Submanifolds of a conformal (k, μ)-contact manifold

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Abstract. In the present paper, we introduce and study conformal (k, μ)-contact manifolds. We obtain several results related to such manifolds and their submanifolds, like invariant and anti-invariant submanifolds.

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1 Introduction

Let (M^{2n}, J, g) be a Hermitian manifold of complex dimension n, where J denotes its complex structure and g is its Hermitian metric. Then (M^{2n}, J, g) is a locally conformal Kähler manifold if there is an open cover $\{U_i\}_{i\in I}$ of M^{2n} and a family $\{f_i\}_{i\in I}$ of C^∞ functions $f_i : U_i \to \mathbb{R}$ such that each local metric $g_i = exp(-f_i)g|Ui$ is Kählerian. Here $g|Ui = \iota_i^* g$ where $\iota_i : U_i \to M^{2n}$ is the inclusion. Also (M^{2n}, J, g) is globally conformal Kähler if there is a C^∞ function $f : M^{2n} \to \mathbb{R}$ such that the metric $exp(f)g$ is Kählerian [6]. In 1955, Libermann [8] initiated the study of locally conformal Kähler manifolds. The geometrical conditions for locally conformal Kähler manifold have been obtained by Vaisman [10] and examples of these locally conformal Kähler manifolds were given by Triceri in 1982 [9]. In 2001, Banaru [3] succeeded to classify the sixteen classes of almost Hermitian Kirichenko's tensors. The locally conformal Kähler manifold is one of the sixteen classes of almost Hermitian manifolds. It is known that there is a close relationship between Kähler and contact metric manifolds because Kählerian structures can be made into contact structures by adding a characteristic vector field ξ. The contact structures consists of Sasakian and non-Sasakian cases. In 1972, K. Kenmotsu introduced a class of contact metric manifolds, called Kenmotsu manifolds, which are not Sasakian [7]. Later in 1995, Blair, Koufogiorgos and Papantoniou [5] introduced the notion of (k, μ)-contact manifold which consists of both Sasakian and non-Sasakian manifolds.

Recently, the author [2] introduced conformal Sasakian manifold and studied submanifolds of the conformal Sasakian manifold and the same author with his colleague studied the submanifolds of conformal Kenmotsu manifold [1].

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Motivated by the above studies, in the present paper we define conformal (k, μ)-contact manifold which are not (k, μ)-contact manifold and study the geometry of invariant (anti-invariant) submanifolds of a conformal (k, μ)-contact manifold. We also find the necessary conditions for the CR-submanifolds to be invariant.

The paper is organized as follows: In Section 2, we recall the notion of (k, μ)-contact manifolds and their submanifolds, which are used for further study. In Section 3, we define the conformal (k, μ)-contact manifold and further we give some basic results on conformal (k, μ)-contact manifold. In Section 4, we obtain the necessary and sufficient condition for the invariant submanifolds of a conformal (k, μ)-contact manifold to be minimal. Section 5 is devoted to study of anti-invariant submanifolds of a conformal (k, μ)-contact manifold and obtain the conditions under which these type submanifolds have a flat normal connection. In last section we find the necessary condition for CR-submanifolds of a conformal (k, μ)-contact manifold to be invariant.

2 Preliminaries

Let M be a $(2n+1)$-dimensional almost contact metric manifold with structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, where $\tilde{\phi}, \tilde{\xi}, \tilde{\eta}$ are tensor fields of type $(1,1)$, $(1,0)$, $(0,1)$ respectively, and $\tilde{g}$ is a Riemannian metric on M satisfying

$$\begin{aligned} \tilde{\phi}^2 &= -I + \tilde{\eta} \otimes \tilde{\xi}, \quad \tilde{\eta}(\tilde{\xi}) = 1, , \quad \tilde{\phi}\tilde{\xi} = 0, \quad \tilde{\eta} \cdot \tilde{\phi} = 0, \\ \tilde{g}(\tilde{\phi}X, \tilde{\phi}Y) &= \tilde{g}(X, Y) - \tilde{\eta}(X)\tilde{\eta}(Y), \quad \eta(X) = g(X, \xi), \end{aligned}$$

for all vector fields X, Y on M. An almost contact metric structure becomes a contact metric structure if

