On the matrix equation for a spin 2 particle in pseudo-Riemannian space-time (II). Separating the variables in spherical coordinates

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Abstract. In the present paper we develop the theory of the massive spin 2 field, extended to the generally covariant theory within the Tetrode-Weyl-Fock-Ivanenko tetrad method. Such an equation is specified in spherical coordinates of the Minkowski space. We separate the variables by diagonalizing the square and the third projection of the total angular momentum; at this, the formalism of Wigner D-function is applied instead of spin-weight harmonics. As a result, we derive the radial system of differential equations of the first order. From these we derive the 2-nd order radial equations for components referring to symmetric tensor and scalar involved in the description of the spin 2 field. The radial system is divided into two more simple subsystems, which describe states with opposite space parities. We find, in closed form, some exact solutions for such subsystems. The restriction in the radial equations to the massless spin 2 field is possible, and the extension of the developed procedure of separating the variables to arbitrary space-time models with spherical symmetry does not require new ideas.

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1 Introduction

After the investigation by Pauli and Fierz [11], [24], the theory of massive and massless fields with spin 2 has attracted much attention ([1],[3]-[10],[12]-[17], [20]-[22],[27]-[28]; also see [2], [19]). Most of the studies were performed in the framework of 2-nd order differential equations. It is known that many specific difficulties may be avoided if from the very beginning we start with 1-st order systems. Apparently, the first

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systematic study of the theory of spin 2 fields within this formalism was performed by F.I. Fedorov [7]. It turns out that this description requires a field function with 30 independent components. This theory was further re-discovered by Regee in [27].

In the present paper we develop the theory of the spin 2 field starting from the matrix equation in Minkowski space-time, and extending it to the generally covariant theory within the Tetrode-Weyl-Fock-Ivanenko tetrad method [23]. Such an equation is specified in spherical coordinates of the Minkowski space, we separate the variables and find some simple solutions of the radial equations. Concerning the general theory, see the previous paper [14].

The extension to arbitrary space-time with spherical symmetry does not require new ideas for separating the variables in the equation.

2 The main equation in spherical coordinates

We shall further consider the matrix equation for spin 2 field in spherical coordinates [25], $x^\alpha = (t, r, \theta, \phi)$ and tetrad $e^\beta_{(b)}$ in Minkowski space

$$
\begin{gathered}
dS^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \quad e^\alpha_{(0)} = (1,0,0,0) \,, \\
e^\alpha_{(3)} = (0,1,0,0) \,, \quad e^\alpha_{(1)} = (0,0,\frac{1}{r},0) \,, \quad e^\alpha_{(2)} = (1,0,0,\frac{1}{r\sin\theta}) \,.
\end{gathered} \tag{2.1}
$$

Corresponding to tetrad (2.1), the Ricci rotation coefficients are

$$
\gamma_{ab0} = 0, \gamma_{ab3} = 0, \gamma_{ab1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{r} \\ 0 & 0 & 0 & 0 \\ 0 & +\frac{1}{r} & 0 & 0 \end{pmatrix}, \gamma_{ab2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & +\frac{\cos\theta}{r\sin\theta} & 0 \\ 0 & -\frac{\cos\theta}{r\sin\theta} & 0 & -\frac{1}{r} \\ 0 & 0 & +\frac{1}{r} & 0 \end{pmatrix}.
$$

Equation (2.1) takes the form (see [25])

$$
\begin{gathered}
\Big\{\Gamma^0 \partial_t + \Big(\Gamma^3 \partial_r + \frac{\Gamma^1 J^{31} + \Gamma^2 J^{32}}{r}\Big) + \frac{1}{r}\Sigma_{\theta,\phi} - m\Big\}\Psi = 0 \,, \\
\Sigma_{\theta,\phi} = \Gamma^1 \partial_\theta + \Gamma^2 \frac{\partial_\phi + J^{12}\cos\theta}{\sin\theta} \,, \quad \Psi = \begin{pmatrix} H \\ H_1 \\ H_2 \\ H_3 \end{pmatrix} \begin{matrix} 1 \\ 4 \\ 10 \\ 24 \end{matrix}
\end{gathered} \tag{2.2}
$$

where H corresponds to the scalar, H_1 corresponds to the vector (4 components), H_2 corresponds to the symmetric tensor (10 components), and H_3 refers to the 3-rank tensor (24 components). We further get the system of equations in block form (see [14]):

$$
\begin{gathered}
(-i\epsilon) G^0 H_1 + G^3 \frac{\partial}{\partial r} H_1 + \frac{G^1 J_1^{31} + G^2 J_1^{32}}{r} H_1 \\
+ \frac{1}{r}\Big(G^1 \partial_\theta - i G^2 \frac{-m + i\cos\theta J_1^{12}}{\sin\theta}\Big) H_1 = mH \,,
\end{gathered} \tag{2.3}
$$

$$(-i\epsilon)\frac{1}{2}\Delta^0 H - \frac{1}{3}K^0(-i\epsilon)H_2 + \frac{1}{2}\Delta^3\frac{\partial}{\partial r}H$$
$$-\frac{1}{3}\Big(K^3\frac{\partial}{\partial r} + \frac{K^1 J_2^{31} + K^2 J_2^{32}}{r}\Big)H_2 + \frac{1}{r}\Big[\frac{1}{2}\Big(\Delta^1\partial_\theta - i\Delta^2\frac{-m}{\sin\theta}\Big)H$$
$$-\frac{1}{3}\Big(K^1\partial_\theta - iK^2\frac{-m + i\cos\theta J_2^{12}}{\sin\theta}\ \Big)H_2\Big] = mH_1\,, \tag{2.4}$$

$$(-i\epsilon)\Lambda^0 H_1 + (-i\epsilon)\frac{1}{2}B^0 H_3 + \Big(\Lambda^3\frac{\partial}{\partial r} + \frac{\Lambda^1 J_1^{31} + \Lambda^2 J_1^{32}}{r}\Big)H_1$$
$$+\frac{1}{2}\Big(B^3\frac{\partial}{\partial r} + \frac{B^1 J_3^{31} + B^2 J_3^{32}}{r}\Big)H_3 + \frac{1}{r}\Big[\Big(\Lambda^1\partial_\theta - i\Lambda^2\frac{-m + i\cos\theta J_1^{12}}{\sin\theta}\Big)H_1$$
$$+\frac{1}{2}\Big(B^1\partial_\theta - iB^2\frac{-m + i\cos\theta J_3^{12}}{\sin\theta}\Big)H_3\Big] = mH_2\,, \tag{2.5}$$

$$(-i\epsilon)F^0 H_2 + F^3\frac{\partial}{\partial r}H_2 + \frac{F^1 J_2^{31} + F^2 J_2^{32}}{r}H_2$$
$$+\frac{1}{r}\Big(F^1\partial_\theta - iF^2\frac{-m + i\cos\theta J_2^{12}}{\sin\theta}\Big)H_2 = mH_3\,. \tag{2.6}$$

3 The separation of variables

We shall further construct the solutions which satisfy the following four groups of equations

$$\vec{J}^2 H = j(j+1)H,\ J_3 H = mH;\quad \vec{J}^2_{(1)}H_1 = j(j+1)H_1,\ J_3^{(1)}H_1 = mH_1;$$
$$\vec{J}^2_{(2)}H_2 = j(j+1)H_2, J_3^{(2)}H_2 = mH_2;\quad \vec{J}^2_{(3)}H_3 = j(j+1)H_3, J_3^{(3)}H_2 = mH_3.$$

To this aim, we apply the Wigner D-functions technique [25] $D^j_{-m,-s_3}(\phi,\theta,0) = D_{-s_3}$, where we take into account the known recurrent relations [29]:

$$\begin{aligned}
\partial_\theta D_0 &= +\frac{1}{2}a\ D_{-1} - \frac{1}{2}a\ D_{+1}, & \frac{-m}{\sin\theta}D_0 &= -\frac{1}{2}a\ D_{-1} - \frac{1}{2}a\ D_{+1},\\
\partial_\theta D_{+1} &= +\frac{1}{2}a\ D_0 - \frac{1}{2}b\ D_{+2}, & \frac{-m-\cos\theta}{\sin\theta}D_{+1} &= -\frac{1}{2}a\ D_0 - \frac{1}{2}b\ D_{+2},\\
\partial_\theta D_{-1} &= +\frac{1}{2}b\ D_{-2} - \frac{1}{2}a\ D_0, & \frac{-m+\cos\theta}{\sin\theta}D_{-1} &= -\frac{1}{2}b\ D_{-2} - \frac{1}{2}a\ D_0,\\
\partial_\theta D_{+2} &= +\frac{1}{2}b\ D_{+1} - \frac{1}{2}c\ D_{+3}, & \frac{-m-2\cos\theta}{\sin\theta}D_{+2} &= -\frac{1}{2}b\ D_{+1} - \frac{1}{2}c\ D_{+3},\\
\partial_\theta D_{-2} &= +\frac{1}{2}c\ D_{-3} - \frac{1}{2}b\ D_{-1}, & \frac{-m+2\cos\theta}{\sin\theta}D_{-2} &= -\frac{1}{2}c\ D_{-3} - \frac{1}{2}b; D_{-1},
\end{aligned} \tag{3.1}$$

where $\sqrt{j(j+1)} = a,\ \sqrt{(j-1)(j+2)} = b,\ \sqrt{(j-2)(j+3)} = c.$

For a correct choice of the substitution for Ψ (see the general approach in [25]), we need to know the eigenvalues of the third projection of the spin operators:

$$S_3^{(1)}H_1 = \sigma H_1,\quad S_3^{(2)}H_2 = \sigma H_2,\quad S_3^{(3)}H_3 = \sigma H_3. \tag{3.2}$$

These eigenvalues are determined by the known diagonal structure of these operators in a cyclic basis [14]. Thus we use the substitutions (which refer to the cyclic basis, while the common multiplier $e^{-i\epsilon t}$ is omitted):

$$H = hD_0, \quad H_1 = \begin{pmatrix} h_0\ D_0 \\ h_1\ D_{-1} \\ h_2\ D_0 \\ h_3\ D_{+1} \end{pmatrix}, \quad H_2 = \begin{pmatrix} f_1\ D_{-2} \\ f_2\ D_0 \\ f_3\ D_{+2} \\ c_1\ D_{+1} \\ c_2\ D_0 \\ c_3\ D_{-1} \\ d_1\ D_{-1} \\ d_2\ D_0 \\ d_3\ D_{+1} \\ f_0\ D_0 \end{pmatrix}, \tag{3.3}$$

$$\varphi_0 = \begin{pmatrix} E_{10}D_{-1} \\ E_{20}D_0 \\ E_{30}D_{+1} \\ B_{10}D_{+1} \\ B_{20}D_0 \\ B_{30}D_{-1} \end{pmatrix}, \varphi_1 = \begin{pmatrix} E_{11}D_{-2} \\ E_{21}D_{-1} \\ E_{31}D_0 \\ B_{11}D_0 \\ B_{21}D_{-1} \\ B_{31}D_{-2} \end{pmatrix}, \varphi_2 = \begin{pmatrix} E_{12}D_{-1} \\ E_{22}D_0 \\ E_{32}D_{+1} \\ B_{12}D_{+1} \\ B_{22}D_0 \\ B_{32}D_{-1} \end{pmatrix}, \varphi_3 = \begin{pmatrix} E_{13}D_0 \\ E_{23}D_{+1} \\ E_{33}D_{+2} \\ B_{13}D_{+2} \\ B_{23}D_{+1} \\ B_{33}D_0 \end{pmatrix}.$$

Let us turn to eq. (2.3). First consider the term

$$(-i\epsilon)G^0 H_1 + G^3 H_1' + \frac{G^1 J_1^{31} - G^2 J_1^{23}}{r} H_1 = \frac{1}{r}\left(-ir\epsilon h_0 - 2h_2 - rh_2'\right) D_0. \tag{3.4}$$

