Lie algebroid structures on the 1-jet bundle of a Jacobi manifold

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Abstract. This work consists in proving that the bundle of jets of order 1 of a Jacobi manifold is a Lie algebroid.

Key words: bundle; jet; Poisson; Jacobi; algebroid.

1 Introduction
The Lie algebroid bracket on the 1-jet bundle of a Jacobi manifold was built without proof by Y. Kerbrat and Z. Souici-Benhammadi in [4]. This construction shows that the Lie algebroid of a contact groupoid is isomorphic to the Lie algebroid of a Jacobi manifold.

In this paper we prove that the bundle of jets of order 1 of a Jacobi manifold is a Lie algebroid. We start by recalling the notions of Poisson and Jacobi manifolds, and of Lie algebroid [1],[3],[5]; an example of Poisson manifold will be given from [3]. The latter will be used for proving our main result.

We will describe from the Poissonification of a Jacobi manifold (see [2, 5]), which allows us to prove that the structure constructed in [4] on the 1-jet bundle of a Jacobi manifold $J^1(M, \mathbb{R})$, is a Lie algebroid structure.

2 Preliminaries

2.1 Jacobi manifolds
Let $M$ be a connected paracompact smooth differentiable manifold of dimension $n$. As well, let $E$ be a vector field on $M$ and let $\Lambda$ be a contravariant skew-symmetric 2-tensor field. We shall further denote $N = \mathcal{C}^\infty(M, \mathbb{R})$.

From the couple $(E, \Lambda)$, we associate the mapping $N \to \chi(M)$ given by:

$$f \to J_f = f.E + \Lambda^\# df,$$
where $\Lambda^\#: \Omega^1(M) \to \chi(M)$ is the associated linear application of $\Lambda$ and where the skew-symmetric bracket on $N$ is given by:

$$\{f, g\} = i_{\Lambda} df \wedge dg + f.E(g) - g.E(f)$$

**Proposition 2.1.** The following equivalence holds true:

$$(2.1) \quad [J_f, J_g] = J[f, g] \iff \{\{f, g, \{h\}\} + \{\{g, h, f\\} + \{h, \{f, g\}\} = 0.$$  

**Definition 2.1.** A triple $(M, \Lambda, E)$ satisfying one of the two equivalent conditions from (2.1), will be called a *Jacobi manifold*.

**Remark 2.2.** 1) We can also define a Jacobi manifold $M$ as a manifold equipped with a bivector $\Lambda$ and a vector field $E$, such that:

$$L_E \Lambda = [E, \Lambda] = 0 \text{ and } [\Lambda, \Lambda] = 2E \wedge \Lambda,$$

where $[.,.]$ is the Schouten bracket (see [1],[6]).

2) If $E = 0$, then the manifold $M$ is a Poisson manifold.

### 2.2 Lie algebroids

**Definition 2.3.** A Lie algebroid on the differentiable manifold $M$ is a triplet $(E, [.,.], \rho)$, where $E \to M$ is a smooth vector bundle over $M$ with a Lie algebra structure $[.,.]$ on the module $E$ of smooth global sections of $E$, and a morphism of vector bundles $\rho: E \to TM$, such that for $s, s' \in \text{Sect}(M, E)$ and $f \in C^\infty(M, \mathbb{R})$, we have

1) $[\rho \circ s, \rho \circ s'] = \rho \circ [s, s']$;

2) $[s, f \cdot s'] = f \cdot [s, s'] + (\rho \circ s)(f) \cdot s'$.

### 2.3 The algebroid of a Poisson manifold ([3])

**Theorem 2.2.** Let $(P, \Lambda)$ be a Poisson manifold. Then $T^*P$ is canonically provided with a Lie algebroid structure, where the morphism of vector bundles of $T^*P$ into $TP$ is $\Lambda^\#$, and where the bracket of two differential forms on $P$ is given by:

$$\{\omega_1, \omega_2\} = i_{\Lambda^\#} d\omega_2 - i_{\Lambda^\#} d\omega_1 + di_\Lambda(\omega_1 \wedge \omega_2)$$