$$\tilde{g}(X, \tilde{\phi}Y) = d\tilde{\eta}(X, Y).$$

Then the 1-form $\tilde{\eta}$ is contact form and $\tilde{\xi}$ is a characteristic vector field. A contact metric manifold is said to be (k, μ)-contact manifold [5], if the relation

$$(\tilde{\nabla}_X \tilde{\phi})Y = \tilde{g}(X + hX, Y)\tilde{\xi} - \tilde{\eta}(Y)(X + hX), \tag{2.1}$$

holds on M, where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{g}$ and h is a tensor field given by $h = \frac{1}{2}\mathfrak{L}_{\tilde{\xi}}\tilde{\phi}$. From the above equation, for a (k, μ)-contact manifold we also have

$$\tilde{\nabla}_X \tilde{\xi} = -\tilde{\phi}X - \tilde{\phi}hX. \tag{2.2}$$

Assume $\acute{M}$ is a submanifold of a (k, μ)-contact manifold M. Let $\acute{g}$ and $\acute{\nabla}$ be the induced Riemannian metric and connections of $\acute{M}$, respectively. Then Gauss and Weingarten formulas of $\acute{M}$ are given respectively, by

$$\tilde{\nabla}_X Y = \acute{\nabla}_X Y + \sigma(X, Y), \quad \tilde{\nabla}_X N = -A_N X + \acute{\nabla}^{\perp}_X N,$$

for all vector fields X, Y on $\acute{M}$, where $\acute{\nabla}^{\perp}$ is the normal connection and A is the shape operator of $\acute{M}$ with respect to the unit normal vector N. Let R and $\acute{R}$ denote the curvature tensor of M and $\acute{M}$, then, Gauss and Ricci equations are given by

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, W) &= \acute{g}(\acute{R}(X, Y)Z, W) - g(\sigma(X, W), \sigma(Y, Z)) + g(\sigma(X, Z), \sigma(Y, W)), \\ \tilde{g}(\tilde{R}(X, Y)N_1, N_2) &= g(\acute{R}^{\perp}(X, Y)N_1, N_2) - \acute{g}([A_1, A_2]X, Y), \end{aligned}$$

for all $X, Y, Z, W \in T\acute{M}$, $N_1, N_2 \in T\acute{M}^{\perp}$ and A_1, A_2 are the shape operators corresponding to N_1, N_2.

3 Conformal (k, μ)-contact manifolds

A smooth manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is called a conformal (k, μ)-contact manifold of a (k, μ)-contact structure $(M^{2n+1}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ if, there is a positive smooth function $f : M^{2n+1} \to \mathbb{R}$ such that

$$\tilde{g} = exp(f)g, \quad \tilde{\phi} = \phi, \quad \tilde{\eta} = (exp(f))^{\frac{1}{2}}\eta, \quad \tilde{\xi} = (exp(-f))^{\frac{1}{2}}\xi.$$

Let M be a conformal (k, μ)-contact manifold, let $\tilde{\nabla}$ and ∇ denote the Riemannian connections of M with respect to the metrics $\tilde{g}$ and g, respectively. Using the Koszul formula, we obtain the following relation between the connections $\tilde{\nabla}$ and ∇

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X, Y)\omega^{\sharp}\}, \tag{3.1}$$

such that $\omega(X) = X(f)$ and $\omega^{\sharp} = gradf$ is a vector field metrically equivalent to 1-form ω, that is, $g(\omega^{\sharp}, X) = \omega(X)$.

Then with a straightforward computation, we infer

$$\begin{aligned} exp(-f)(\tilde{R}(X, Y, Z, W)) &= R(X, Y, Z, W) + \frac{1}{2}\{B(X, Z)g(Y, W) - B(Y, Z) \\ &\quad g(X, W) + B(Y, W)g(X, Z) - B(X, W)g(Y, Z)\} \\ &\quad + \frac{1}{4}\|\omega^{\sharp}\|^2\{g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\}, \end{aligned} \tag{3.2}$$

for all vector fields X, Y, Z, W on M, where $B = \nabla\omega - \frac{1}{2}\omega \otimes \omega$ and R, $\tilde{R}$ are the curvature tensors of M related to connections of ∇ and $\tilde{\nabla}$, respectively. Furthermore, by the relations, (2.1), (2.2) and (3.1), we get