Then consider the angular term

$$\Big(G^1 \partial_\theta - iG^2 \frac{-m + i\cos\theta J_1^{12}}{\sin\theta}\Big) H_1$$

$$= \Big\{ \frac{1}{\sqrt{2}}(010-1)\partial_\theta + \frac{1}{\sqrt{2}}(0101)\frac{-m+\cos\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}{\sin\theta} \Big\} \begin{pmatrix} h_0\ D_0 \\ h_1\ D_{-1} \\ h_2\ D_0 \\ h_3\ D_{+1} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(h_1 \partial_\theta D_{-1} - h_3 \partial_\theta D_{+1}) + \frac{1}{\sqrt{2}} \frac{-m(h_1 D_{-1} + h_3 D_{+1}) + \cos\theta (h_1 D_{-1} - h_3 D_{+1})}{\sin\theta}$$

$$= \frac{1}{\sqrt{2}} h_1 \Big[\partial_\theta D_{-1} + \frac{-m+\cos\theta}{\sin\theta} D_{-1}\Big] - \frac{1}{\sqrt{2}} h_3 \Big[\partial_\theta D_{+1} - \frac{-m+\cos\theta}{\sin\theta} D_{-}\Big].$$

Then, taking in mind the identities

$$\partial_\theta D_{-1} + \frac{-m+\cos\theta}{\sin\theta} D_{-1} = \frac{b}{2} D_{-2} - \frac{a}{2} D_0 - \frac{b}{2} D_{-2} - \frac{a}{2} D_0 = -aD_0,$$

$$\partial_\theta D_{+1} - \frac{-m+\cos\theta}{\sin\theta} D_{-} = \frac{a}{2} D_0 - \frac{b}{2} D_{+2} + \frac{a}{2} D_0 + \frac{b}{2} D_{+2} = +aD_0,$$

we get

$$\frac{1}{r}\Big(G^1\partial_\theta - iG^2\frac{-m+i\cos\theta J_1^{12}}{\sin\theta}\Big)H_1 = -\frac{1}{r}\frac{a}{\sqrt{2}}(h_1+h_3)D_0. \tag{3.5}$$

Thus, eq. (2.3) leads to the radial equation

$$-i\epsilon h_0 - \frac{2}{r}h_2 - h_2' - \frac{a}{r\sqrt{2}}(h_1+h_3) = mh. \tag{3.6}$$

Now we turn to eq. (2.4). First we consider the term

$$(-i\epsilon)\frac{1}{2}\Delta^0 H - (-i\epsilon)\frac{1}{3}K^0 H_2 + \frac{1}{2}\Delta^3\frac{\partial}{\partial r}H - \frac{1}{3}(K^3\frac{\partial}{\partial r} + \frac{K^1J_2^{31}+K^2J_2^{32}}{r})H_2$$

$$= \frac{1}{3r}\left|\begin{array}{c} \frac{1}{2}D_0\left(-3ir\epsilon h + 4d_2 + 2r\left(i\epsilon f_0 + d_2'\right)\right) \\ D_{-1}\left(3c_3 + r\left(i\epsilon d_1 + c_3'\right)\right) \\ \frac{1}{2}D_0\left(4c_2 + 2ir\epsilon d_2 + 4f_2 + 3rh' + 2rf_2'\right) \\ D_{+1}\left(3c_1 + r\left(i\epsilon d_3 + c_1'\right)\right) \end{array}\right|.$$

Then we consider the first angular term

$$\frac{1}{2}\Big(\Delta^1\partial_\theta - i\Delta^2\frac{-m}{\sin\theta}\Big)H = \frac{1}{2}\frac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}\partial_\theta + \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\frac{-m}{\sin\theta}\right)hD_0$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}}h\begin{pmatrix}0\\-(\partial_\theta+\frac{m}{\sin\theta})D_0\\0\\(\partial_\theta-\frac{m}{\sin\theta})D_0\end{pmatrix} = \frac{1}{2}\frac{1}{\sqrt{2}}h\begin{pmatrix}0\\aD_{-1}\\0\\-aD_{+1}\end{pmatrix}. \tag{3.7}$$

Now we consider the second angular term

$$\Big(K^1\partial_\theta - iK^2\frac{-m+i\cos\theta J_2^{12}}{\sin\theta}\Big)H_2.$$

For its first summand we obtain

$$K^1\partial_\theta H_2 = -\frac{1}{\sqrt{2}}\begin{pmatrix}-d_1\partial_\theta D_{-1}+d_3\partial_\theta D_{+1}\\-f_1\partial_\theta D_{-2}+c_2\partial_\theta D_0\\c_1\partial_\theta D_{+1}-c_3\partial_\theta D_{-1}\\f_3\partial_\theta D_{+2}-c_2\partial_\theta D_0\end{pmatrix}$$

$$= -\frac{1}{\sqrt{2}}\begin{pmatrix}-d_1(\frac{b}{2}D_{-2}-\frac{a}{2}D_0)+d_3(\frac{b}{2}D_0-\frac{b}{2}D_{+2})\\-f_1(\frac{c}{2}D_{-3}-\frac{b}{2}D_{-1})+c_2(\frac{a}{2}D_{-1}-\frac{a}{2}D_{+1})\\c_1(\frac{a}{2}D_0-\frac{b}{2}D_{+2})-c_3(\frac{b}{2}D_{-2}-\frac{a}{2}D_0)\\f_3(\frac{b}{2}D_{+1}-\frac{c}{2}D_{+3})-c_2(\frac{a}{2}D_{-1}-\frac{a}{2}D_{+1})\end{pmatrix};$$

for the second summand we derive

$$-iK^2\frac{-m+i\cos\theta J_2^{12}}{\sin\theta}$$

$$= -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sin\theta} \begin{pmatrix} -mf_1 D_{-2} + 2\cos\theta D_{-2} f_1 \\ -mf_2 D_0 + 0 \\ -mf_3 D_{+2} - 2\cos\theta D_{+2} f_3 \\ -mc_1 D_{+1} - \cos\theta D_{+1} c_1 \\ -mc_2 D_0 + 0 \\ -mc_3 D_{-1} + \cos\theta D_{-1} c_3 \\ -md_1 D_{-1} + \cos\theta D_{-1} d_1 \\ -md_2 D_0 + 0 \\ -md_3 D_{+1} - \cos\theta D_{+1} d_3 \\ -mf_0 D_0 + 0 \end{pmatrix}$$

$$= -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1(-\frac{c}{2}D_{-3} - \frac{b}{2}D_{-1}) \\ f_2(-\frac{a}{2}D_{-1} - \frac{a}{2}D_{+1}) \\ f_3(-\frac{b}{2}D_{+1} - \frac{c}{2}D_{+3}) \\ c_1(-\frac{a}{2}D_0 - \frac{b}{2}D_{+2}) \\ c_2(-\frac{a}{2}D_{-1} - \frac{a}{2}D_{+1}) \\ c_3(-\frac{b}{2}D_{-2} - \frac{a}{2}D_0) \\ d_1(-\frac{b}{2}D_{-2} - \frac{a}{2}D_0) \\ d_2(-\frac{a}{2}D_{-1} - \frac{a}{2}D_{+1}) \\ d_3(-\frac{a}{2}D_0 - \frac{b}{2}D_{+2}) \\ f_0(-\frac{a}{2}D_{-1} - \frac{a}{2}D_{+1}) \end{pmatrix}$$

$$= -\frac{1}{\sqrt{2}} \begin{pmatrix} d_1(\frac{b}{2}D_{-2} + \frac{a}{2}D_0) + d_3(\frac{b}{2}D_0 + \frac{b}{2}D_{+2}) \\ f_1(\frac{c}{2}D_{-3} + \frac{b}{2}D_{-1}) + c_2(\frac{a}{2}D_{-1} + \frac{a}{2}D_{+1}) \\ c_1(\frac{a}{2}D_0 + \frac{b}{2}D_{+2}) + c_3(\frac{b}{2}D_{-2} + \frac{a}{2}D_0) \\ f_3(\frac{b}{2}D_{+1} + \frac{c}{2}D_{+3}) + c_2(\frac{a}{2}D_{-1} + \frac{a}{2}D_{+1}) \end{pmatrix}.$$

Summing up the last two relations, we find the expression for the second angular term,

$$\text{(3.8)} \qquad \Big(K^1\partial_\theta - iK^2 \frac{-m + i\cos\theta J_2^{12}}{\sin\theta}\Big) H_2 = -\frac{1}{\sqrt{2}} \begin{pmatrix} (ad_1 + bd_3)D_0 \\ (bf_1 + ac_2)D_{-1} \\ (ac_1 + ac_3)D_0 \\ (bf_3 + ac_2)D_{+1} \end{pmatrix}.$$

Therefore, eq. (2.4) takes the form

$$\frac{1}{6r} \begin{pmatrix} D_0\left(-3ir\epsilon h + \sqrt{2}ad_1 + \sqrt{2}bd_3 + 4d_2 + 2ir\epsilon f_0 + 2rd_2'\right) \\ \frac{1}{2}D_{-1}\left(\sqrt{2}a\left(3h + 2c_2\right) + 2\left(\sqrt{2}bf_1 + 6c_3 + 2r\left(i\epsilon d_1 + c_3'\right)\right)\right) \\ D_0\left(\sqrt{2}a\left(c_1 + c_3\right) + 4c_2 + 2ir\epsilon d_2 + 4f_2 + 3rh' + 2rf_2'\right) \\ \frac{1}{2}D_{+1}\left(\sqrt{2}a\left(2c_2 - 3h\right) + 2\left(\sqrt{2}bf_3 + 6c_1 + 2r\left(i\epsilon d_3 + c_1'\right)\right)\right) \end{pmatrix} = m \begin{pmatrix} D_0 h_0 \\ D_{-1} h_1 \\ D_0 h_2 \\ D_{+1} h_3 \end{pmatrix},$$

which yield the following four radial equations

$$\text{(3.9)} \qquad \begin{aligned} \frac{1}{6r}\left[\left(-3ir\epsilon h + \sqrt{2}ad_1 + \sqrt{2}bd_3 + 4d_2 + 2ir\epsilon f_0 + 2rd_2'\right)\right] &= mh_0, \\ \frac{1}{6r}\left[\frac{1}{2}\left(\sqrt{2}a\left(3h + 2c_2\right) + 2\left(\sqrt{2}bf_1 + 6c_3 + 2r\left(i\epsilon d_1 + c_3'\right)\right)\right)\right] &= mh_1, \\ \frac{1}{6r}\left[\left(\sqrt{2}a\left(c_1 + c_3\right) + 4c_2 + 2ir\epsilon d_2 + 4f_2 + 3rh' + 2rf_2'\right)\right] &= mh_2, \\ \frac{1}{6r}\left[\frac{1}{2}\left(\sqrt{2}a\left(2c_2 - 3h\right) + 2\left(\sqrt{2}bf_3 + 6c_1 + 2r\left(i\epsilon d_3 + c_1'\right)\right)\right)\right] &= mh_3. \end{aligned}$$

