
$$+\Sigma^2\Big(\dot{\Sigma}^2\Big(1-2eKr^2\Big(eKr^2+2\Big)\Big)+2r\dot{\Sigma}\ddot{\Sigma}\Big)+2r\Sigma\Big(r\ddot{\Sigma}+\dot{\Sigma}\Big)\Big(e^2r^2\Big(\Phi^2-f^2\Big)-\Big(\dot{\Sigma}\Big)^2\Big)$$
$$+e^2K^2r^2\Sigma^4\left(eKr^2+2\right)^2+2eKr\Sigma^3\left(eKr^2+2\right)\Big(r\ddot{\Sigma}+\dot{\Sigma}\Big)\Big]^{1/2}. \tag{2.6}$$

The next step is to construct a Lagrangian function L for the phase space $\{x_i,\ y_i\}$, such that the formulas for coefficients Q^i (1.2) hold true, and the dynamics of the system is defined by the equations (2.1). We will search for the function L in the form

$$L=g_{ij}(r)y^iy^j+b_j(r,x)y^j\ , \tag{2.7}$$

assuming the diagonal structure of the metrical tensor:

$$g_{ij}(r)=\begin{vmatrix} g_{11}(r) & 0 & 0\\ 0 & g_{22}(r) & 0\\ 0 & 0 & g_{33}(r)\end{vmatrix}. \tag{2.8}$$

In this case, by substituting (2.7)–(2.8) into (1.2), we derive

$$\begin{aligned} Q^1&=\frac{1}{4g_{11}}\Big(2\dot{g}_{11}y^1+\frac{\partial b_1}{\partial r}+\Big(\frac{\partial b_1}{\partial x^2}-\frac{\partial b_2}{\partial x^1}\Big)y^2+\Big(\frac{\partial b_1}{\partial x^3}-\frac{\partial b_3}{\partial x^1}\Big)y^3\Big),\\ Q^2&=\frac{1}{4g_{22}}\Big(2\dot{g}_{22}y^2+\frac{\partial b_2}{\partial r}+\Big(\frac{\partial b_2}{\partial x^1}-\frac{\partial b_1}{\partial x^2}\Big)y^1+\Big(\frac{\partial b_2}{\partial x^3}-\frac{\partial b_3}{\partial x^2}\Big)y^3\Big),\\ Q^3&=\frac{1}{4g_{33}}\Big(2\dot{g}_{33}y^3+\frac{\partial b_3}{\partial r}+\Big(\frac{\partial b_3}{\partial x^1}-\frac{\partial b_1}{\partial x^3}\Big)y^1+\Big(\frac{\partial b_3}{\partial x^2}-\frac{\partial b_2}{\partial x^3}\Big)y^2\Big).\end{aligned} \tag{2.9}$$

Equating the terms from (2.2) to the corresponding terms from (2.9) and taking in mind that $x^1=\Phi$, $x^2=f$, $x^3=K$, we obtain the system of equations with respect to the variables $g_{ij}(r)$ and $b_j(r,x)$:

$$\begin{cases} \frac{\dot{g}_{11}}{g_{11}}=\frac{4}{r}-\frac{\dot{\Sigma}}{\Sigma}, \quad \frac{\dot{g}_{22}}{g_{22}}=\frac{4}{r}-\frac{\dot{\Sigma}}{\Sigma}, \quad \frac{\dot{g}_{33}}{g_{33}}=\frac{4}{r}-\frac{\dot{\Sigma}}{\Sigma},\\ \frac{\partial b_1}{\partial f}-\frac{\partial b_2}{\partial \Phi}=0, \quad \frac{\partial b_1}{\partial K}-\frac{\partial b_3}{\partial \Phi}=0, \quad \frac{\partial b_2}{\partial K}-\frac{\partial b_3}{\partial f}=0,\\ \frac{1}{4g_1}\frac{\partial b_1}{\partial r}=-e\Phi K\left(2+er^2K\right)-\frac{1}{2r}\frac{\dot{\Sigma}}{\Sigma}\Phi,\\ \frac{1}{4g_2}\frac{\partial b_2}{\partial r}=-efK\left(2+er^2K\right)-\frac{1}{2r}\frac{\dot{\Sigma}}{\Sigma}f,\\ \frac{1}{2g_3}\frac{\partial b_3}{\partial r}=-eK^2\left(3+er^2K\right)+\frac{2}{r}\frac{\dot{\Sigma}}{\Sigma}K-\frac{e(\Phi^2-f^2)(1+er^2K)}{\Sigma^2}. \end{cases} \tag{2.10}$$

Its solution is given by the formulas

$$g_{11}=\frac{C_1r^4}{\Sigma},\quad g_{22}=-\frac{C_1r^4}{\Sigma},\quad g_{33}=2C_1r^4\Sigma,$$

$$b_1=B_1(\Phi,f,K)-2C_1\Phi\int\frac{4eKr^4\Sigma(r)+2e^2K^2r^6\Sigma(r)+r^3\dot{\Sigma}(r)}{\Sigma(r)^2}dr,$$

$$b_2=B_2(\Phi,f,K)+2C_1f\int\frac{4eKr^4\Sigma(r)+2e^2K^2r^6\Sigma(r)+r^3\dot{\Sigma}(r)}{\Sigma(r)^2}dr,$$

$$
\begin{aligned}
b_3 = B_3(\Phi, f, K) - 4C_1 \cdot \int \tfrac{1}{\Sigma(r)^2}[e(\Phi^2 - f^2)r^4(1 + eKr^2) \\
+eK^2\Sigma(r)^2 r^4(3 + eKr^2) - 2Kr^3\Sigma(r)\dot{\Sigma}(r)]dr,
\end{aligned}
\tag{2.11}
$$

where C_1 is an arbitrary constant, and the functions $B_i(\Phi, f, K)$ obey the following constraints

$$
\frac{\partial B_1}{\partial f} - \frac{\partial B_2}{\partial \Phi} = 0, \qquad \frac{\partial B_2}{\partial K} - \frac{\partial B_3}{\partial f} = 0, \qquad \frac{\partial B_1}{\partial K} - \frac{\partial B_3}{\partial \Phi} = 0.
\tag{2.12}
$$

In accordance with the known theorem, from (2.12) we conclude that this 3-dimensional vector field B_j can be presented as a gradient of some scalar function $\varphi(x^1, x^2, x^3)$:

$$
B_1 = \frac{\partial \varphi}{\partial x^1}, \qquad B_2 = \frac{\partial \varphi}{\partial x^2}, \qquad B_3 = \frac{\partial \varphi}{\partial x^3}, \qquad B_i = \mathrm{grad}\, \varphi\,.
\tag{2.13}
$$

Therefore, there exists some gauge freedom in choosing the Lagrange function L which refers to the considered differential system.