### 2.4 The Poissonnification of a Jacobi manifold

Let $(M, \Lambda, E)$ be a Jacobi manifold, and let $P = \mathbb{R}_+^* \times M = [0, +\infty[ \times M$, and $(x^0, x) \in P$. We define on $P$ a 2- contravariant skew-symmetric tensor $\tilde{\Lambda}$, by:

$$(2.2) \quad \tilde{\Lambda} = \frac{\partial}{\partial x^0} \wedge E_x + \frac{1}{x^0} \Lambda_x$$

or, in local coordinates,

$$(2.3) \quad \tilde{\Lambda}^{0i} = -\Lambda^i 0 = E^i; \quad \tilde{\Lambda}^{ij} = \frac{1}{x^0} \Lambda^{ij}.$$  

From [2] and [5], we have:

**Proposition 2.3.** $(P, \tilde{\Lambda})$ is a Poisson manifold.

**Definition 2.4.** $(P, \tilde{\Lambda})$ is called the Poissonification of the Jacobi manifold $(M, \Lambda, E)$.  

*Lie algebroid structure*
3 The Lie algebroid structure on the 1-jet bundle of a Jacobi manifold

Let \((M, \Lambda, E)\) be a Jacobi manifold and let \((P, \tilde{\Lambda})\) be its Poissonnification. We denote by \(J^1(M, \mathbb{R}) = \mathbb{R} \times T^*M\) the vector bundle over \(TM\) of 1- jets of real functions on \(M\).

In the following we prove that the Lie algebroid structure on the 1-jet bundle \(J^1(M, \mathbb{R})\) stated in [4] is indeed a Lie algebroid structure.

We shall further denote by \(J^1(M, \mathbb{R})\) the module of sections \(C^\infty(M, \mathbb{R}) \times \Omega^1(M)\) of \(J^1(M, \mathbb{R})\). Let \(\omega \in J^1(M, \mathbb{R})\). Then \(\omega = (\omega_0, \omega_1)\), where \(\omega_0 \in C^\infty(M, \mathbb{R})\) and \(\omega_1 \in \Omega^1(M)\). We define \(\phi : J^1(M, \mathbb{R}) \rightarrow \Omega^1(P)\) by:

\[
\omega = (\omega_0, \omega_1) \rightarrow \phi(\omega) = \tilde{\omega} = \omega_0 dx^0 + x^0 \omega_1, \tag{3.1}
\]

where \(x^0 : P \rightarrow \mathbb{R}\) is the natural coordinate mapping over the \(\mathbb{R}^*_+\) factor.

From the Jacobi structure, we trivially deduce a vector bundle morphism \(\rho : J^1(M, \mathbb{R}) \rightarrow TM\), or \(\rho : J^1(M, \mathbb{R}) \rightarrow \chi(M)\), given such that for any \(\omega = (\omega_0, \omega_1) \in J^1(M, \mathbb{R})\), we have:

\[
\rho \circ \omega = \omega_0, E + \Lambda^\# \omega_1. \tag{3.2}
\]

We denote \(\rho \circ \omega\) by \(\rho(\omega)\), and we define [4] a bracket on \(J^1(M, \mathbb{R})\) denoted by \([\omega, \omega']\), given by

\[
\omega \in J^1(M, \mathbb{R}) \Rightarrow \omega = (\omega_0, \omega_1),
\]

\[
\omega' \in J^1(M, \mathbb{R}) \Rightarrow \omega' = (\omega_0', \omega_1').
\]

**Proposition 3.1.** For any pair of sections \(\omega, \omega'\) of \(J^1(M, \mathbb{R})\), there exists a single section \([\omega, \omega']\) of \(J^1(M, \mathbb{R})\) such that:

\[
\phi([\omega, \omega']) = \{\phi(\omega), \phi(\omega')\}
\]

where \(\phi\) is given by formula (3.1). Moreover \([\omega, \omega'] = ([\omega, \omega']_0, [\omega, \omega']_1)\), with:

\[
[\omega, \omega']_0 = -i_\Lambda(\omega_1 \wedge \omega_1') + i_{\rho(\omega)}d\omega_0' - i_{\rho(\omega')}d\omega_0, \tag{3.3}
\]

\[
[\omega, \omega']_1 = i_{\rho(\omega)}d\omega_1' - i_{\rho(\omega')}d\omega_1 + i_E \omega_1(d\omega_0' - \omega_1') - i_E \omega_1'(d\omega_0 - \omega_1) + d(\omega_0, i_E \omega_1' - \omega_0', i_E \omega_1 + i_\Lambda(\omega_1 \wedge \omega_1')). \tag{3.4}
\]