Now we turn to eq. (2.5). We first consider the term

$$-i\epsilon\Lambda^0 H_1 - i\epsilon\frac{1}{2}B^0 H_3 + \left(\Lambda^3\frac{\partial}{\partial r} + \frac{\Lambda^1 J_1^{31} + \Lambda^2 J_1^{32}}{r}\right)H_1 + \frac{1}{2}\left(B^3\frac{\partial}{\partial r} + \frac{B^1 J_3^{31} + B^2 J_3^{32}}{r}\right)H_3 = \frac{1}{r}$$

$$\times \begin{pmatrix} D_{-2}\left(B_{31} + r\left(B'_{31} - i\epsilon E_{11}\right)\right) \\ \frac{1}{4}D_0\left(2B_{11} - 2B_{33} + 2\left(E_{20} - 2h_2\right) + r\left(-i\epsilon\left(E_{13} + 3E_{22} + E_{31} + 2h_0\right) - B'_{11} + B'_{33} + E'_{20} + 6h'_2\right)\right) \\ -D_2\left(B_{13} + r\left(i\epsilon E_{33} + B'_{13}\right)\right) \\ \frac{1}{2}D_1\left(B_{23} - 2h_3 + r\left(-i\epsilon\left(E_{23} + E_{32}\right) - B'_{12} + 2h'_3\right)\right) \\ -\frac{1}{4}D_0\left(2\left(E_{20} + 2h_2\right) + r\left(i\epsilon\left(E_{13} + E_{22} + E_{31} - 2h_0\right) + B'_{11} - B'_{33} + E'_{20} - 2h'_2\right)\right) \\ -\frac{1}{2}D_{-1}\left(B_{21} + 2h_1 + ir\left(\epsilon\left(E_{12} + E_{21}\right) + i\left(B'_{32} + 2h'_1\right)\right)\right) \\ \frac{1}{2}D_{-1}\left(B_{30} + E_{12} + 2E_{21} + r\left(-i\epsilon\left(E_{10} + 2h_1\right) + B'_{30} + E'_{21}\right)\right) \\ \frac{1}{2}D_0\left(E_{13} + 2E_{22} + E_{31} + r\left(-i\epsilon\left(E_{20} + 2h_2\right) + E'_{22} + 2h'_0\right)\right) \\ \frac{1}{2}D_1\left(-B_{10} + 2E_{23} + E_{32} + r\left(-i\epsilon\left(E_{30} + 2h_3\right) - B'_{10} + E'_{23}\right)\right) \\ \frac{1}{4}D_0\left(2B_{11} - 2B_{33} + 6E_{20} + 4h_2 + r\left(i\epsilon\left(E_{13} - E_{22} + E_{31} - 6h_0\right) + B'_{11} - B'_{33} + 3E'_{20} + 2h'_2\right)\right) \end{pmatrix}.$$

Then we consider the first angular term

$$\text{(3.10)} \qquad \left(\Lambda^1\partial_\theta - i\Lambda^2\frac{-m + i\cos\theta J_1^{12}}{\sin\theta}\right)H_1 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 2bh_1 D_{-2} \\ \frac{1}{2}a\left(h_1 + h_3\right)D_0 \\ 2bh_3 D_{+2} \\ ah_2 D_{+1} \\ \frac{1}{2}a\left(h_1 + h_3\right)D_0 \\ ah_2 D_{-1} \\ ah_0 D_{-1} \\ 0 \\ ah_0 D_{+1} \\ -\frac{1}{2}a\left(h_1 + h_3\right)D_0 \end{pmatrix}.$$

Now we consider the second angular term, and omitting the details we write down the result,

(3.11)

$$\frac{1}{2}\left(B^1\partial_\theta - iB^2\frac{-m + i\cos\theta J_3^{12}}{\sin\theta}\right)H_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -bD_{-2}B_{21} \\ \frac{1}{4}aD_0\left(3B_{12} + B_{21} - B_{23} - 3B_{32} + E_{10} + E_{30}\right) \\ bD_{+2}B_{23} \\ \frac{1}{2}D_{+1}\left(bB_{13} + a\left(B_{22} - B_{33}\right)\right) \\ \frac{1}{4}aD_0\left(B_{12} + B_{21} - B_{23} - B_{32} - E_{10} - E_{30}\right) \\ \frac{1}{2}D_{-1}\left(aB_{11} - aB_{22} - bB_{31}\right) \\ \frac{1}{2}D_{-1}\left(-aB_{20} + bE_{11} + aE_{31}\right) \\ \frac{1}{2}aD_0\left(B_{10} - B_{30} + E_{12} + E_{32}\right) \\ \frac{1}{2}D_{+1}\left(a\left(B_{20} + E_{13}\right) + bE_{33}\right) \\ \frac{1}{4}aD_0\left(B_{12} - B_{21} + B_{23} - B_{32} + 3\left(E_{10} + E_{30}\right)\right) \end{pmatrix}.$$

Therefore, eq. (2.5) takes the form

$$
\frac{1}{r}\begin{pmatrix}
D_{-2}\left(B_{31}+r\left(B'_{31}-i\epsilon E_{11}\right)\right)\\
\frac{1}{4}D_0\left(2B_{11}-2B_{33}+2\left(E_{20}-2h_2\right)+r\left(-i\epsilon\left(E_{13}+3E_{22}+E_{31}+2h_0\right)-B'_{11}+B'_{33}+E'_{20}+6h'_2\right)\right)\\
-D_2\left(B_{13}+r\left(i\epsilon E_{33}+B'_{13}\right)\right)\\
\frac{1}{2}D_1\left(B_{23}-2h_3+r\left(-i\epsilon\left(E_{23}+E_{32}\right)-B'_{12}+2h'_3\right)\right)\\
-\frac{1}{4}D_0\left(2\left(E_{20}+2h_2\right)+r\left(i\epsilon\left(E_{13}+E_{22}+E_{31}-2h_0\right)+B'_{11}-B'_{33}+E'_{20}-2h'_2\right)\right)\\
-\frac{1}{2}D_{-1}\left(B_{21}+2h_1+ir\left(\epsilon\left(E_{12}+E_{21}\right)+i\left(B'_{32}+2h'_1\right)\right)\right)\\
\frac{1}{2}D_{-1}\left(B_{30}+E_{12}+2E_{21}+r\left(-i\epsilon\left(E_{10}+2h_1\right)+B'_{30}+E'_{21}\right)\right)\\
\frac{1}{2}D_0\left(E_{13}+2E_{22}+E_{31}+r\left(-i\epsilon\left(E_{20}+2h_2\right)+E'_{22}+2h'_0\right)\right)\\
\frac{1}{2}D_1\left(-B_{10}+2E_{23}+E_{32}+r\left(-i\epsilon\left(E_{30}+2h_3\right)-B'_{10}+E'_{23}\right)\right)\\
\frac{1}{4}D_0\left(2B_{11}-2B_{33}+6E_{20}+4h_2+r\left(i\epsilon\left(E_{13}-E_{22}+E_{31}-6h_0\right)+B'_{11}-B'_{33}+3E'_{20}+2h'_2\right)\right)
\end{pmatrix}
$$

$$
+\frac{1}{\sqrt{2}r}\left[-\begin{pmatrix}
2bh_1D_{-2}\\
\frac{1}{2}a\left(h_1+h_3\right)D_0\\
2bh_3D_{+2}\\
ah_2D_{+1}\\
\frac{1}{2}a\left(h_1+h_3\right)D_0\\
ah_2D_{-1}\\
ah_0D_{-1}\\
0\\
ah_0D_{+1}\\
-\frac{1}{2}a\left(h_1+h_3\right)D_0
\end{pmatrix}+\begin{pmatrix}
-bD_{-2}B_{21}\\
\frac{1}{4}aD_0\left(3B_{12}+B_{21}-B_{23}-3B_{32}+E_{10}+E_{30}\right)\\
bD_{+2}B_{23}\\
\frac{1}{2}D_{+1}\left(bB_{13}+a\left(B_{22}-B_{33}\right)\right)\\
\frac{1}{4}aD_0\left(B_{12}+B_{21}-B_{23}-B_{32}-E_{10}-E_{30}\right)\\
\frac{1}{2}D_{-1}\left(aB_{11}-aB_{22}-bB_{31}\right)\\
\frac{1}{2}D_{-1}\left(-aB_{20}+bE_{11}+aE_{31}\right)\\
\frac{1}{2}aD_0\left(B_{10}-B_{30}+E_{12}+E_{32}\right)\\
\frac{1}{2}D_{+1}\left(a\left(B_{20}+E_{13}\right)+bE_{33}\right)\\
\frac{1}{4}aD_0\left(B_{12}-B_{21}+B_{23}-B_{32}+3\left(E_{10}+E_{30}\right)\right)
\end{pmatrix}\right]=m\begin{pmatrix}
D_{-2}f_1\\
D_0f_2\\
D_{+2}f_3\\
c_1D_{+1}\\
c_2D_0\\
c_3D_{-1}\\
d_1D_{-1}\\
d_2D_0\\
d_3D_{+1}\\
D_0f_0
\end{pmatrix}. \tag{3.12}
$$

Finally, we consider eq. (2.6). Omitting all the details, we write down the result:

(3.13)

$$
\frac{1}{r}\begin{pmatrix}
\frac{1}{6}D_{-1}\left(6c_3-4ir\epsilon d_1+\sqrt{2}a\left(c_2+3f_0\right)+\sqrt{2}bf_1+2rc'_3\right)\\
\frac{1}{6}D_0\left(4c_2+\sqrt{2}a\left(c_1+c_3\right)+4f_2+2r\left(-2i\epsilon d_2-3f'_0+f'_2\right)\right)\\
\frac{1}{6}D_{+1}\left(6c_1-4ir\epsilon d_3+\sqrt{2}a\left(c_2+3f_0\right)+\sqrt{2}bf_3+2rc'_1\right)\\
rD_{+1}\left(\frac{ad_2}{\sqrt{2}r}+\frac{d_3}{r}+d'_3\right)\\
\frac{aD_0(d_3-d_1)}{\sqrt{2}}\\
-\frac{1}{2}D_{-1}\left(2d_1+\sqrt{2}ad_2+2rd'_1\right)\\
rD_{-2}\left(\frac{bd_1}{\sqrt{2}r}-i\epsilon f_1\right)\\
rD_{-1}\left(-i\epsilon c_3-d'_1\right)\\
\frac{1}{6}D_0\left(2d_2+\sqrt{2}a\left(2d_1-d_3\right)-2ir\epsilon\left(3c_2+f_0\right)-2rd'_2\right)\\
\frac{1}{6}D_0\left(2\left(c_2-ir\epsilon d_2+f_2+3rc'_2-rf'_2\right)-\sqrt{2}a\left(c_1-2c_3\right)\right)\\
\frac{1}{3}D_{-1}\left(2\sqrt{2}ac_2+ir\epsilon d_1-\sqrt{2}bf_1+rc'_3\right)\\
-\frac{1}{2}D_{-2}\left(\sqrt{2}bc_3+2f_1+2rf'_1\right)\\
rD_{-1}\left(-i\epsilon c_3+\frac{d_1}{r}+\frac{ad_2}{\sqrt{2}r}\right)\\
\frac{1}{6}D_0\left(4d_2+\sqrt{2}a\left(d_1+d_3\right)+2ir\epsilon\left(f_0-3f_2\right)-4rd'_2\right)\\
rD_{+1}\left(-i\epsilon c_1+\frac{ad_2}{\sqrt{2}r}+\frac{d_3}{r}\right)\\
-\frac{1}{6}D_{+1}\left(-6c_1+2ir\epsilon d_3+\sqrt{2}a\left(c_2-3f_2\right)+\sqrt{2}bf_3-4rc'_1\right)\\
\frac{aD_0(c_1-c_3)}{\sqrt{2}}\\
\frac{1}{6}D_{-1}\left(-6c_3+2ir\epsilon d_1+\sqrt{2}bf_1+\sqrt{2}a\left(c_2-3f_2\right)-4rc'_3\right)\\
-\frac{1}{6}D_0\left(-2d_2+\sqrt{2}a\left(d_1-2d_3\right)+2ir\epsilon\left(3c_2+f_0\right)+2rd'_2\right)\\
rD_{+1}\left(-i\epsilon c_1-d'_3\right)\\
rD_{+2}\left(\frac{bd_3}{\sqrt{2}r}-i\epsilon f_3\right)\\
rD_{+2}\left(\frac{bc_1}{\sqrt{2}r}+\frac{f_3}{r}+f'_3\right)\\
-\frac{1}{3}D_{+1}\left(2\sqrt{2}ac_2+ir\epsilon d_3-\sqrt{2}bf_3+rc'_1\right)\\
\frac{1}{6}D_0\left(\sqrt{2}a\left(c_3-2c_1\right)-2\left(c_2-ir\epsilon d_2+f_2+3rc'_2-rf'_2\right)\right)
\end{pmatrix}=m\begin{pmatrix}
D_{-1}E_{10}\\
D_0E_{20}\\
D_{+1}E_{30}\\
D_{+1}B_{10}\\
D_0B_{20}\\
D_{-1}B_{30}\\
D_{-2}E_{11}\\
D_{-1}E_{21}\\
D_0E_{31}\\
D_0B_{11}\\
D_{-1}B_{21}\\
D_{-2}B_{31}\\
D_{-1}E_{12}\\
D_0E_{22}\\
D_{+1}E_{32}\\
D_{+1}B_{12}\\
D_0B_{22}\\
D_{-1}B_{32}\\
D_0E_{13}\\
D_{+1}E_{23}\\
D_{+2}E_{33}\\
D_{+2}B_{13}\\
D_{+1}B_{23}\\
D_0B_{33}
\end{pmatrix}.
$$