2.1 The Euclidean space

Now, we consider the case of Euclidean space, $\Sigma(r) = 1$, $\dot{\Sigma}(r) = 0$. The expressions for Q^i become simpler

$$
\begin{aligned}
Q^1 = -eK\Phi\left(eKr^2 + 2\right) + \frac{2\dot{\Phi}}{r}, \qquad Q^2 = -efK\left(eKr^2 + 2\right) + \frac{2\dot{f}}{r}, \\
Q^3 = \frac{1}{2}e\left(f^2 - \Phi^2\right)\left(eKr^2 + 1\right) - \frac{1}{2}eK^2\left(eKr^2 + 3\right) + \frac{2\dot{K}}{r}.
\end{aligned}
\tag{2.14}
$$

The first and second invariants are get simpler, as well:

$$
\begin{aligned}
\varepsilon^1 = 2eK\Phi\left(eKr^2 + 2\right) - \frac{2}{r}\dot{\Phi}, \qquad \varepsilon^2 = 2eKf\left(eKr^2 + 2\right) - \frac{2}{r}\dot{f}, \\
\varepsilon^3 = e\left(\Phi^2 - f^2\right)\left(eKr^2 + 1\right) + eK^2\left(eKr^2 + 3\right) - \frac{2}{r}\dot{K};
\end{aligned}
\tag{2.15}
$$

(2.16)

$$
P^i_{\ j} = \left|\begin{array}{ccc}
-\frac{2\left(eKr^2+1\right)^2}{r^2} & 0 & -4e\Phi\left(eKr^2 + 1\right) \\
0 & -\frac{2\left(eKr^2+1\right)^2}{r^2} & -4ef\left(eKr^2 + 1\right) \\
-2e\Phi\left(eKr^2 + 1\right) & 2ef\left(eKr^2 + 1\right) & e^2r^2(f^2 - \Phi^2 - 3K^2) - 6eK - \frac{2}{r^2}
\end{array}\right|.
$$

The eigenvalues Λ_i of the second invariant are given by the formulas

$$
\begin{aligned}
&\Lambda_1 = -\tfrac{2\left(eKr^2+1\right)^2}{r^2}, \\
&\Lambda_{2,3} = -\tfrac{2}{r^2} - 5eK + \tfrac{1}{2}e^2r^2(-f^2 + 5K^2 + \Phi^2) \\
&\qquad \pm \left[e^2f^4r^3 + K^2(2 + er^2K)^2 - 2f^2(16 + 17er^2K(2 + er^2K))\right. \\
&\qquad \left. + \Phi^2(32 + 68er^2K - 2e^2r^4(f^2 - 17K^2)) + e^2r^4\Phi^4\right]^{1/2}.
\end{aligned}
\tag{2.17}
$$

The Lagrangian function L has the following explicit form

$$L = g_{ij}(r)y^i y^j + b_j(r,x)y^j \,, \tag{2.18}$$

where the metric tensor is

$$g_{ij} = \begin{vmatrix} C_1 r^4 & 0 & 0 \\ 0 & -C_1 r^4 & 0 \\ 0 & 0 & 2C_1 r^4 \end{vmatrix},$$

and the functions $b_j(r,x)$ are determined by the expressions

$$\begin{aligned} b_1 &= B_1(\Phi,f,K) - 4eC_1\Phi K r^5\Big(\tfrac{2}{5} + \tfrac{1}{7}er^2K\Big),\\ b_2 &= B_2(\Phi,f,K) + 4eC_1 f K r^5\Big(\tfrac{2}{5} + \tfrac{1}{7}er^2K\Big),\\ b_3 &= B_3(\Phi,f,K) - 4eC_1 r^5\Big(\Big(\tfrac{1}{5} + \tfrac{1}{7}er^2K\Big)\Big(\Phi^2 - f^2\Big) + K^2\Big(\tfrac{1}{7}er^2K + \tfrac{3}{5}\Big)\Big). \end{aligned} \tag{2.19}$$

2.2 The Riemannian space

Let us consider a case of the spherical Riemannian space:

$$\Sigma(r) = 1 + r^2/4, \quad \dot{\Sigma}(r) = r/2. \tag{2.20}$$

Here, the quantities Q^i are given by the formulas

$$\begin{aligned} Q^1 &= -eK\Phi\left(eKr^2+2\right) + \frac{2\dot{\Phi}}{r} - \frac{\Phi + r\dot{\Phi}}{4+r^2}, \qquad Q^2 = -efK\left(eKr^2+2\right) + \frac{2\dot{f}}{r} - \frac{f + r\dot{f}}{4+r^2},\\ Q^3 &= \frac{8e\left(f^2-\Phi^2\right)\left(eKr^2+1\right)}{(4+r^2)^2} - \frac{1}{2}eK^2\left(eKr^2+3\right) + \frac{2\dot{K}}{r} + \frac{2K + r\dot{K}}{4+r^2}\,. \end{aligned} \tag{2.21}$$

The first and the second invariants are

$$\begin{aligned} \varepsilon^1 &= 2eK\Phi\left(eKr^2+2\right) + \frac{2\Phi}{4+r^2} - \frac{8+r^2}{r(4+r^2)}\dot{\Phi},\\ \varepsilon^2 &= 2eKf\left(eKr^2+2\right) + \frac{2f}{4+r^2} - \frac{8+r^2}{r(4+r^2)}\dot{f},\\ \varepsilon^3 &= \frac{16e\left(\Phi^2-f^2\right)\left(eKr^2+1\right)}{(4+r^2)^2} + eK^2\left(eKr^2+3\right) - \frac{4K}{4+r^2} - \frac{8+3r^2}{r(4+r^2)}\dot{K}; \end{aligned}$$

$$P^i_j = \begin{vmatrix} -\frac{2\left(eKr^2+1\right)^2}{r^2} + \frac{12}{(4+r^2)^2} & 0 & -4e\Phi\left(eKr^2+1\right) \\ 0 & -\frac{2\left(eKr^2+1\right)^2}{r^2} + \frac{12}{(4+r^2)^2} & -4ef\left(eKr^2+1\right) \\ -32e\Phi\frac{eKr^2+1}{(4+r^2)^2} & 32ef\frac{eKr^2+1}{(4+r^2)^2} & \tau \end{vmatrix},$$

where $\tau = -\frac{2}{r^2} - \frac{4}{(4+r^2)^2} - 6eK - 3e^2r^2K^2 + \frac{16e^2r^2(f^2-\Phi^2)}{(4+r^2)^2}$. The eigenvalues Λ_i of the second invariant are given by the formulas