**Proof.** For \(P = \mathbb{R}^*_+ \times M = [0, +\infty[ \times M\) and let \((x^0, x') = (x^\alpha)\) be the coordinates of \(P\). Then, the Poisson tensor \(\tilde{\Lambda}\) is given by the following formulas:

\[
i_{\tilde{\Lambda}}(dx^0 \wedge \alpha) = i_E \alpha
\]

where \(\alpha \in \Omega^1(M)\), and

\[
i_{\tilde{\Lambda}}(\alpha_1 \wedge \alpha_2) = \frac{1}{x^0}i_\Lambda(\alpha_1 \wedge \alpha_2),
\]

where

\[
i_\Lambda(\alpha_1 \wedge \alpha_2) = i_{\Lambda^\# \alpha_1} \alpha_2 \text{ and } \alpha_1, \alpha_2 \in \Omega^1(M).
\]
We put
\[ \tilde{\omega} = \phi(\omega) = \omega_0 dx^0 + x^0 \omega_1 \]
\[ \tilde{\omega}' = \phi(\omega') = \omega_0' dx^0 + x^0 \omega_1'. \]

Then
\[ (3.5) \quad \tilde{\Lambda}^\# \tilde{\omega} = -x^0 i_{E \omega_1} \frac{1}{x_0} + \rho(\omega), \]
where \( \rho(\omega) = \omega_0, E + \Lambda^\# \omega_1 \) is independent of \( x^0 \), since
\[ \{ \tilde{\omega}, \tilde{\omega}' \} = i_{\Lambda^\#} d \tilde{\omega}' - i_{\Lambda^\#} \omega d \tilde{\omega} + di_{\Lambda} (\omega \wedge \tilde{\omega}'). \]

We search a section \( \omega'' \) of \( J^1(M, \mathbb{R}) \) such that
\[ \tilde{\omega}'' = \{ \tilde{\omega}, \tilde{\omega}' \} \]
\[ \tilde{\omega}'' = \phi(\omega'') = \phi([\omega, \omega']) = \{ \tilde{\omega}, \tilde{\omega}' \}. \]

So we have to search for \( \omega_0'' \) and \( \omega_1'' \) such that
\[ \tilde{\omega}'' = \omega_0'' dx^0 + x^0 \omega_1'' \]
\[ \tilde{\omega}' = \phi(\omega') = \omega_0' dx^0 + x^0 \omega_1' \implies d \tilde{\omega}' = dx^0 \wedge (\omega_1' - d \omega_0') + x^0. d \omega_1'. \]

We then derive the formulas:
\[ i_{\tilde{\Lambda}^\#} \omega d \tilde{\omega}' = -x^0 i_{E \omega_1} (\omega_1' - d \omega_0') - i_{\rho(\omega)} (\omega_1' - d \omega_0'). dx^0 + x^0 i_{\rho(\omega)} d \omega_1' \]
\[ i_{\tilde{\Lambda}} (\tilde{\omega} \wedge \tilde{\omega}') = x^0 (i_{E \omega_1'} - \omega_0' i_{E \omega_1} + i_{\Lambda} (\omega_1 \wedge \omega_1')). \]

By reporting in the Lie algebroid the bracket of the Poisson manifold \((P, \tilde{\Lambda})\), we get:
\[ \{ \tilde{\omega}, \tilde{\omega}' \} = [i_{\rho(\omega)} (d \omega_0' - \omega_1')] - i_{\rho(\omega')}(d \omega_0 - \omega_1) + i_{\Lambda} (\omega_1 \wedge \omega_1') + \omega_0 i_{E \omega_1'} - \omega_0' i_{E \omega_1} + x^0 [i_{\rho(\omega')} d \omega_1' + i_{E \omega_1} (d \omega_0' - \omega_1') - i_{\rho(\omega)} d \omega_1' - i_{E \omega_1'} (d \omega_0 - \omega_1) + d(\omega_0 i_{E \omega_1'} - \omega_0' i_{E \omega_1} + i_{\Lambda} (\omega_1 \wedge \omega_1')). \]