We write down the separate equations of the system[1]

$$H \qquad -i\epsilon h_0 - 2\frac{1}{r}h_2 - h_2' - \frac{a}{r\sqrt{2}}(h_1 + h_3) = mh; \tag{3.14}$$

$$\begin{aligned}
&\frac{1}{6r}\left(-3ir\epsilon h + \sqrt{2}ad_1 + \sqrt{2}bd_3 + 4d_2 + 2ir\epsilon f_0 + 2rd_2'\right) = mh_0,\\
&\frac{1}{6r}\left(\sqrt{2}a\,(c_1 + c_3) + 4c_2 + 2ir\epsilon d_2 + 4f_2 + 3rh' + 2rf_2'\right) = mh_2,\\
&\frac{1}{12r}\left[\sqrt{2}a\,(2c_2 + 3h) + 2\left(\sqrt{2}bf_1 + 6c_3 + 2r\,(i\epsilon d_1 + c_3')\right)\right] = mh_1,\\
&\frac{1}{12r}\left[\sqrt{2}a\,(2c_2 - 3h) + 2\left(\sqrt{2}bf_3 + 6c_1 + 2r\,(i\epsilon d_3 + c_1')\right)\right] = mh_3;
\end{aligned} \tag{3.15}$$

$$-\frac{1}{2r}\left(\sqrt{2}b\,(B_{21} + 2h_1) - 2\,(B_{31} + r\,(B_{31}' - i\epsilon E_{11}))\right) = mf_1,$$

$$\frac{1}{2r}\left(\sqrt{2}b\,(B_{23} - 2h_3) - 2(B_{13} + r\,(B_{13}' + i\epsilon E_{33}))\right) = mf_3,$$

$$\frac{1}{8r}\Big[4B_{11} + \sqrt{2}a\,(3B_{12} + B_{21} - B_{23} - 3B_{32} + E_{10} + E_{30} - 2\,(h_1 + h_3))$$
$$-2\,(2B_{33} - 2E_{20} + 4h_2 + r\,(i\epsilon\,(E_{13} + 3E_{22} + E_{31} + 2h_0) + B_{11}' - B_{33}' - E_{20}' - 6h_2'))\Big] = mf_2,$$

$$\frac{1}{4r}\left(\sqrt{2}bB_{13} + \sqrt{2}a\,(B_{22} - B_{33} - 2h_2) + 2\,(B_{23} - 2h_3 + r\,(-i\epsilon\,(E_{23} + E_{32}) - B_{12}' + 2h_3'))\right) = mc_1,$$

$$\frac{1}{4r}\left(-\sqrt{2}bB_{31} + \sqrt{2}a\,(-B_{22} + B_{11} - 2h_2) + 2(-B_{21} - 2h_1 + r\,(-i\epsilon\,(E_{12} + E_{21}) + B_{32}' + 2h_1')\right) = mc_3,$$

$$-\frac{1}{8r}\left(\sqrt{2}a\,(-B_{12} - B_{21} + B_{23} + B_{32} + E_{10} + E_{30} + 2\,(h_1 + h_3)) + 2\,(2\,(E_{20} + 2h_2)\right.$$
$$\left. +\ r\,(i\epsilon\,(E_{13} + E_{22} + E_{31} - 2h_0) + B_{11}' - B_{33}' + E_{20}' - 2h_2'))\right)\, mc_2,$$

$$\frac{1}{4r}\left(2B_{30} + \sqrt{2}bE_{11} + 2E_{12} + 4E_{21} + \sqrt{2}a\,(-B_{20} + E_{31} - 2h_0) + 2r\,(-i\epsilon\,(E_{10} + 2h_1) + B_{30}' + E_{21}')\right) = md_1,$$

$$\frac{1}{4r}\left(-2B_{10} + 4E_{23} + 2E_{32} + \sqrt{2}bE_{33} + \sqrt{2}a\,(B_{20} + E_{13} - 2h_0) + 2r\,(-i\epsilon\,(E_{30} + 2h_3) - B_{10}' + E_{23}')\right) = md_3,$$

$$\frac{1}{4r}\left(\sqrt{2}a\,(B_{10} - B_{30} + E_{12} + E_{32}) + 2\,(E_{13} + 2E_{22} + E_{31} + r\,(-i\epsilon\,(E_{20} + 2h_2) + E_{22}' + 2h_0'))\right) md_2,$$

$$\frac{1}{8r}\left(4B_{11} + \sqrt{2}a\,(B_{12} - B_{21} + B_{23} - B_{32} + 3E_{10} + 3E_{30} + 2\,(h_1 + h_3)) + 2\,(-2B_{33} + 6E_{20} + 4h_2\right.$$
$$\left. +\ r\,(i\epsilon\,(E_{13} - E_{22} + E_{31} - 6h_0) + B_{11}' - B_{33}' + 3E_{20}' + 2h_2'))\right) = mf_0;$$

$$\begin{aligned}
&\frac{1}{6r}\left(6c_3 - 4ir\epsilon d_1 + \sqrt{2}a\,(c_2 + 3f_0) + \sqrt{2}bf_1 + 2rc_3'\right) = mE_{10},\\
&\frac{1}{6r}\left(6c_1 - 4ir\epsilon d_3 + \sqrt{2}a\,(c_2 + 3f_0) + \sqrt{2}bf_3 + 2rc_1'\right) = mE_{30},\\
&\left(\frac{ad_2}{\sqrt{2}r} + \frac{d_3}{r} + d_3'\right) = mB_{10}, \quad -\left(\frac{ad_2}{\sqrt{2}r} + \frac{d_1}{r} + d_1'\right) = mB_{30},\\
&\frac{1}{6r}\left(4c_2 + \sqrt{2}a\,(c_1 + c_3) + 4f_2 + 2r\,(-2i\epsilon d_2 - 3f_0' + f_2')\right) = mmE_{20}, \quad \frac{a}{\sqrt{2}r}\,(d_3 - d_1) = mB_{20},\\
&\left(-i\epsilon c_3 + \frac{d_1}{r} + \frac{ad_2}{\sqrt{2}r}\right) = mE_{12}, \quad \left(-i\epsilon c_1 + \frac{d_3}{r} + \frac{ad_2}{\sqrt{2}r}\right) = mE_{32},
\end{aligned} \tag{3.16}$$

---

[1] We re-group the equations in a special way, which is convenient for taking into account additional restrictions due to the diagonalization of the space reflection operator.

$$\frac{1}{6r}\Big(4d_2+\sqrt{2}a\,(d_1+d_3)+2ir\epsilon\,(f_0-3f_2)-4rd_2'\Big)=mE_{22},\quad \frac{a}{\sqrt{2}r}\,(c_1-c_3)=mB_{22},$$
$$-\frac{1}{6r}\Big(-6c_1+2ir\epsilon d_3+\sqrt{2}a\,(c_2-3f_2)+\sqrt{2}bf_3-4rc_1'\Big)=mB_{12},$$
$$\frac{1}{6r}\Big(-6c_3+2ir\epsilon d_1+\sqrt{2}a\,(c_2-3f_2)+\sqrt{2}bf_1-4rc_3'\Big)=mB_{32},$$
$$(\frac{bd_1}{\sqrt{2}r}-i\epsilon f_1)=mE_{11},\quad (\frac{bd_3}{\sqrt{2}r}-i\epsilon f_3)=mE_{33},$$
$$(-i\epsilon c_3-d_1')=mE_{21},\quad (-i\epsilon c_1-d_3')=mE_{23},$$
$$\frac{1}{6r}\Big(2d_2+\sqrt{2}a\,(2d_1-d_3)-2ir\epsilon\,(3c_2+f_0)-2rd_2'\Big)=mmE_{31},$$
$$\frac{1}{6r}\Big(2d_2+\sqrt{2}a\,(2d_3-d_1)-2ir\epsilon\,(3c_2+f_0)-2rd_2'\Big)=mE_{13},$$
$$\frac{1}{3r}\Big(2\sqrt{2}ac_2+ir\epsilon d_1-\sqrt{2}bf_1+rc_3'\Big)=mB_{21},$$
$$-\frac{1}{3r}\Big(2\sqrt{2}ac_2+ir\epsilon d_3-\sqrt{2}bf_3+rc_1'\Big)=mB_{23},$$
$$-(\frac{bc_3}{\sqrt{2}r}+\frac{f_1}{r}+f_1')=mB_{31},\quad (\frac{bc_1}{\sqrt{2}r}+\frac{f_3}{r}+f_3')=mB_{13},$$
$$\frac{1}{6r}\Big(2\,(c_2-ir\epsilon d_2+f_2+3rc_2'-rf_2')-\sqrt{2}a\,(c_1-2c_3)\Big)=mB_{11},$$
$$-\frac{1}{6r}\Big(2\,(c_2-ir\epsilon d_2+f_2+3rc_2'-rf_2')-\sqrt{2}a\,(c_3-2c_1)\Big)=mB_{33}.$$

4 Diagonalization of the space reflection operator

As known, the diagonalization of the space reflection operator permits to reduce the general radial system to simpler subsystems. We start with the explicit form of this operator in Cartesian coordinates and tetrads,

$$\hat{P}H=PH,\quad \hat{P}\,(t,x,y,z)=(t,-x,-y,-z),$$
$$\hat{\Pi}_C\hat{P}H_1=PH_1,\quad \hat{\Pi}_C=\left(\begin{array}{cccc}1&.&.&.\\ .&-1&.&.\\ .&.&-1&.\\ .&.&.&-1\end{array}\right), \tag{4.1}$$
$$(\hat{\Pi}_C\otimes\hat{\Pi}_C)\hat{P}H_2=PH_2,\quad (\hat{\Pi}_C\otimes\hat{\Pi}_C\otimes\hat{\Pi}_C)\hat{P}H_3=PH_3.$$

The transformation of the constituents of the function Ψ from the Cartesian tetrad to the spherical one is performed according to the rules [14]:

$$H_1=L\,H_1^C,\quad H_2=(L\otimes L)H_2^C,\quad H_3=(L\otimes L\otimes L\otimes L)H_3^C,$$
$$L=L(\theta,\phi)=\left(\begin{array}{cccc}1&0&0&0\\ 0&\cos\theta\cos\phi&\cos\theta\sin\phi&-\sin\theta\\ 0&-\sin\phi&\cos\phi&0\\ 0&\sin\theta\cos\phi&\sin\theta\sin\phi&\cos\theta\end{array}\right). \tag{4.2}$$