(2.22)

$$\Lambda_1 = -\frac{2(eKr^2+1)^2}{r^2} + \frac{12}{(4+r^2)^2},$$

$$\Lambda_{2,3} = \frac{1}{2r^2(r^2+4)^2} \{r^2e^2(16r^2(f^2-\Phi^2) - 5K^2(r^3+4r)^2) - 10eKr^2(r^2+4)^2$$
$$-4r^2(r^2+6) - 64 \pm r^2 \left[e^4K^4(r^3+4r)^4 + 64Ke(r^2+4)^2(17e^2r^2(\Phi^2-f^2)+1)\right.$$
$$+4e^2K^2(r^2+4)^2(r^4(136e^2(\Phi^2-f^2)+1) + 16(r^2+1))$$
$$\left.+256e^2(\Phi^2-f^2)(r^4(e^2(\Phi^2-f^2)+2) + 18r^2 + 32) + 4e^3K^3r^2(r^2+4)^4 + 256\right]^{1/2}\}.$$

The Lagrangian function L can be found in the form

$$L = g_{ij}(r)y^iy^j + b_j(r,x)y^j\,, \tag{2.23}$$

where the metric tensor is

$$g_{ij} = C_1 \begin{vmatrix} \frac{r^4}{4+r^2} & 0 & 0 \\ 0 & -\frac{r^4}{4+r^2} & 0 \\ 0 & 0 & \frac{1}{8}r^4(4+r^2) \end{vmatrix},$$

and the functions $b_j(r,x)$ are determined by the following expressions

$$\begin{aligned} b_1 &= B_1(\Phi,f,K) - 4C_1\Phi\Big\{\frac{2r}{4+r^2} + \frac{1}{5}e^2r^5K^2 - \frac{2}{3}er^3K(2eK-1) \\ &\quad +r(4eK-1)^2 + (3-16eK+32e^2K^2)\arctan\frac{2}{r}\Big\}, \\ b_2 &= B_2(\Phi,f,K) + 4C_1f\Big\{\frac{2r}{4+r^2} + \frac{1}{5}e^2r^5K^2 - \frac{2}{3}er^3K(2eK-1) \\ &\quad +r(4eK-1)^2 + (3-16eK+32e^2K^2)\arctan\frac{2}{r}\Big\}, \\ b_3 &= B_3(\Phi,f,K) - \frac{1}{4}C_1\Big\{\frac{1}{9}e^2r^9K^3 + \frac{4}{5}r^5K(3eK-1) + \frac{1}{7}er^7K^2(3+4eK) \\ &\quad +\frac{16}{15}e(\Phi^2-f^2)\Big(3er^5K - 5\Big(-12r + r^3 - 24\arctan\frac{2}{r}\Big)\Big(4eK-1\Big)\Big)\Big\}. \end{aligned} \tag{2.24}$$

2.3 The Lobachevsky space

Let us consider the Lobachevsky model:

$$\Sigma(r) = 1 - r^2/4, \quad \dot{\Sigma}(r) = -r/2. \tag{2.25}$$

In this case, the quantities Q^i equal

(2.26)

$$Q^1 = -eK\Phi\left(eKr^2+2\right) + \frac{2\dot{\Phi}}{r} - \frac{\Phi + r\dot{\Phi}}{r^2-4}, \qquad Q^2 = -efK\left(eKr^2+2\right) + \frac{2\dot{f}}{r} - \frac{f + r\dot{f}}{r^2-4},$$

$$Q^3 = \frac{8e\left(f^2-\Phi^2\right)\left(eKr^2+1\right)}{(r^2-4)^2} - \frac{1}{2}eK^2\left(eKr^2+3\right) + \frac{2\dot{K}}{r} + \frac{2K + r\dot{K}}{r^2-4}\,.$$

The first and second invariants are

$$\varepsilon^1 = 2eK\Phi\left(eKr^2+2\right) + \frac{2\Phi}{r^2-4} - \frac{r^2-8}{r(r^2-4)}\dot{\Phi},$$

$$\varepsilon^2 = 2eKf\left(eKr^2+2\right) + \frac{2f}{r^2-4} - \frac{r^2-8}{r(r^2-4)}\dot{f}, \tag{2.27}$$

$$\varepsilon^3 = \frac{16e\left(\Phi^2-f^2\right)\left(eKr^2+1\right)}{(r^2-4)^2} + eK^2\left(eKr^2+3\right) - \frac{4K}{r^2-4} - \frac{3r^2-8}{r(r^2-4)}\dot{K};$$

(2.28)

$$P^i_j = \begin{vmatrix} -\frac{2\left(eKr^2+1\right)^2}{r^2} - \frac{12}{(r^2-4)^2} & 0 & -4e\Phi\left(eKr^2+1\right) \\ 0 & -\frac{2\left(eKr^2+1\right)^2}{r^2} - \frac{12}{(r^2-4)^2} & -4ef\left(eKr^2+1\right) \\ -32e\Phi\frac{eKr^2+1}{(r^2-4)^2} & 32ef\frac{eKr^2+1}{(r^2-4)^2} & \tau \end{vmatrix},$$

where $\tau = -\frac{2}{r^2} + \frac{4}{(r^2-4)^2} - 6eK - 3e^2r^2K^2 + \frac{16e^2r^2(f^2-\Phi^2)}{(r^2-4)^2}$. The eigenvalues Λ_i of the second invariant are given by the formulas

(2.29)

$$\Lambda_1 = -\frac{2(eKr^2+1)^2}{r^2} - \frac{12}{(r^2-4)^2},$$

$$\begin{aligned}\Lambda_{2,3} = \frac{1}{2r^2(r^2-4)^2}\{&r^2e^2(16r^2(f^2-\Phi^2) - 5K^2(r^3-4r)^2) - 10eKr^2(r^2-4)^2 \\ &-4r^2(r^2-6) - 64 \pm r^2[e^4K^4(r^3-4r)^4 + 64Ke(r^2-4)^2(17e^2r^2(\Phi^2-f^2)-1) \\ &+4e^2K^2(r^2-4)^2(r^4(136e^2(\Phi^2-f^2)+1) - 16(r^2-1)) \\ &+256e^2(\Phi^2-f^2)(r^4(e^2(\Phi^2-f^2)+2) - 18r^2+32) + 4e^3K^3r^2(r^2-4)^4 + 256]^{1/2}\}.\end{aligned}$$