Then (3.3) and (3.4) lead to:
\[ [\omega, \omega']_0 = -i_{\Lambda} (\omega_1 \wedge \omega_1') + i_{\rho(\omega)} d \omega_0' - i_{\rho(\omega')} d \omega_0, \]
\[ [\omega, \omega']_1 = i_{\rho(\omega)} d \omega_1' - i_{\rho(\omega')} d \omega_1 + i_{E \omega_1} (d \omega_0' - \omega_1') - i_{E \omega_1'} (d \omega_0 - \omega_1) + d(\omega_0 i_{E \omega_1'} - \omega_0' i_{E \omega_1} + i_{\Lambda} (\omega_1 \wedge \omega_1')), \]
which proves the claim. \( \square \)

**Theorem 3.2.** Let \( (J^1(M, \mathbb{R}) \rightarrow TM, [\cdot, \cdot], \rho) \) be a Lie algebroid, where \([\cdot, \cdot]\) is defined by formulas (3.3), (3.4) and let \( \rho \) be given by formula (3.2) in

\[
\begin{array}{ccc}
J^1(M, \mathbb{R}) & \xrightarrow{\rho} & TM \\
\downarrow & & \downarrow \\
M & \rightarrow & TM
\end{array}
\]
Proof. 1) It is obvious that the bracket $[,]$ is bilinear and skew-symmetric, and it verifies Jacobi identity since $\phi$ is injective.

2) We check that $\rho$ defines a Lie algebra morphism. Let $\omega, \omega' \in J^1(M, \mathbb{R})$. Using the formula (3.5) on the Poissonnification $(P, \tilde{\Lambda})$ of the Jacobi manifold $(M, \Lambda, E)$, we get:

$$\tilde{\Lambda}^\# \phi([\omega, \omega']) = -x^0_i E[\omega, \omega'_1] \frac{\partial}{\partial x^0} + \rho([\omega, \omega'])$$

$$= \tilde{\Lambda}^\# \{\phi(\omega), \phi(\omega')\}$$

$$= [\tilde{\Lambda}^\# \phi(\omega), \tilde{\Lambda}^\# \phi(\omega')]$$

$$= [-x^0_i E\omega_1 \frac{\partial}{\partial x^0} + \rho(\omega), -x^0_i E\omega'_1 \frac{\partial}{\partial x^0} + \rho(\omega')]$$

Since

$$[\omega, \omega']_1 = \omega''_1 = i_{\rho(\omega)} d\omega'_1 - i_{\rho(\omega')} d\omega_1 + i_{E\omega_1} (d\omega'_1 - \omega_1') - i_{E\omega'_1} (d\omega_0 - \omega_1') + d(\omega_0, i_{E\omega_1'} - \omega'_1, i_{E\omega'_1} + i_{\Lambda}(\omega_1 \wedge \omega_1'))$$

from (3.4) and

$$[\rho(\omega'), x^0_i E\omega_1 \frac{\partial}{\partial x^0}] = i_{\rho(\omega')} df_i E\omega_1 x^0 \frac{\partial}{\partial x^0},$$

we infer:

$$\rho([\omega, \omega']) = [\rho(\omega), \rho(\omega')].$$

3) Let’s check that we have:

$$[\omega, f.\omega'] = f.[\omega, \omega'] + i_{\rho(\omega')} df.\omega',$$

for $\omega, \omega' \in J^1(M, \mathbb{R})$ and $f \in C^\infty(M, \mathbb{R})$. We place ourselves again on the Poissonnification $(P, \tilde{\Lambda})$,

$$\rho([\omega, f.\omega']) = \{\phi(\omega), \phi(f.\omega')\}$$

$$= \{\phi(\omega), f.\phi(\omega')\}$$

$$= f.\{\phi(\omega), \phi(\omega')\} + i_{\tilde{\Lambda}^\# \phi(\omega')} df.\phi(\omega')$$

$$= f.\phi([\omega, \omega']) + i_{\rho(\omega)} df.\phi(\omega')$$

$$= \phi(f.[\omega, \omega']) + i_{\rho(\omega)} df.\omega'$$

From the injectivity of $\phi$, we deduce:

$$[\omega, f.\omega'] = f.[\omega, \omega'] + i_{\rho(\omega)} df.\omega'.$$

we therefore conclude that $(J^1(M, \mathbb{R}) \to TM, [,], \rho)$ is a Lie algebroid, and we call it the algebroid of the Jacobi manifold $(M, \Lambda, E)$.

$\Box$
References


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