Correspondingly, the eigenvalue equations transform as follows

$$\Pi_C\hat{P}H_1^C=PH_1\implies L\Pi_C\hat{P}L^{-1}H_1=PH_1,$$

$$(\Pi_C \otimes \Pi_C)\hat{P}H_2^C = PH_2^C \implies (L\otimes L)(\Pi_C \otimes \Pi_C)\hat{P}(L^{-1}\otimes L^{-1})H_2 = PH_2,$$
$$(\Pi_C\otimes\Pi_C\otimes\Pi_C)\hat{P}H_3^C = PH_3^C \implies (L\otimes L\otimes L)(\Pi_C\otimes\Pi_C\otimes\Pi_C)\hat{P}(L^{-1}\otimes L^{-1}\otimes L^{-1})H_3 = PH_3.$$

We are to take into account the identities

$$\hat{P}L(\theta,\phi) = \hat{P}L(\pi-\theta,\phi+\pi), \quad \hat{P}D^j_{-m,\sigma}(\phi,\theta,0) = (-1)^j D^j_{-m,-\sigma}(\phi,\theta,0),$$
$$\cos\theta \Longrightarrow -\cos\theta, \quad \sin\theta \Longrightarrow +\sin\theta, \quad \cos\phi \Longrightarrow -\cos\phi, \quad \sin\phi \Longrightarrow -\sin\phi,$$
$$L(\pi-\theta,\phi+\pi) = L_-(\theta,\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & \sin\phi & -\cos\phi & 0 \\ 0 & -\sin\theta\cos\phi & -\sin\theta\sin\phi & -\cos\theta \end{pmatrix} = L_-(\theta,\phi).$$

Therefore, related to spherical tetrad basis, the above eigenvalue equations read

$$\hat{P}H = PH, \quad [L\Pi_C L_-^{-1}]\,\hat{P}H_1 = PH_1, \quad [L\Pi_C L_-^{-1}\otimes L\Pi_C L_-^{-1}]\,\hat{P}H_2 = PH_2,$$
$$[L\Pi_C L_-^{-1}\otimes L\Pi_C L_-^{-1}\otimes L\Pi_C L_-^{-1}]\,\hat{P}H_3 = PH_3, \quad L\Pi_C L_-^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, we shall transform these equations to the cyclic basis (we shall add for this time the bar-symbol over all the quantities which refer to the cyclic basis, but in this symbol will be omitted afterwards):

$$\bar{H}_1 = UH_1, \quad \bar{H}_2 = (U\otimes U)H_2, \quad \bar{H}_3 = (U\otimes U\otimes)H_3,$$

(4.3)
$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

this results in

(4.4)
$$\begin{aligned} [U(L\Pi_C L_-^{-1})U^{-1}]\,\hat{P}\bar{H}_1 &= P\bar{H}_1, \\ [U(L\Pi_C L_-^{-1})U^{-1}\otimes U(L\Pi_C L_-^{-1})U^{-1}]\,\hat{P}\bar{H}_2 &= P\bar{H}_2, \\ [U(L\Pi_C L_-^{-1})U^{-1}\otimes U(L\Pi_C L_-^{-1})U^{-1}\otimes U(L\Pi_C L_-^{-1})U^{-1}]\,\hat{P}\bar{H}_3 &= P\bar{H}_3; \end{aligned}$$

where the matrix $\bar{\Pi}$ in cyclic basis is

(4.5)
$$\bar{\Pi} = U(L\Pi_C L_-^{-1})U^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

First let us find the eigenvalue states of the space reflection operator for the function $\bar{H}_1$:

$$\bar{\Pi}\hat{P}\bar{H}_1 = P\bar{H}_1 \quad \Longrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} (-1)^j \begin{pmatrix} h_0 D_0 \\ h_1 D_{+1} \\ h_2 D_0 \\ h_3 D_{-1} \end{pmatrix} = P \begin{pmatrix} h_0 D_0 \\ h_1 D_{-1} \\ h_2 D_0 \\ h_3 D_{+1} \end{pmatrix},$$

whence it follows (we use the notation $(-1)^j = \alpha, \alpha^2 = 1$)

$$\alpha h_0 = P h_0, \quad \alpha h_3 = P h_1, \quad \alpha h_2 = P h_2, \quad \alpha h_1 = P h_3. \tag{4.6}$$

In this way we find two types of eigenvalues and relevant restrictions:

$$\begin{array}{lll} P = -\alpha = (-1)^{j+1}, & h_0 = 0, \quad h_2 = 0, & h_3 = -h_1; \\ P = +\alpha = (-1)^{j}, & & h_3 = +h_1. \end{array} \tag{4.7}$$

Now let us construct the eigenvalue states for $\bar{H}_2$, $(\bar{\Pi} \otimes \bar{\Pi})\hat{P}\bar{H}_2 = P\bar{H}_2$. Taking in mind the identity

$$\left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \left(\begin{array}{rrrr} f_0 D_0 & d_1 D_{+1} & d_2 D_0 & d_3 D_{-1} \\ d_1 D_{+1} & f_1 D_{+2} & c_3 D_{+1} & c_2 D_0 \\ d_2 D_0 & c_3 D_{+1} & f_2 D_0 & c_1 D_{-1} \\ d_3 D_{-1} & c_2 D_0 & c_1 D_{-1} & f_3 D_{-2} \end{array} \right) \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right)$$

$$= (-1)^j P \left(\begin{array}{rrrr} f_0 D_0 & d_1 D_{-1} & d_2 D_0 & d_3 D_{+1} \\ d_1 D_{-1} & f_1 D_{-2} & c_3 D_{-1} & c_2 D_0 \\ d_2 D_0 & c_3 D_{-1} & f_2 D_0 & c_1 D_{+1} \\ d_3 D_{+1} & c_2 D_0 & c_1 D_{+1} & f_3 D_{+2} \end{array} \right),$$

we obtain

$$\left(\begin{array}{rrrr} f_0 D_0 & d_3 D_{-1} & d_2 D_0 & d_1 D_{+1} \\ d_3 D_{-1} & f_3 D_{-2} & c_1 D_{-1} & c_2 D_0 \\ d_2 D_0 & c_1 D_{-1} & f_2 D_0 & c_3 D_{+1} \\ d_1 D_{+1} & c_2 D_0 & c_3 D_{+1} & f_1 D_{+2} \end{array} \right) = \alpha P \left(\begin{array}{rrrr} f_0 D_0 & d_1 D_{-1} & d_2 D_0 & d_3 D_{+1} \\ d_1 D_{-1} & f_1 D_{-2} & c_3 D_{-1} & c_2 D_0 \\ d_2 D_0 & c_3 D_{-1} & f_2 D_0 & c_1 D_{+1} \\ d_3 D_{+1} & c_2 D_0 & c_1 D_{+1} & f_3 D_{+2} \end{array} \right),$$

whence there follows the algebraic system

$$\begin{array}{c} f_0 = \alpha P f_0, \\ f_1 = \alpha P f_3, \quad f_2 = \alpha P f_2, \quad f_3 = \alpha P f_1, \\ c_1 = \alpha P c_3, \quad c_2 = \alpha P c_2, \quad c_3 = \alpha P c_1, \\ d_1 = \alpha P d_3, \quad d_2 = \alpha P d_2, \quad d_3 = \alpha P d_1. \end{array}$$

Its solutions are:

$$P = -\alpha = (-1)^{j+1}, \quad f_0 = 0, f_2 = 0, f_3 = -f_1, c_2 = 0, c_3 = -c_1, d_2 = 0, d_3 = -d_1;$$

$$P = +\alpha = (-1)^{j}, \quad f_0, \ f_2, \ f_3 = +f_1, \ c_2, \ c_3 = +c_1, \ d_2, \ d_3 = d_1.$$

Now consider the third eigenvalue equation

$$(\bar{\Pi} \otimes \bar{\Pi} \otimes \bar{\Pi})\hat{P}\bar{H}_3 = P\bar{H}_3 \quad \Longrightarrow \quad \bar{\Pi}_{aa'}\bar{\Pi}_{bb'}\bar{\Pi}_{cc'}\bar{\varphi}_{[a'b']c'} = \bar{\Pi}_{cc'}[\bar{\Pi}_{aa'}\bar{\varphi}_{[a'b']c'}\bar{\Pi}_{bb'}];$$

here we have four subcases:

$$c = 0, \quad \bar{\Pi}\hat{P}\bar{\varphi}_0\tilde{\bar{\Pi}} = P\bar{\varphi}_0, \quad c = 2, \quad \bar{\Pi}\hat{P}\bar{\varphi}_2\tilde{\bar{\Pi}} = P\bar{\varphi}_2,$$

$$c = 3, \quad \bar{\Pi}\hat{P}\bar{\varphi}_1\tilde{\bar{\Pi}} = P\bar{\varphi}_3, \quad c = 1, \quad \bar{\Pi}\hat{P}\bar{\varphi}_3\tilde{\bar{\Pi}} = P\bar{\varphi}_1.$$

This equation at $c = 0$ yields

$$\bar{\Pi}\hat{P}\bar{\varphi}_0\tilde{\bar{\Pi}} = \begin{pmatrix} 0 & \bar{E}_{30}D_{-1} & \bar{E}_{20}D_0 & \bar{E}_{10}D_{+1} \\ -\bar{E}_{30}D_{-1} & 0 & -\bar{B}_{10}D_{-1} & \bar{B}_{20}D_0 \\ -\bar{E}_{20}D_0 & \bar{B}_{10}D_{-1} & 0 & -\bar{B}_{30}D_{+1} \\ -\bar{E}_{10}D_{+1} & -\bar{B}_{20}D_0 & \bar{B}_{30}D_{+1} & 0 \end{pmatrix} (-1)^j$$

$$= P\begin{pmatrix} 0 & \bar{E}_{10}D_{-1} & \bar{E}_{20}D_0 & \bar{E}_{30}D_{+1} \\ -\bar{E}_{10}D_{-1} & 0 & \bar{B}_{30}D_{-1} & -\bar{B}_{20}D_0 \\ -\bar{E}_{20}D_0 & -\bar{B}_{30}D_{-1} & 0 & \bar{B}_{10}D_{+1} \\ -\bar{E}_{30}D_{+1} & \bar{B}_{20}D_0 & -\bar{B}_{10}D_{+1} & 0 \end{pmatrix};$$

whence it follows

$$\bar{E}_{30} = \alpha P\bar{E}_{10}, \bar{E}_{20} = \alpha P\bar{E}_{20}, \bar{E}_{10} = \alpha P\bar{E}_{30}, \bar{B}_{30} = -\alpha P\bar{B}_{10}, \bar{B}_{20} = -\alpha P\bar{B}_{20}, \bar{B}_{10} = -\alpha P\bar{B}_{30}.$$

Depending on the parity, we have

$P = -\alpha = (-1)^{j+1}$,

$$\bar{E}_{30} = -\bar{E}_{10}, \bar{E}_{20} = -\bar{E}_{20}, \bar{E}_{10} = -\bar{E}_{30},$$
$$\bar{B}_{30} = +\bar{B}_{10}, \bar{B}_{20} = +\bar{B}_{20}, \bar{B}_{10} = +\bar{B}_{30};$$

$P = +\alpha = (-1)^j$,

$$\bar{E}_{30} = +\bar{E}_{10}, \bar{E}_{20} = +\bar{E}_{20}, \bar{E}_{10} = +\bar{E}_{30}, \quad \bar{B}_{30} = -\bar{B}_{10}, \bar{B}_{20} = -\bar{B}_{20}, \bar{B}_{10} = -\bar{B}_{30}.$$

Similarly, we find the results for cases $c = 1$, 2, 3. Collecting these results, we write down the following restrictions for states with different parities:

$P = -\alpha = (-1)^{j+1}$,

$$\begin{aligned} \bar{h} = 0, \quad \bar{h}_0 = 0,\ \bar{h}_2 = 0,\ \bar{h}_3 = -\bar{h}_1, \\ \bar{f}_0 = 0, \bar{f}_2 = 0, \bar{c}_2 = 0, \bar{d}_2 = 0, \bar{f}_3 = -\bar{f}_1, \bar{c}_3 = -\bar{c}_1, \bar{d}_3 = -\bar{d}_1, \\ \bar{E}_{30} = -\bar{E}_{10},\ \bar{E}_{20} = 0,\ \bar{B}_{30} = +\bar{B}_{10}, \\ \bar{E}_{32} = -\bar{E}_{12},\ \bar{E}_{22} = 0,\ \bar{B}_{32} = +\bar{B}_{12}, \\ \bar{E}_{13} = -\bar{E}_{31},\ \bar{E}_{23} = -\bar{E}_{21},\ \bar{E}_{33} = -\bar{E}_{11}, \\ \bar{B}_{13} = +\bar{B}_{31},\ \bar{B}_{23} = +\bar{B}_{21},\ \bar{B}_{33} = +\bar{B}_{11}; \end{aligned} \tag{4.8}$$

$P = +\alpha = (-1)^j$,

$$\begin{aligned} \bar{h}_3 = +\bar{h}_1, \quad \bar{f}_3 = +\bar{f}_1, \quad \bar{c}_3 = +\bar{c}_1, \quad \bar{d}_3 = +\bar{d}_1, \\ \bar{B}_{30} = -\bar{B}_{10}, \quad \bar{B}_{20} = 0, \quad \bar{E}_{30} = +\bar{E}_{10}, \\ \bar{B}_{32} = -\bar{B}_{12}, \quad \bar{B}_{22} = 0, \quad \bar{E}_{32} = +\bar{E}_{12}, \\ \bar{B}_{13} = -\bar{B}_{31}, \quad \bar{B}_{23} = -\bar{B}_{21}, \quad \bar{B}_{33} = -\bar{B}_{11}, \\ \bar{E}_{13} = +\bar{E}_{31}, \quad \bar{E}_{23} = +\bar{E}_{21}, \quad \bar{E}_{33} = +\bar{E}_{11}. \end{aligned} \tag{4.9}$$

In what follows, for brevity, we shall omit the bar-symbol over functions.

5 Parity and radial equations

Let us turn back to the complete radial system and take into account restrictions related to parity. In this way we derive two independent subsystems:

The parity $P = (-1)^{j+1}$,

$$\bar{h} = 0, \quad d_1 = 0, \quad \frac{1}{12}[2(\sqrt{2}bf_1 - 6c_1 + 2r(i\epsilon d_1 - ic_1')] = mh_1,$$

$$-\frac{1}{2r}\{\sqrt{2}b(B_{21} + 2h_1) - 2[B_{31} + r(B_{31}' - i\epsilon E_{11})]\} = mf_1,$$

$$\frac{1}{4}\{\sqrt{2}bB_{31} + \sqrt{2}a(B_{22} - B_{11} + 0) + 2[B_{21} + 2h_1 + r(-i\epsilon(-E_{21} - E_{12}) - B_{12}' - 2h_1')]\} = mc_1,$$

$$\frac{1}{4r}\{2B_{10} + \sqrt{2}bE_{11} + 2E_{12} + 4E_{21} + \sqrt{2}a(-B_{20} + E_{31}) + 2r[-i\epsilon(E_{10} + 2h_1) + B_{10}' + E_{21}']\} = md_1,$$

$$\frac{1}{6r}\{-6c_1 + \sqrt{2}bf_1 - 2rc_1'\} = mE_{10},$$

$$0 = mB_{10}, \quad +i\epsilon c_1 = mE_{12}, \quad \frac{a}{\sqrt{2}r}2c_1 = mB_{22}, \quad \frac{1}{6r}\{+6c_1 + 2bf_1 + 4rc_1'\} = mB_{12},$$

$$-i\epsilon f_1 = mB_{11}, +i\epsilon c_1 = mE_{21}, 0 = mE_{31}, \frac{1}{3}(-\sqrt{2}bf_1 - rc_1') = mB_{21}, \frac{bc_1}{\sqrt{2}r} - \frac{f_1}{r} - f_1' = mB_{31}.$$

The parity $P = (-1)^j$:

$$h = 0, \quad -i\epsilon h_0 - 2\frac{1}{r}h_2 - h_2' - \frac{2a}{r\sqrt{2}}h_1 = 0;$$

$$\frac{1}{12r}\left[\sqrt{2}a2c_2 + 2\left(\sqrt{2}bf_1 + 6c_1 + 2r\left(i\epsilon d_1 + c_1'\right)\right)\right] = mh_1; ,$$

$$\frac{1}{6r}\left((a+b)\sqrt{2}d_1 + 4d_2 + 2ir\epsilon f_0 + 2rd_2'\right) = mh_0,$$

$$\frac{1}{6r}\left(2a\sqrt{2}c_1 + 4c_2 + 4f_2 + 2ir\epsilon d_2 + 2rf_2'\right) = mh_2,$$

$$\frac{1}{2r}\left(\sqrt{2}b\left(-B_{21} - 2h_1\right) - 2(-B_{31} + r\left(-B_{31}' + i\epsilon E_{11}\right))\right) = mf_1,$$

$$\frac{1}{8r}\Big[4B_{11} + \sqrt{2}a\left(3B_{12} + B_{21} + B_{21} + 3B_{12} + E_{10} + E_{10} - 2\left(h_1 + h_1\right)\right)$$
$$-2\left(-2B_{11} - 2E_{20} + 4h_2 + r\left(i\epsilon\left(E_{31} + 3E_{22} + E_{31} + 2h_0\right) + B_{11}' + B_{11}' - E_{20}' - 6h_2'\right)\right)\Big] = mf_2,$$

$$\frac{1}{4r}\left(-\sqrt{2}bB_{31} + \sqrt{2}a\left(+B_{11} - 2h_2\right) + 2\left(-B_{21} - 2h_1 + r\left(-i\epsilon\left(E_{21} + E_{12}\right) - B_{12}' + 2h_1'\right)\right)\right) = mc_1,$$

$$-\frac{1}{8r}\Big(\sqrt{2}a\left(-B_{12} - B_{21} - B_{21} - B_{12} + E_{10} + E_{10} + 2\left(h_1 + h_1\right)\right) + 2\left(2\left(E_{20} + 2h_2\right)\right.$$
$$\left.+\ r\left(i\epsilon\left(E_{31} + E_{22} + E_{31} - 2h_0\right) + B_{11}' + B_{11}' + E_{20}' - 2h_2'\right)\right)\Big) = mc_2,$$

$$\frac{1}{4r}\left(-2B_{10} + \sqrt{2}bE_{11} + 2E_{12} + 4E_{21} + \sqrt{2}a\left(0 + E_{31} - 2h_0\right) + 2r\left(-i\epsilon\left(E_{10} + 2h_1\right) - B_{10}' + E_{21}'\right)\right) = md_1,$$

$$\frac{1}{4r}\left(\sqrt{2}a\left(B_{10} + B_{10} + E_{12} + E_{12}\right) + 2\left(E_{31} + 2E_{22} + E_{31} + r\left(-i\epsilon\left(E_{20} + 2h_2\right) + E_{22}' + 2h_0'\right)\right)\right) = md_2,$$

$$\frac{1}{8r}\Big(4B_{11} + \sqrt{2}a\left(B_{12} - B_{21} - B_{21} + B_{12} + 3E_{10} + 3E_{10} + 2\left(h_1 + h_1\right)\right) + 2\left(2B_{11} + 6E_{20} + 4h_2\right.$$

$$+ \; r\,(i\epsilon\,(E_{31} - E_{22} + E_{31} - 6h_0) + B'_{11} + B'_{11} + 3E'_{20} + 2h'_2))) = mf_0;$$

$$\frac{1}{6r}\left(6c_1 - 4ir\epsilon d_1 + \sqrt{2}a\,(c_2 + 3f_0) + \sqrt{2}bf_1 + 2rc'_1\right) = mE_{10},$$

$$\frac{ad_2}{\sqrt{2}r} + \frac{d_1}{r} + d'_1 = mB_{10}, \quad -i\epsilon c_1 + \frac{d_1}{r} + \frac{ad_2}{\sqrt{2}r} = mE_{12},$$

$$\frac{1}{6r}\left(4c_2 + \sqrt{2}a\,(c_1 + c_1) + 4f_2 + 2r\,(-2i\epsilon d_2 - 3f'_0 + f'_2)\right) = mE_{20},$$

$$\frac{1}{6r}\left(4d_2 + \sqrt{2}a\,(d_1 + d_1) + 2ir\epsilon\,(f_0 - 3f_2) - 4rd'_2\right) mE_{22},$$

$$-\frac{1}{6r}\left(-6c_1 + 2ir\epsilon d_1 + \sqrt{2}a\,(c_2 - 3f_2) + \sqrt{2}bf_1 - 4rc'_1\right) = mB_{12},$$

$$\frac{bd_1}{\sqrt{2}r} - i\epsilon f_1 = mE_{11}, \quad -i\epsilon c_1 - d'_1 = mE_{21},$$

$$\frac{1}{6r}\left(2d_2 + \sqrt{2}a\,(2d_1 - d_1) - 2ir\epsilon\,(3c_2 + f_0) - 2rd'_2\right) = mE_{31},$$

$$\frac{1}{3r}\left(2\sqrt{2}ac_2 + ir\epsilon d_1 - \sqrt{2}bf_1 + rc'_1\right) = mB_{21}, \quad -(\frac{bc_1}{\sqrt{2}r} + \frac{f_1}{r} + f'_1) = mB_{31},$$

$$\frac{1}{6r}\left(2\,(c_2 - ir\epsilon d_2 + f_2 + 3rc'_2 - rf'_2) - \sqrt{2}a\,(c_1 - 2c_1)\right) = mB_{11}.$$

6 Second order radial equations

In accordance with the Pauli– Fierz approach, let us eliminate the variables related to tensors of the first and third rank. In this way we derive two different second order systems:

$P = -\alpha = (-1)^{j+1}$,

$$-\epsilon^2 f_1 r^2 - f''_1 r^2 + \sqrt{2}bc'_1 r - 2f'_1 r + \sqrt{2}bc_1 = m^2 r^2 f_1,$$

$$-\frac{bf'_1 r}{\sqrt{2}} + \frac{1}{4}\left(3a^2 + b^2 - 4r^2\epsilon^2 - 6\right) c_1 = m^2 r^2 c_1,$$

$$i\epsilon c'_1 r^2 + 3i\epsilon c_1 r - \frac{ib\epsilon f_1 r}{\sqrt{2}} = 0, \quad d_1 = 0.$$

We may note that this cannot describe the states with $j = 0$, because all equations turn into identities $0 \equiv 0$. By eliminating the variables c_1, c'_1, we derive an equation for f_1,

$$f''_1 + \frac{1}{r}\left(2 + \frac{8b^2}{3a^2 + b^2 - 2[2r^2(m^2 + \epsilon^2) + 3]}\right) f'_1 + \left(\epsilon^2 + m^2 - \frac{b^2}{r^2}\right) f_1 = 0, \quad (m = iM).$$

Allowing for

$$a = \sqrt{j\,(j+1)}, \quad b = \sqrt{(j-1)(j+2)}, \quad x = -\frac{\epsilon^2 - M^2}{4} r^2,$$

we get the following equation

$$f_1'' + \left(\frac{5}{2x} - \frac{4}{j^2+j-2+4\,x} \right) f_1' + \left(-\frac{1}{x} - \frac{1}{4}\,\frac{j^2+j-2}{x^2} \right) f_1 = 0\,.$$