The Lagrangian function L is

$$L = g_{ij}(r)y^iy^j + b_j(r,x)y^j\,, \tag{2.30}$$

where

$$g_{ij} = C_1\begin{vmatrix} -\frac{r^4}{r^2-4} & 0 & 0 \\ 0 & \frac{r^4}{r^2-4} & 0 \\ 0 & 0 & -\frac{1}{8}r^4(r^2-4) \end{vmatrix},$$

and the functions $b_j(r,x)$ are determined by the formulas:

$$\begin{aligned} b_1 &= B_1(\Phi,f,K) + 4C_1\Phi\Big\{-\frac{2r}{r^2-4} + \frac{1}{5}e^2r^5K^2 + \frac{2}{3}er^3K(2eK+1) \\ &\qquad +r(4eK+1)^2 - (3+16eK+32e^2K^2)\text{arctanh}\frac{r}{2}\Big\}, \\ b_2 &= B_2(\Phi,f,K) - 4C_1f\Big\{-\frac{2r}{r^2-4} + \frac{1}{5}e^2r^5K^2 + \frac{2}{3}er^3K(2eK+1) \\ &\qquad +r(4eK+1)^2 - (3+16eK+32e^2K^2)\text{arctanh}\frac{r}{2}\Big\}, \\ b_3 &= B_3(\Phi,f,K) + \frac{1}{4}C_1\Big\{\frac{1}{9}e^2r^9K^3 - \frac{4}{5}r^5K(3eK+1) + \frac{1}{7}er^7K^2(3-4eK) \\ &\qquad +\frac{16}{15}e(\Phi^2-f^2)\Big(3er^5K + 5\Big(12r+r^3-24\text{arctanh}\frac{r}{2}\Big)\Big(4eK+1\Big)\Big)\Big\}. \end{aligned} \tag{2.31}$$

From the physical point of view the state of a quantum-mechanical system can be qualitatively described by the behavior of the radial parts of wavefunctions in the vicinity of physically meaningful points: $r \to 0$, $r \to \infty$ and singular points of the system if any. The Jacobi stability allows to estimate the divergence(convergence) of the radial functions near these points. To do this, however, one should know the explicit forms of the functions Φ, f, K entering the eigenvalues (2.17), (2.22), (2.29) and corresponding Lagrangians. Further we study their behavior in the Bogomol'nyi-Prasad-Sommerfield limit.

2.4 The BPS-monopole solution in Euclidean space

The following exact solutions of the radial Yang – Mills equations in the BPS-limit are known (for simplicity we restrict ourselves to the pure monopole case, by setting $f(r) = 0$):

$$\ddot{\Phi} + \frac{4}{r}\dot{\Phi} - 2e\Phi\,(2 + er^2K)\,K = 0\,,$$
$$\ddot{K} + \frac{4\dot{K}}{r} - e\Phi^2(1 + er^2K) - eK^2(3 + er^2K) = 0. \tag{2.32}$$

Instead of $\Phi(r)$ and $K(r)$, it is more convenient to use the variables f_1 and f_2:

$$1 + er^2K(r) = rf_1(r),\ 1 + er^2\Phi(r) = rf_2(r). \tag{2.33}$$

In this way, we transform equations (2.32) to the simpler form

$$\begin{aligned} 2(\dot{f}_2 + f_1^2) + r(\ddot{f}_2 - 2f_1^2 f_2) = 0,\\ 2(\dot{f}_1 + f_1 f_2) + r(\ddot{f}_1 - f_1 f_2^2 - f_1^3) = 0. \end{aligned} \tag{2.34}$$

In terms of these variables, the quantities Q^i (1.2) and the first KCC-invariants ε (1.3) read

$$Q_f^1 = -\frac{1}{2}f_1(f_1^2 + f_2^2) + \frac{1}{r}(\dot{f}_1 + f_1 f_2),\ \ Q_f^2 = \frac{1}{r}(f_1^2 + \dot{f}_2) - f_1^2 f_2; \tag{2.35}$$

$$\varepsilon_f^1 = -\frac{2f_2 f_1}{r} + f_1^3 + f_2^2 f_1 - \frac{\dot{f}_1}{r},\ \ \varepsilon_f^2 = -\frac{2f_1^2}{r} + 2f_2 f_1^2 - \frac{\dot{f}_2}{r}. \tag{2.36}$$

The second invariant P_j^i (1.3) is given by the formula

$$P_j^i = \begin{vmatrix} (\frac{2}{r}f_2 - f_2^2 - 3f_1^2) & (\frac{2}{r}f_1 - 2f_1 f_2) \\ (\frac{4}{r}f_1 - 4f_1 f_2) & -2f_1^2 \end{vmatrix}. \tag{2.37}$$

The eigenvalues of the second invariant are

$$\Lambda_{f\ (1,2)} = \frac{1}{2r}(\pm\sqrt{f_1^4 r^2 + f_1^2(34 f_2 r(f_2 r - 2) + 32) + f_2^2(f_2 r - 2)^2} - (5f_1^2 + f_2^2)r + 2f_2). \tag{2.38}$$

To analyze the behavior of these eigenvalues, we shall apply the known solutions of equations (2.34) in the BPS-limit. To this end, we search for functions f_1 and f_2 which obey the following 4 equations:

$$\begin{gathered} \frac{df_2}{dr} + f_1^2 = 0\,, \qquad \frac{df_2}{dr} - 2\,f_1^2\,f_2 = 0\,, \\ \frac{df_1}{dr} + f_1\,f_2 = 0\,, \quad \frac{df_1}{dr} - f_1\,f_2^2 - f_1^3 = 0\,. \end{gathered} \tag{2.39}$$

It can be readily seen that the second and the fourth equations are consequences of the first and the third ones, so we have only two independent equations:

$$\begin{cases} \frac{df_1}{dr} = -f_1 f_2 \\ \frac{df_2}{dr} = -f_1^2 \end{cases} \Rightarrow \begin{cases} f_2 = -\frac{1}{f_1}\frac{df_1}{dr} \\ \frac{d}{dr}\left(\frac{1}{f_1}\frac{df_1}{dr}\right) = f_1^2. \end{cases} \tag{2.40}$$

Therefore, the problem reduces to a single equation

$$\left(\frac{f_1'}{f_1}\right)' = f_1^2 \qquad \Longrightarrow (\ln f_1)'' = f_1^2\,,$$

the last is equivalent to

$$\frac{d}{dr}\,\left[\,(\ln f_1)'\,\right]^2 = \frac{d}{dr}\,f_1^2\,,$$

whence we obtain

$$(\ln f_1)' = \pm\sqrt{f_1^2 + c} \;\;\Longrightarrow\;\; \int \frac{d\,f_1}{f_1\sqrt{c+f_1^2}} = \pm\,(r + \text{const})\,.$$

Depending on the sign of the constant c, we have three different types of solutions:

$$\begin{array}{lll} I \quad c=0 & f_1 = \pm\dfrac{A}{Ar+B}\,, & f_2 = \dfrac{A}{Ar+B}\,; \\ II \quad c<0 & f_1 = \pm\dfrac{A}{\text{sh}\,(Ar+B)}\,, & f_2 = \dfrac{A}{\text{th}\,(Ar+B)}\,; \\ III \quad c>0 & f_1 = \pm\dfrac{A}{\sin\,(Ar+B)}\,, & f_2 = \dfrac{A}{\text{tg}\,(Ar+B)}\,, \end{array} \tag{2.41}$$

where A and B are arbitrary constants. For physical reasons, we should set $B = 0$, and then the solutions exhibit only one singular point, $r = 0$. For generality, we follow below all the three possibilities: $B = 0, B < 0$, and $B > 0$.

It should be noted that only the relative sign of the parameters A and B is significant; for this reason, we shall consider only the two cases:

$$A > 0, B > 0, \quad \text{and} \quad A > 0, B < 0\,.$$

In case I, the eigenvalues (2.38) have the form

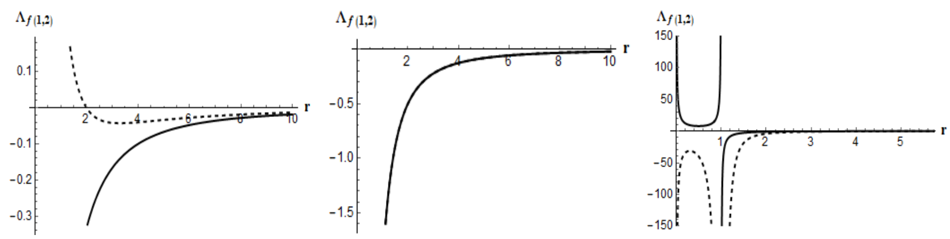
$$\Lambda_{f\,1} = -\frac{2A}{Ar^2+Br}, \quad \Lambda_{f\,2} = -\frac{2A(Ar-2B)}{r(Ar+B)^2}; \tag{2.42}$$

$$B \neq 0, \quad r \to 0, \quad \Lambda_{f\,1} \to -2A/Br, \quad \Lambda_{f\,2} \to +4A/Br; \quad r \to \infty, \quad \Lambda_{f\,1} \to -2/r^2, \quad \Lambda_{f\,2} \to -2/r^2. \tag{2.43}$$

When $B \neq 0$, the behavior of $\Lambda_{f\,(1,2)}(r)$ (2.42) is illustrated in figure 1a,c.

When $B = 0$, the solutions and the eigenvalues simplify,

$$f_1 = \pm\frac{1}{r}, \quad f_2 = \frac{1}{r}, \quad \Lambda_{f\,1} = \Lambda_{f\,2} = -\frac{2}{r^2} < 0\,.$$



a) b) c)

Figure 1: The dependencies of eigenvalues $\Lambda_{f\,1}(r)$ (solid line) and $\Lambda_{f\,2}$ (dashed line) in Euclidean space for the solution of the type I at $B = 1$ (a), $B = 0$ (b) and $B = -1$ (c). $A = 1$.

Now, let us consider the solutions of type II. In this case, the eigenvalues (2.38) take the form (while using the notation csch $= 1/\sinh$)

$$\begin{aligned} \Lambda_{f\,(1,2)} = \pm\frac{A}{2r}\Big[A^2r^2\mathrm{csch}^4(Ar+B) + \coth^2(Ar+B)\left[Ar\coth(Ar+B)-2\right]^2 \\ + \left[34Ar\coth(Ar+B)\left[Ar\coth(Ar+B)-2\right]+32\right]\mathrm{csch}^2(Ar+B)\Big]^{1/2} \\ -\frac{1}{2}A^2\left[6\mathrm{csch}^2(Ar+B)+1\right] + \frac{A\coth(Ar+B)}{r}. \end{aligned} \tag{2.44}$$

They become much simpler when $B = 0$:

$$\Lambda_{f\,(1,2)} = \pm\frac{A}{2r}\Big[A^2r^2 + \mathrm{csch}^2(Ar)[2Ar\coth(Ar)[-\cosh(2Ar)$$

$$+18Ar\coth(Ar) - 35] + 36] + 4\Big]^{1/2} - 3A^2\mathrm{csch}^2(Ar) - \frac{A^2}{2} + \frac{A\coth(Ar)}{r}.$$

Let us specify their behavior near the points $r = 0, \infty$. When $B = 0$, we have

$$r \to 0 \quad \Lambda_{f\,1} = \Lambda_{f\,2} = -\frac{2}{r^2}, \qquad r \to \infty \quad \Lambda_{f\,1} = -A^2, \quad \Lambda_{f\,2} = \frac{A}{r}. \tag{2.45}$$

If $B \neq 0$, we have

$$r \to 0 \qquad \Lambda_{f\ (1,2)} = \frac{A}{r}\left(\coth B \pm \sqrt{\coth^2 B + 8\mathrm{csch}^2 B}\right),$$
$$r \to \infty \quad \Lambda_{f\ 1} = -A^2, \quad \Lambda_{f\ 2} = \frac{A}{r}.$$

In case III, the eigenvalues (2.38)

$$\begin{aligned} \Lambda_{f\ (1,2)} = \pm\frac{A}{2r}\Big[A^2r^2\csc^4(Ar+B) + \cot^2(Ar+B)\left[Ar\cot(Ar+B)-2\right]^2 \\ + \left[34Ar\cot(Ar+B)\left[Ar\cot(Ar+B)-2\right]+32\right]\csc^2(Ar+B)\Big]^{1/2} \\ -3A^2\csc^2(Ar+B) + \frac{A^2}{2} + \frac{A\cot(Ar+B)}{r}; \end{aligned} \tag{2.46}$$

get simpler if $B = 0$:

$$\Lambda_{f\ (1,2)} = \pm\frac{A}{2r}\Big[A^2r^2\csc^4 Ar + \cot^2 Ar\left[Ar\cot Ar - 2\right]^2$$
$$+\left[34Ar\cot Ar\left[Ar\cot Ar-2\right]+32\right]\csc^2 Ar\Big]^{1/2} - 3A^2\csc^2 Ar + \frac{A^2}{2} + \frac{A\cot Ar}{r}.$$

The expression of Lagrangian is found as follows:

$$\begin{aligned} L = \left(2\dot{f}_1^2 + \dot{f}_2^2\right)r^2 + \frac{2}{3}f_1r^2\Big(-2\left(f_1^2+f_2^2\right)\dot{f}_1 r \\ + f_1\dot{f}_2\left(3-2f_2r\right) + 6f_2\dot{f}_1\Big) \\ + \dot{f}_1 B_1\left(f_1,f_2\right) + \dot{f}_2 B_2\left(f_1,f_2\right); \end{aligned} \tag{2.47}$$

where the functions B_1 and B_2 obey the following restriction

$$\frac{\partial B_1}{\partial f_2} - \frac{\partial B_2}{\partial f_1} = 0.$$

So, the vector field B_1, B_2 can be presented as a gradient of some scalar function $B_i = \mathrm{grad}\,\varphi(f_1, f_2)$.