Making the additional change of the independent variable

$$\gamma = \frac{j^2+j-2}{4}, \quad z = -\frac{x}{\gamma}, \quad \frac{d}{dx} = -\frac{1}{\gamma}\frac{d}{dz}, \quad z = +\frac{\epsilon^2 - M^2}{j^2+j-2} r^2 \in (0, +\infty),$$

we get

$$\left[\frac{d^2}{dz^2} + \left(\frac{5/2}{z} - \frac{1}{z-1} \right) \frac{d}{dz} + \left(\frac{1}{z} - \frac{\gamma}{z^2} \right) \right] f_1 = 0.$$

Making the substitution $f_1(z) = z^{\rho} e^{\sigma z} F(Z)$, we obtain the equation

$$\frac{d^2F}{dz^2} + \left(2\,\sigma + \frac{1}{2}\,\frac{4\,\rho + 5}{z} + \frac{1}{1-z} \right) \frac{dF}{dz} +$$

$$+ \left(\sigma^2 + \frac{1}{2}\,\frac{4\,\rho\,\sigma + 5\,\sigma + 2\,\rho + 2\,\gamma}{z} + \frac{1}{2}\,\frac{2\,\rho^2 + 3\,\rho - 2\,\gamma}{z^2} + \frac{\rho + \sigma}{1-z} \right) F = 0\,.$$

By imposing the restrictions

$$\sigma^2 = 0\,, \quad 2\,\rho^2 + 3\,\rho - 2\,\gamma = 0 \qquad \Longrightarrow$$

$$\sigma = 0\,, \qquad \rho = -\frac{3}{4} \pm \frac{1}{4}\,\sqrt{16\gamma + 9} = -\frac{3}{4} \pm \frac{2j+1}{4} = \frac{j-1}{2}, -\frac{j+2}{2}\,,$$

we arrive at the confluent Heun equation

$$\frac{d^2F}{dz^2} + \left(\frac{1}{2}\,\frac{4\,\rho + 5}{z} + \frac{1}{1-z} \right) \frac{dF}{dz} + \left(\frac{\rho + \gamma}{z} + \frac{\rho}{1-z} \right) F = 0\,.$$

The most interesting is the regular solution $f_1(z) = z^{(j-1)/2} F(Z)$.

There exists a simpler possibility. Let us consider the function f_1 from the third equation, find c_1' from the second equation, whence the first equation leads an equation for c_1:

$$\frac{d^2c_1}{dr^2} + \frac{4}{r}\,\frac{dc_1}{dr} + \left(\epsilon^2 + m^2 - \frac{j^2+j-2}{r^2} \right) c_1 = 0\,,$$

which is the Bessel function.

For states with parity $P = (-1)^j$, we get the system for the variables $f_1, f_2, c_1, c_2, d_1, d_2, f_0$:

1) $$-2\left(a^2+2\right) c_2 + d_1 \left(-3i\sqrt{2}ar\epsilon - i\sqrt{2}br\epsilon \right) - 2abf_1 - 4\sqrt{2}arc_1' - 8\sqrt{2}ac_1 - 4rc_2' - 4ir^2\epsilon d_2'$$

$$-8ir\epsilon d_2 + 2r^2\epsilon^2 f_0 - 2r^2 f_2'' - 8rf_2' - 4f_2 = 0,$$

2) $$-\epsilon^2 f_1 r^2 - f_1'' r^2 - i\sqrt{2}b\epsilon d_1 r - \sqrt{2}bc_1' r - 2f_1' r - \sqrt{2}bc_1 - abc_2 = m^2 r^2 f_1,$$

$$3) \qquad i\epsilon d_2' r^2 - \frac{1}{2} c_2'' r^2 - \frac{1}{4} f_0'' r^2 + \frac{3}{4} f_2'' r^2 - 2i\epsilon d_2 r + \sqrt{2} a c_1' r + 2c_2' r - \frac{1}{2} f_0' r + \frac{1}{2} f_2' r$$

$$+\frac{ac_1}{2\sqrt{2}} - \frac{1}{2}\left(r^2\epsilon^2+2\right)c_2 - \frac{ac_3}{2\sqrt{2}} + \frac{1}{12}\left(-11i\sqrt{2}ar\epsilon - i\sqrt{2}br\epsilon\right)d_1 + \frac{1}{12}\left(3a^2+3r^2\epsilon^2\right)f_0$$

$$-\frac{1}{2}abf_1 + \frac{1}{12}\left(9a^2-9r^2\epsilon^2-12\right)f_2 = m^2r^2f_2,$$

$$4) \qquad i\epsilon d_2' r^2 - \frac{1}{2} c_2'' r^2 + \frac{1}{4} f_0'' r^2 + \frac{1}{4} f_2'' r^2 + \frac{1}{2} f_0' r - \frac{1}{2} f_2' r$$

$$-\frac{3ac_1}{2\sqrt{2}} - \frac{1}{2}\left(r^2\epsilon^2+2\right)c_2 - \frac{ac_3}{2\sqrt{2}} + \frac{1}{12}\left(i\sqrt{2}br\epsilon - i\sqrt{2}ar\epsilon\right)d_1 + \frac{1}{12}\left(-3a^2-3r^2\epsilon^2\right)f_0 - \frac{1}{2}abf_1$$

$$+\frac{1}{12}\left(3a^2-3r^2\epsilon^2-12\right)f_2 = m^2r^2c_2,$$

$$5) \qquad i\epsilon d_1' r^2 - i\epsilon d_1 r - \frac{ia\epsilon d_2 r}{\sqrt{2}} + \frac{ac_2' r}{\sqrt{2}} + \frac{bf_1' r}{\sqrt{2}} - \frac{af_2' r}{\sqrt{2}} + \frac{1}{4}\left(-a^2+b^2-4r^2\epsilon^2-6\right)c_1 - \sqrt{2}ac_2 = m^2r^2c_1,$$

$$6) \qquad -i\epsilon c_1' r^2 - d_1'' r^2 - 3i\epsilon c_1 r - \frac{ia\epsilon c_2 r}{\sqrt{2}} - \frac{ia\epsilon f_0 r}{\sqrt{2}} - \frac{ib\epsilon f_1 r}{\sqrt{2}} - 2d_1' r - \frac{ad_2' r}{\sqrt{2}} + \frac{1}{12}(6-(a-b)(a+3b))d_1 = m^2r^2d_1,$$

$$7) \qquad d_2a^2 - i\sqrt{2}r\epsilon c_1 a - 2ir\epsilon c_2 + \frac{(7a-b)d_1}{3\sqrt{2}} - 2ir\epsilon f_2 + \frac{(5a+b)rd_1'}{3\sqrt{2}} + ir^2\epsilon f_0' - ir^2\epsilon f_2' = m^2r^2d_2,$$

$$8) \qquad -i\epsilon d_2' r^2 + \tfrac{1}{2} c_2'' r^2 - \tfrac{3}{4} f_0'' r^2 + \tfrac{1}{4} f_2'' r^2 - 2i\epsilon d_2 r + \sqrt{2} a c_1' r + 2c_2' r - \tfrac{3}{2} f_0' r$$

$$+\frac{3}{2}f_2' r + \frac{7ac_1}{2\sqrt{2}} + \frac{1}{2}\left(r^2\epsilon^2+2\right)c_2 + \frac{ac_3}{2\sqrt{2}} + \frac{1}{4}\left(-3i\sqrt{2}ar\epsilon - i\sqrt{2}br\epsilon\right)d_1 + \frac{1}{4}\left(3a^2+3r^2\epsilon^2\right)f_0$$

$$+\frac{1}{2}abf_1 + \frac{1}{4}\left(a^2-r^2\epsilon^2+4\right)f_2 = m^2r^2f_0.$$

7 The state with minimal $j = 0$

Let us consider the simplest case $j = 0$. To this end, in the complete first order system of radial equations we take into account the following restrictions on radial functions

$$h_1 = 0, h_3 = 0, f_1 = 0, f_3 = 0, c_1 = 0, c_3 = 0, d_1 = 0, d_3 = 0,$$

$$\begin{array}{llll} E_{10} = 0 & E_{11} = 0 & E_{12} = 0 & E_{23} = 0 \\ E_{30} = 0 & E_{21} = 0 & E_{32} = 0 & E_{33} = 0 \\ B_{10} = 0 & B_{21} = 0 & B_{12} = 0 & B_{13} = 0 \\ B_{30} = 0 & B_{31} = 0 & B_{32} = 0 & B_{23} = 0 \end{array} \qquad (7.1)$$

In this way we get 13 equations for 13 variables,

$$h\,,\ h_0\,,\ h_2\,,\ d_2\,,\ f_0\,,\ f_2\,,\ c_2\,,\ B_{11}\,,\ B_{33}\,,\ E_{20}\,,\ E_{13}\,,\ E_{22}\,,\ E_{31}\,:$$

$$-i\epsilon\, h_0 - \frac{2\, h_2}{r} - h_2' = mh\,,$$

$$\frac{1}{6\,r}\,\left(-3\, ir\epsilon\, h + 4\, d_2 + 2\, ir\epsilon\, f_0 + 2\, r\, d_2'\right) = mh_0\,,$$

$$\frac{1}{6\,r}\,\left(4\, c_2 + 2\, ir\epsilon\, d_2 + 4\, f_2 + 3\, r\, h' + 2\, r\, f_2'\right) = mh_2\,,$$

$$\frac{1}{8r}\,\left\{4B_{11} - 4B_{33} + 4E_{20} - 8h_2 - 2r\left[i\epsilon\,(E_{13} + 3E_{22} + E_{31} + 2h_0) + B_{11}' - B_{33}' - E_{20}' - 6h_2'\right]\right\} = mf_2\,,$$

$$-\frac{1}{8\,r}\,\left\{4\, E_{20} + 8\, h_2 + 2\, r\left[i\epsilon\,\left(E_{13} + E_{22} + E_{31} - 2\, h_0\right) + B_{11}' - B_{33}' + E_{20}' - 2\, h_2'\right]\right\} = mc_2\,,$$

$$\frac{1}{4\,r}\,\left\{2\, E_{13} + 4\, E_{22} + 2\, E_{31} + 2\, r\left[-i\epsilon\,\left(E_{20} + 2\, h_2\right) + E_{22}' + 2\, h_0'\right]\right\} = md_2\,,$$

$$\frac{1}{8r}\left\{4B_{11} - 4B_{33} + 12E_{20} + 8h_2 + 2r\left[i\epsilon\,(E_{13} - E_{22} + E_{31} - 6h_0) + B_{11}' - B_{33}' + 3E_{20}' + 2h_2'\right]\right\} = mf_0\,,$$

$$\frac{1}{6\,r}\,\left[4\, c_2 + 4\, f_2 + 2\, r\,(-2\, i\epsilon\, d_2 - 3\, f_0' + f_2')\right] = mE_{20}\,,$$

$$\frac{1}{6\,r}\,\left[4\, d_2 + 2\, ir\epsilon\,(f_0 - 3\, f_2) - 4\, r\, d_2'\right] = mE_{22}\,,$$

$$\frac{1}{6\,r}\,\left[2\, d_2 - 2\, ir\epsilon\,\left(3\, c_2 + f_0\right) - 2\, r\, d_2'\right] = mE_{31}\,,$$

$$\frac{1}{6\,r}\,\left[2\, d_2 - 2\, ir\epsilon\,\left(3\, c_2 + f_0\right) - 2\, r\, d_2'\right] = mE_{13}\,,$$

$$\frac{1}{6\,r}\,\left(2\, c_2 - 2\, ir\epsilon\, d_2 + 2\, f_2 + 6\, r\, c_2' - 2\, r\, f_2'\right) = mB_{11}\,,$$

$$-\frac{1}{6\,r}\,\left(2\, c_2 - 2\, ir\epsilon\, d_2 + 2\, f_2 + 6\, r\, c_2' - 2\, r\, f_2'\right) = mB_{33}\,.$$