The substitution of explicit solutions (2.41) for f_1 and f_2 leads to the following results:

$$I \quad L = \frac{A^4r^2(Ar-3B)}{(Ar+B)^5},$$

$$II \quad L = \frac{1}{6}A^4r^2\mathrm{csch}^5(Ar+B)(2Ar(11\cosh(Ar+B)$$
$$+\cosh(3(Ar+B))) - 3(3\sinh(Ar+B) + \sinh(3(Ar+B)))),$$

$$III \quad L = \frac{1}{6}A^4r^2\csc^5(Ar+B)(2Ar(11\cos(Ar+B)$$
$$+\cos(3(Ar+B))) - 3(3\sin(Ar+B) + \sin(3(Ar+B)))).$$

When $B = 0$, the formulas becomes simpler

$$I \quad L = \frac{1}{r^2},$$

$$II \quad L = \frac{1}{6} A^4 r^2 \operatorname{csch}^5(Ar)(2Ar(11\cosh(Ar) + \cosh(3Ar)) - 3(3\sinh(Ar) + \sinh(3Ar))),$$

$$III \quad L = \frac{1}{6} A^4 r^2 \csc^5(Ar)(2Ar(11\cos(Ar) + \cos(3Ar)) - 3(3\sin(Ar) + \sin(3Ar))).$$

Their behavior in the physically meaningful case I is illustrated by figure 2.

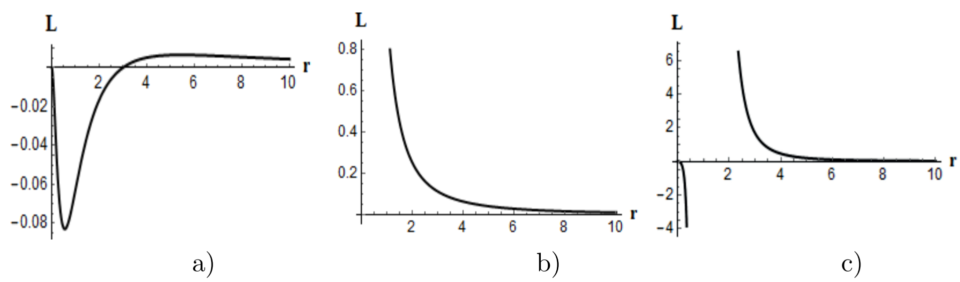


Figure 2: The dependencies of Lagrangians on radial coordinate r, in Euclidean space for the solution of the type I; $A = 1$, $B = +1$ (a), $B = 0$ (b) and $B = -1$ (c).

3 Geometrizing the doublet problem

3.1 Non-relativistic approximation, the case $j = 0$

The radial system of equations which has been obtained from Dirac equation for the quantum-mechanical problem of an isotopic doublet of Dirac particles on the background of the external non-Abelian field W, reads as follows (in non-relativistic approximation; for the case of the quantum number of the total angular momentum $j = 0$) [20] [17] [16]:

$$\Big(\frac{d}{dr} + \frac{W}{r}\Big) f + 2M\, g = 0\,, \quad \Big(\frac{d}{dr} - \frac{W}{r}\Big) g - E\, f = 0\,.$$

We differentiate these equations, and substitute the expressions for the derivatives f' and g' from the initial equations. In this way, we get

$$\begin{aligned} &\Big[\frac{d^2}{dr^2} + 2ME + \Big(\frac{W}{r}\Big)' - \frac{W^2}{r^2}\Big] f(r) = 0, \\ &\Big[\frac{d^2}{dr^2} + 2ME - \Big(\frac{W}{r}\Big)' - \frac{W^2}{r^2}\Big] g(r) = 0. \end{aligned} \tag{3.1}$$

It is easy to find the explicit form of the coefficients Q^i (we apply the notations $x^1 \equiv f$, $x^2 \equiv g$):

$$Q^1 = \Big[ME + \frac{1}{2}\Big(\frac{W}{r}\Big)' - \frac{1}{2}\frac{W^2}{r^2}\Big]x^1, \tag{3.2}$$

$$Q^2 = \Big[ME - \frac{1}{2}\Big(\frac{W}{r}\Big)' - \frac{1}{2}\frac{W^2}{r^2}\Big]x^2. \tag{3.3}$$

The calculation of the second invariant gives

$$P^i_j = \begin{vmatrix} 2ME + \big(\frac{W}{r}\big)' - \frac{W^2}{r^2} & 0 \\ 0 & 2ME - \big(\frac{W}{r}\big)' - \frac{W^2}{r^2} \end{vmatrix} . \tag{3.4}$$

The diagonal elements of the matrix (3.4) are the eigenvalues of the second invariant. Let us study behavior of the eigenvalue at the singular points $r = 0$, $r = \infty$ for the following six different variants of the function $W(r)$:

$$W = \pm 1 \, , \quad W = \pm \, \frac{Ar}{\text{sh } Ar} \, , \quad W = \pm \, \frac{Ar}{\sin \ Ar} \, . \tag{3.5}$$

Each pair of the solution related to $W(r)$ has the same two eigenvalues; hence the corresponding eigenvalues are

$$\begin{aligned} W = \pm 1 \, , \qquad & \Lambda_1 = 2EM, \ \Lambda_2 = 2EM - \frac{2}{r^2}; \\ W = \pm \, \frac{Ar}{\text{sh } Ar}, \qquad & \Lambda_{1,2} = 2EM + \frac{A^2}{1 \pm \cosh(Ar)}, \\ W = \pm \, \frac{Ar}{\sin \ Ar}, \qquad & \Lambda_{1,2} = 2EM - \frac{A^2}{1 \pm \cos(Ar)}. \end{aligned}$$

Near the singular points, in the case $W = \pm 1$, the eigenvalues behave as follows:

$$\begin{aligned} r \to 0, \quad \Lambda_1 = 2EM, \quad \Lambda_2 = -\frac{2}{r^2}; \\ r \to \infty, \quad \Lambda_{1,2} = 2EM, \end{aligned} \tag{3.6}$$

while in the case $W = \pm \, \frac{Ar}{\text{sh } Ar}$ the eigenvalues behave as:

$$\begin{aligned} r \to 0, \quad \Lambda_1 = 2EM + \frac{1}{2}A^2, \quad \Lambda_2 = -\frac{2}{r^2}; \\ r \to \infty, \quad \Lambda_{1,2} = 2EM. \end{aligned} \tag{3.7}$$

Now we will construct a Lagrangian function for the differential system (3.1) in the following form:

$$\begin{aligned} L = g_{ij}(r) y^i y^j + b_j(r, x) y^j, \\ g_{ij}(r) = \begin{vmatrix} g_{11}(r) & 0 \\ 0 & g_{22}(r) \end{vmatrix} . \end{aligned} \tag{3.8}$$

The coefficients Q^i take the form

$$Q^1 = \frac{1}{2g_{11}}\frac{dg_{11}}{dr}y^1 + \frac{1}{4g_{11}}\Big(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\Big)y^2 + \frac{1}{4g_{11}}\frac{\partial b_1}{\partial r},$$
$$Q^2 = \frac{1}{2g_{22}}\frac{dg_{22}}{dr}y^2 - \frac{1}{4g_{22}}\Big(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\Big)y^1 + \frac{1}{4g_{22}}\frac{\partial b_2}{\partial r}. \quad (3.9)$$

Following to the same consideration as has been performed for the monopole problem, we equate the coefficient (3.9) and (3.3), producing the equations:

$$\frac{dg_{11}}{dr} = 0, \quad \frac{dg_{22}}{dr} = 0, \quad \frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1} = 0,$$
$$\frac{1}{4g_{11}}\frac{\partial b_1}{\partial r} = x^1\Big[ME + \frac{1}{2}\Big(\frac{W}{r}\Big)' - \frac{1}{2}\frac{W^2}{r^2}\Big],$$
$$\frac{1}{4g_{22}}\frac{\partial b_2}{\partial r} = x^2\Big[ME - \frac{1}{2}\Big(\frac{W}{r}\Big)' - \frac{1}{2}\frac{W^2}{r^2}\Big]. \quad (3.10)$$

The solution of this system is

$$g_{11} = C_1, \qquad g_{22} = C_2,$$
$$b_1 = B_1(x^i) + 2C_1x^1\Big(2MEr + \frac{W}{r} - \int \frac{W^2}{r^2}dr\Big),$$
$$b_2 = B_2(x^i) + 2C_2x^2\Big(2MEr - \frac{W}{r} - \int \frac{W^2}{r^2}dr\Big)$$

with the restriction

$$\frac{\partial B_1}{\partial x^2} - \frac{\partial B_2}{\partial x^1} = 0.$$

For particular cases of the monopole solutions (3.5), we get the Lagrangians (we choose the constants C_1, C_2 equal to 1):

$W = \pm 1$,

$$L = (y^1)^2 + (y^2)^2 + 2\Big(2EMr + \frac{1 \pm 1}{r}\Big)x^1y^1$$
$$+2\Big(2EMr + \frac{1 \mp 1}{r}\Big)x^2y^2 + B_1y^1 + B_2y^2;$$

$W = \frac{Ar}{\sinh\ Ar}$,

$$L = (y^1)^2 + (y^2)^2 + 2\Big(2EMr + 2A\ \coth\ \frac{AM}{2}\Big)x^1y^1$$
$$+2\Big(2EMr + 2A\ \tanh\ \frac{AM}{2}\Big)x^2y^2 + B_1y^1 + B_2y^2;$$

$W = -\frac{Ar}{\sinh\ Ar}$,

$$L = (y^1)^2 + (y^2)^2 + 2\Big(2EMr + 2A\ \tanh\ \frac{AM}{2}\Big)x^1y^1$$
$$+2\Big(2EMr + 2A\ \coth\ \frac{AM}{2}\Big)x^2y^2 + B_1y^1 + B_2y^2;$$

$W = \frac{Ar}{\sin\ Ar}$,

$$L = (y^1)^2 + (y^2)^2 + 2\Big(2EMr + 2A\ \cot\ \frac{AM}{2}\Big)x^1y^1$$

$$+2\Big(2EMr + 2A\ \tan\ \frac{AM}{2}\Big)x^2y^2 + B_1y^1 + B_2y^2;$$

$W = -\frac{Ar}{\sin\ Ar}$,

$$L = (y^1)^2 + (y^2)^2 + 2\Big(2EMr + 2A\ \tan\ \frac{AM}{2}\Big)x^1y^1$$

$$+2\Big(2EMr + 2A\ \cot\ \frac{AM}{2}\Big)x^2y^2 + B_1y^1 + B_2y^2.$$

The Hamiltonian can be found from the Lagrangian according the formula (see [15])

$$H = y^i\frac{\partial L}{\partial y^i} - L. \tag{3.11}$$

Using the explicit form of Lagrangian, one obtains:

$$H = C_1(y^1)^2 + C_2(y^2)^2 = C_1\Big(\frac{df}{dr}\Big)^2 + C_2\Big(\frac{dg}{dr}\Big)^2 \tag{3.12}$$

for all the monopole solutions. This structure (3.12) is formally similar to expression for the kinetic energy of a free particle in classical mechanics.

3.2 Non-relativistic approximation, the case $j > 0$

Now we consider the non-relativistic radial system of equations for the case $j > 0$ [17]

$$\frac{d}{dr}f + EF + \frac{\nu}{r}g = 0\ ,\quad \frac{d}{dr}F + \frac{\nu}{r}G - 2Mf = 0\ ,$$
$$-\frac{d}{dr}g + EG - \frac{\nu}{r}f + \frac{W}{r}g = 0\ ,\quad \frac{d}{dr}G + \frac{\nu}{r}F + \frac{W}{r}G + 2Mg = 0\ .$$

Using the new notations $x^1 \equiv f$, $x^2 \equiv g$, $x^3 \equiv F$, $x^4 \equiv G$, and performing the same procedure as in the previous case, we find the coefficients Q^i:

$$\begin{aligned} Q^1 &= -\Big(\tfrac{\nu^2}{2r^2} - EM\Big)x^1 + \tfrac{\nu(W-1)}{2r^2}x^2,\\ Q^2 &= -\Big(\tfrac{\nu^2}{2r^2} - EM + \tfrac{W^2}{2r^2} + \tfrac{1}{2}\Big(\tfrac{W}{r}\Big)'\Big)x^2 + \tfrac{\nu(W-1)}{2r^2}x^1,\\ Q^3 &= -\Big(\tfrac{\nu^2}{2r^2} - EM\Big)x^3 - \tfrac{\nu(W+1)}{2r^2}x^4,\\ Q^4 &= -\Big(\tfrac{\nu^2}{2r^2} - EM + \tfrac{W^2}{2r^2} - \tfrac{1}{2}\Big(\tfrac{W}{r}\Big)'\Big)x^4 - \tfrac{\nu(W+1)}{2r^2}x^3. \end{aligned} \tag{3.13}$$

The second invariant reads

(3.14)

$$P^i_j = \left| \begin{array}{cccc} 2EM - \frac{\nu^2}{r^2} & \frac{\nu(W-1)}{r^2} & 0 & 0 \\ \frac{\nu(W-1)}{r^2} & 2EM - \frac{\nu^2}{r^2} - \frac{W^2}{r^2} - \left(\frac{W}{r}\right)' & 0 & 0 \\ 0 & 0 & 2EM - \frac{\nu^2}{r^2} & -\frac{\nu(W+1)}{r^2} \\ 0 & 0 & -\frac{\nu(W+1)}{r^2} & 2EM - \frac{\nu^2}{r^2} - \frac{W^2}{r^2} + \left(\frac{W}{r}\right)' \end{array} \right| .$$