With the help of the six last equations, we eliminate the variables with two indices:

$$-i\epsilon\, h_0 - \frac{2\, h_2}{r} - h_2' = mh\,,$$

$$\frac{1}{6\,r}\,\left(-3\, ir\epsilon\, h + 4\, d_2 + 2\, ir\epsilon\, f_0 + 2\, r\, d_2'\right) = mh_0\,,$$

$$\frac{1}{6\,r}\,\left(4\, c_2 + 2\, ir\epsilon\, d_2 + 4\, f_2 + 3\, r\, h' + 2\, r\, f_2'\right) = mh_2\,,$$

$$\frac{1}{3}\,\frac{8 - 6\,\epsilon^2 r^2}{r^2}\, c_2 - \frac{16}{3}\,\frac{i\epsilon}{r}\, d_2 + \frac{1}{3}\,\epsilon^2 f_0 + \frac{1}{3}\,\frac{-9\,\epsilon^2 r^2 + 8}{r^2}\, f_2 - 2\, i\epsilon m\, h_0 - \frac{4\, m}{r}\, h_2$$
$$+\frac{4}{r}\, c_2' + \frac{8}{3}\, i\epsilon\, d_2' - \frac{2}{r}\, f_0' - \frac{2}{3}\,\frac{1}{r}\, f_2' + 6\, m\, h_2' - 2\, c_2'' - f_0'' + f_2'' = 4\, m^2 f_2\,,$$

$$-6\,\epsilon^2 c_2 - \epsilon^2 f_0 - 3\,\epsilon^2 f_2 + 6\, i\epsilon\, mh_0 - \frac{12\, m}{r}\, h_2$$
$$-\frac{4}{r}\, c_2' + 8\, i\epsilon\, d_2' + \frac{6}{r}\, f_0' - \frac{6}{r}\, f_2' + 6\, m\, h_2' - 6\, c_2'' + 3\, f_0'' + f_2'' = 12\, m^2 c_2\,,$$

$$\frac{-4\, i\epsilon}{r}\, c_2 + \frac{\left(2 - \epsilon^2 r^2\right)}{r^2}\, d_2 - \frac{4\, i\epsilon}{r}\, f_2 - 3\, i\epsilon\, m\, h_2 - \frac{2}{r}\, d_2' + 2\, i\epsilon\, f_0' - 2\, i\epsilon\, f_2' + 3\, m\, h_0' - d_2'' = 3\, m^2 d_2\,,$$

$$\frac{6\,\epsilon^2 r^2+8}{r^2}\,c_2-\frac{16\,i\epsilon}{r}\,d_2+3\,\epsilon^2 f_0+\frac{-3\,\epsilon^2 r^2+8}{r^2}\,f_2-18\,i\epsilon\,m\,h_0+\frac{12\,m}{r}\,h_2$$
$$+\frac{20}{r}\,c_2'-8\,i\epsilon\,d_2'-\frac{18}{r}\,f_0'+\frac{10}{r}\,f_2'+6\,m\,h_2'+6\,c_2''-9\,f_0''+f_2''=12\,m^2 f_0\,.$$

One can readily verify that the functions h, h_0, h_2 identically vanish, and the system becomes simpler:

$$4\,d_2+2\,ir\epsilon\,f_0+2\,r\,d_2'=0\,, \qquad (1')$$

$$4\,c_2+2\,ir\epsilon\,d_2+4\,f_2+2\,r\,f_2'=0\,, \qquad (2')$$

$$\frac{1}{3}\,\frac{8-6\,\epsilon^2 r^2}{r^2}\,c_2-\frac{16}{3}\,\frac{i\epsilon}{r}\,d_2+\frac{1}{3}\,\epsilon^2 f_0+\frac{1}{3}\,\frac{-9\,\epsilon^2 r^2+8}{r^2}\,f_2$$
$$+\frac{4}{r}\,c_2'+\frac{8}{3}\,i\epsilon\,d_2'-\frac{2}{r}\,f_0'-\frac{2}{3}\,\frac{1}{r}\,f_2'-2\,c_2''-f_0''+f_2''=4\,m^2 f_2\,, \qquad (3')$$

$$-6\,\epsilon^2 c_2-\epsilon^2 f_0-3\,\epsilon^2 f_2-\frac{4}{r}\,c_2'+8\,i\epsilon\,d_2'+\frac{6}{r}\,f_0'-\frac{6}{r}\,f_2'-6\,c_2''+3\,f_0''+f_2''=12\,m^2 c_2\,, \qquad (4')$$

$$\frac{-4\,i\epsilon}{r}\,c_2+\frac{\left(2-\epsilon^2 r^2\right)}{r^2}\,d_2-\frac{4\,i\epsilon}{r}\,f_2-\frac{2}{r}\,d_2'+2\,i\epsilon\,f_0'-2\,i\epsilon\,f_2'-d_2''=3\,m^2 d_2\,, \qquad (5')$$

$$\frac{6\,\epsilon^2 r^2+8}{r^2}\,c_2-\frac{16\,i\epsilon}{r}\,d_2+3\,\epsilon^2 f_0+\frac{-3\,\epsilon^2 r^2+8}{r^2}\,f_2$$
$$+\frac{20}{r}\,c_2'-8\,i\epsilon\,d_2'-\frac{18}{r}\,f_0'+\frac{10}{r}\,f_2'+6\,c_2''-9\,f_0''+f_2''=12\,m^2 f_0\,. \qquad (6')$$

We extract the variables f_0 from eq. (1′), and the variable c_2 from eq. (2′), and substitute them into (5′); this yields:

$$d_2''+\frac{2}{r}\,d_2'+\left(\epsilon^2+m^2-\frac{2}{r^2}\right)d_2=0 \quad (m=iM).$$

The remaining functions are determined through d_2 by the formulas

$$c_2=\frac{-im^2 r\,d_2'-2\,i\epsilon^2 d_2-im^2 d_2}{2\,r\epsilon\,m^2+2\,r\epsilon^3},\quad f_0=\frac{ir\,d_2'+2\,id_2}{r\epsilon},\quad f_2=\frac{-i\epsilon^2 r\,d_2'-im^2 d_2}{-r\epsilon\,m^2-r\epsilon^3}\,.$$

As a primary function, one can take c_2, which yields

$$c_2''+\left[\frac{4}{r}-\frac{2\,rm^4}{m^4 r^2+4\,\epsilon^2-2\,m^2}\right]c_2'+\left[\epsilon^2+m^2-2\,\frac{m^2\left(-2\,\epsilon^2+m^2\right)}{m^4 r^2+4\,\epsilon^2-2\,m^2}\right]c_2=0\,,$$

$$d_2=-\frac{2\,i\epsilon\,r^2 m^2}{m^4 r^2+4\,\epsilon^2-2\,m^2}\,c_2'+\frac{-2\,i\epsilon\,rm^2+4\,i\epsilon^3 r}{m^4 r^2+4\,\epsilon^2-2\,m^2}\,c_2\,,$$
$$f_0=\frac{-4\,\epsilon^2 r+2\,m^2 r}{m^4 r^2+4\,\epsilon^2-2\,m^2}\,c_2'+\frac{-12\,\epsilon^2-2\,m^2\epsilon^2 r^2+6\,m^2-2\,m^4 r^2(r)}{m^4 r^2+4\,\epsilon^2-2\,m^2}\,c_2\,,$$
$$f_2=+\frac{-4\,\epsilon^2 r+2\,m^2 r}{m^4 r^2+4\,\epsilon^2-2\,m^2}\,c_2'+\frac{2\,m^2-2\,m^2\epsilon^2 r^2-4\,\epsilon^2}{m^4 r^2+4\,\epsilon^2-2\,m^2}\,c_2\,.$$

By taking f_0 as a primary function, we obtain

$$f_0''+\frac{2}{r}\,f_0'+\left(\epsilon^2+m^2\right)f_0=0\,,$$

$$c_2 = \tfrac{1}{2}\,\tfrac{2\,\epsilon^2 - m^2}{(\epsilon^2+m^2)^2 r}\, f_0' - \tfrac{1}{2}\,\tfrac{m^2}{\epsilon^2+m^2}\, f_0\,,$$

$$d_2 = \tfrac{i\epsilon}{\epsilon^2+m^2}\, f_0'\,, \quad f_2 = \tfrac{2\,\epsilon^2 - m^2}{(\epsilon^2+m^2)^2 r}\, f_0' + \tfrac{\epsilon^2}{\epsilon^2+m^2}\, f_0\,.$$

As a primary function one can take f_2, and then we obtain

$$f_2'' + \left[\frac{4}{r} - \frac{2\,r\epsilon^4}{m^2 - 2\,\epsilon^2 + \epsilon^4 r^2}\right] f_2' + \left[\epsilon^2 + m^2 + 2\,\frac{\epsilon^2\left(-2\,\epsilon^2 + m^2\right)}{m^2 - 2\,\epsilon^2 + \epsilon^4 r^2}\right] f_2 = 0\,,$$

$$c_2 = \frac{2\,\epsilon^2 r - m^2 r}{2\,m^2 - 4\,\epsilon^2 + 2\,\epsilon^4 r^2}\, f_2' + \frac{-2\,m^2 + 4\,\epsilon^2 - m^2\epsilon^2 r^2}{2\,m^2 - 4\,\epsilon^2 + 2\,\epsilon^4 r^2}\, f_2\,,$$

$$d_2 = \frac{i\epsilon^3 r^2}{m^2 - 2\,\epsilon^2 + \epsilon^4 r^2}\, f_2' + \frac{-i\epsilon\, r m^2 + 2\,i\epsilon^3 r}{m^2 - 2\,\epsilon^2 + \epsilon^4 r^2}\, f_2\,,$$

$$f_0 = \frac{-2\,\epsilon^2 r + m^2 r}{m^2 - 2\,\epsilon^2 + \epsilon^4 r^2}\, f_2' + \frac{\epsilon^4 r^2 + m^2\epsilon^2 r^2 - 6\,\epsilon^2 + 3\,m^2}{m^2 - 2\,\epsilon^2 + \epsilon^4 r^2}\, f_2\,.$$

Evidently, these are different representations for the same solution, which describes the spin 2 particle with $j = 0$.

8 Conclusions

The matrix tetrad based equation for the spin 2 particle has been specified in spherical coordinates of the Minkowski space. After separating the variables with the use of the total angular momentum and space reflection operators, we have derived two independent systems of radial equations. Some simple solutions are found in explicit form.

It should be stressed that extension to an arbitrary space-time with spherical symmetry

$$dS^2 = e^{\nu}(dt)^2 - e^{\mu}(dr)^2 - r^2[(d\theta)^2 + \sin^2\theta(d\phi)^2], \tag{8.1}$$

does not require new ideas. This is due to the structure of the main equation for such models,

$$\begin{aligned}&\Big[\Gamma^0\big(e^{-\nu/2}\partial_t + \frac{1}{2}\frac{\partial\nu}{\partial r}e^{-\mu/2}J^{03}\Big) + \Gamma^3\Big(e^{-\mu/2}\partial_r + \frac{1}{2}\frac{\partial\mu}{\partial t}e^{-\nu/2}J^{03}\Big)\\ &\quad + \frac{1}{r}e^{-\mu/2}\Big(\Gamma^1 J^{12} + \Gamma^2 J^{23}\Big) + \frac{1}{r}\Sigma_{\theta,\phi} - m\Big]\Phi(x) = 0,\end{aligned} \tag{8.2}$$

where we assume the tetrad in the form

$$e^{\beta}_{(0)} = (e^{-\nu/2}, 0, 0, 0),\ e^{\beta}_{(3)} = (0, e^{-\mu/2}, 0, 0),\ e^{\beta}_{(1)} = (0, 0, \frac{1}{r}, 0),\ e^{\beta}_{(2)} = (0, 0, 0, \frac{1}{r\sin\theta}).$$

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