The four eigenvalues of the invariant are

$$\Lambda_{1,2} = 2EM + \frac{1}{2}\Big(\frac{W}{r}\Big)' - \frac{\nu^2}{r^2} - \frac{W^2}{2r^2} \pm \sqrt{\frac{\nu^2(W+1)^2}{r^4} + \frac{1}{4}\Big(\frac{W^2}{r^2} - \Big(\frac{W}{r}\Big)'\Big)^2},$$

$$\Lambda_{3,4} = 2EM - \frac{1}{2}\Big(\frac{W}{r}\Big)' - \frac{\nu^2}{r^2} - \frac{W^2}{2r^2} \pm \sqrt{\frac{\nu^2(W-1)^2}{r^4} + \frac{1}{4}\Big(\frac{W^2}{r^2} + \Big(\frac{W}{r}\Big)'\Big)^2}.$$

For the solution $W = \pm 1$ (3.5), the eigenvalues take the form:

$$\Lambda_{1,2} = 2EM - \frac{\nu^2 + 1 \pm \sqrt{4\nu^2+1}}{r^2}, \quad \Lambda_{3,4} = 2EM - \frac{\nu^2}{r^2}.$$

Near the singular points $r = 0$ and $r = \infty$, they behave as follows:

$$r \to 0, \ \Lambda_{1,2} = -\frac{\nu^2 + 1 \pm \sqrt{4\nu^2+1}}{r^2}, \ \Lambda_{3,4} = -\frac{\nu^2}{r^2};$$
$$r \to \infty, \quad \Lambda_{1,2,3,4} = 2EM.$$

The other solutions for W lead to rather complicated expressions for eigenvalues, which cannot be analyzed analytically.

The Lagrangian function is searched in the form

$$L = g_{ij}(r) y^i y^j + b_j(r,x) y^j,$$

$$(3.15) \qquad g_{ij}(r) = \left| \begin{array}{cccc} g_{11}(r) & 0 & 0 & 0 \\ 0 & g_{22}(r) & 0 & 0 \\ 0 & 0 & g_{33}(r) & 0 \\ 0 & 0 & 0 & g_{44}(r) \end{array} \right| .$$

One get the metric tensor coefficients explicitly as follows:

$$g_{11} = C_1, \quad g_{22} = C_2, \quad g_{33} = C_3, \quad g_{44} = C_4,$$

$$b_1 = B_1(x^i) + 2C_1\Big(2EMrx^1 + \frac{\nu^2}{r}x^1 + \frac{\nu}{r}x^2 + \nu x^2 \int \frac{W}{r^2}dr\Big),$$

$$b_2 = B_2(x^i) + 2C_2\Big(2EMrx^2 + \frac{\nu^2}{r}x^2 + \frac{\nu}{r}x^1 - \frac{W}{r}x^2 + \nu x^1 \int \frac{W}{r^2}dr - x^2 \int \frac{W^2}{r^2}dr\Big),$$

$$b_3 = B_3(x^i) + 2C_3\Big(2EMrx^3 + \frac{\nu^2}{r}x^3 + \frac{\nu}{r}x^4 - \nu x^4 \int \frac{W}{r^2}dr\Big),$$

$$b_4 = B_4(x^i) + 2C_4\Big(2EMrx^4 + \frac{\nu^2}{r}x^4 + \frac{\nu}{r}x^3 + \frac{W}{r}x^4 + \nu x^3 \int \frac{W}{r^2}dr - x^4 \int \frac{W^2}{r^2}dr\Big).$$

The gauge condition reads

$$\frac{\partial B_i}{\partial x^j} - \frac{\partial B_j}{\partial x^i} = 0.$$

For the monopole solution $W = \pm 1$ (3.5), we obtain the following Lagrangians (while setting $C_1 = C_2 = C_3 = C_4 = 1$):

$W = 1$,

$$L = (y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2$$
$$+ \frac{2}{r}\Big(2EMr^2 + \nu^2\Big)\Big(x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4\Big)$$
$$+ \frac{4\nu}{r}\Big(x^4y^3 + x^3y^4\Big) + \frac{4x^4y^4}{r} + B_1y^1 + B_2y^2 + B_3y^3 + B_4y^4;$$

$W = -1$,

$$L = (y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2$$
$$+ \frac{2}{r}\Big(2EMr^2 + \nu^2\Big)\Big(x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4\Big)$$
$$+ \frac{4\nu}{r}\Big(x^2y^1 + x^1y^2\Big) + \frac{4x^2y^2}{r} + B_1y^1 + B_2y^2 + B_3y^3 + B_4y^4.$$

The corresponding Hamiltonian is the same for all the monopole solutions:

$$H = C_1\Big(\frac{df}{dr}\Big)^2 + C_2\Big(\frac{dg}{dr}\Big)^2 + C_3\Big(\frac{dF}{dr}\Big)^2 + C_4\Big(\frac{dG}{dr}\Big)^2.$$

4 Conclusions

We apply the geometrical approach based on the Kosambi-Cartan-Chen theory to study the Bogomol'nyi - Prasad - Sommerfield monopole in spaces of constant curvature (Euclidian, Riemannian, and Lobachevsky spaces) and the doublet of Dirac fermions in non-Abelian monopole potential. We start with the radial system derived from the Yang - Mills equation for pure monopole substitution, describe its solutions in BPS-limit. Then we calculate the first and the second KCC-invariants. The Lagrangian function related to the relevant system of nonlinear differential equations is found in explicit form. It is shown that the Lagrangian has the arbitrariness up to certain terms, which may be considered as a specific gauge freedom. In the case of Euclidean space, we have analyzed the behavior of the eigenvalues of the second invariants as functions of radial variable, taking into account the explicit solution of differential equation system in BPS-limit. The same is done for monopole solutions on the background of spherical Riemannian and hyperbolic Lobachevsky geometries.

The geometrization of the problem of the doublet of Dirac fermions in the non-Abelian monopole potential has been performed for the case of non-relativistic approximation at different quantum numbers of the total angular momentum, $j = 0$ and $j > 0$. The second KCC-invarint has been calculated and the Lagrangians have been found for all solutions of the Yang - Mills equation in BPS-limit.

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Authors' addresses:

Nina Krylova
Belarusian State Agrarian Technical University,
99 Nezavisimosti Ave., 220023 Minsk, Belarus.
E-mail: nina-kr@tut.by

Elena Ovsiyuk
Mozyr State Pedagogical University named after I.P. Shamyakin,
28 Studencheskaya Str, Mozyr 247760, Belarus.
E-mail: e.ovsiyuk@mail.ru

Vladimir Balan
University Politehnica of Bucharest,
Splaiul Independentei 313, RO-060042, Bucharest, Romania.
E-mail: vladimir.balan@upb.ro

Viktor Red'kov
B. I. Stepanov Institute of Physics of NAS of Belarus,
68 Nezavisimosti Ave., 220072 Minsk, Belarus.
E-mail: v.redkov@dragon.bas-net.by