$$\begin{aligned} (\nabla_X \phi)Y &= (exp(f))^{\frac{1}{2}}\{g(X + hX, Y)\xi - \eta(Y)(X + hX)\} \\ &\quad - \frac{1}{2}\{\omega(\phi Y)X - \omega(Y)\phi X + g(X, Y)\phi\omega^{\sharp} - g(X, \phi Y)\omega^{\sharp}\} \\ \nabla_X \xi &= -(exp(f))^{\frac{1}{2}}\{\phi X + \phi hX\} + \frac{1}{2}\{\eta(X)\omega^{\sharp} - \omega(\xi)X\}, \end{aligned} \tag{3.3}$$

for all vector fields X, Y on M. Now assume that $\acute{M}$ is a submanifold of a conformal (k, μ)-contact manifold M, and that $\acute{\nabla}$, $\acute{R}$ are the connection, curvature tensor on $\acute{M}$, respectively, and $\acute{g}$ is an induced metric on $\acute{M}$.

We set

$$PX = tan(\phi X), \quad FX = nor(\phi X), \quad tN = tan(\phi N), \quad fN = nor(\phi N),$$

for any $X \in T\acute{M}$ and $N \in T\acute{M}^{\perp}$. Then, using Gauss and Weingarten formulas and considering (3.3), we obtain the following relations:

$$
\begin{aligned}
(\acute{\nabla}_X P)Y &= A_{FY}X + t\sigma(X,Y) + (exp(f))^{\frac{1}{2}}\{g(X+hX,Y)\xi - \eta(Y)(X+hX)\} \\
&\quad -\frac{1}{2}\{\omega(\phi Y)X - \omega(Y)PX + g(X,Y)(\phi\omega^\sharp)^\top - g(X,\phi Y)(\omega^\sharp)^\top\}, \qquad (3.4)
\end{aligned}
$$

$$
(\acute{\nabla}_X F)Y = f\sigma(X,Y) - \sigma(X,PY) + \frac{1}{2}\{\omega(Y)FX - g(X,Y)F\omega^\sharp + g(X,PY)\omega^{\sharp\perp}\}, \qquad (3.5)
$$

$$
\begin{aligned}
(\acute{\nabla}_X t)N &= A_{fN}X - PA_N X - \frac{1}{2}\{\omega(\phi N)X - \omega(N)PX + g(X,\phi N)(\omega^\sharp)^\top\}, \qquad (3.6)\\
(\acute{\nabla}_X f)N &= -\sigma(X,tN) - FA_N X + \frac{1}{2}\{\omega(N)FX + g(X,\phi N)(\omega^\sharp)^\perp\},
\end{aligned}
$$

for all $X, Y \in T\acute{M}$ and $N \in T\acute{M}^\perp$, such that ξ is tangent to $\acute{M}$.

We need the equations of Gauss and Ricci equations of $\acute{M} \subseteq (M,\phi,\xi,\eta,g)$,

$$
\begin{aligned}
exp(-f)\tilde{g}(\tilde{R}(X,Y)Z,W) &= \acute{g}(\acute{R}(X,Y)Z,W) - g(\sigma(X,W),\sigma(Y,Z)) \\
&\quad + g(\sigma(Y,W),\sigma(X,Z)) + \frac{1}{2}\{(B\wedge\acute{g} + \acute{g}\wedge B)(X,Y,Z,W)\} \\
&\quad + \frac{1}{4}\|\omega^\sharp\|^2\{g\wedge g(X,Y,Z,W)\}, \\
g(R(X,Y)N_1,N_2) &= g(\acute{R}^\perp(X,Y)N_1,N_2) - \acute{g}([A_1,A_2]X,Y),
\end{aligned}
$$

for all $X, Y, Z, W \in T\acute{M}$ and $N_1, N_2 \in T\acute{M}^\perp$, where the wedge product of the tensor fields A and B on $\acute{M}$ is given by

$$
(A\wedge B)(X,Y,Z,W) = A(X,Z)B(Y,W) - A(Y,Z)B(X,W),
$$

for all $X, Y, Z, W \in T\acute{M}$.

4 Invariant submanifolds

A submanifold $\acute{M}^m$ of a conformal (k,μ)-contact manifold $(M^{2n+1},\phi,\xi,\eta,g)$ is called an invariant submanifold if $\phi T_p\acute{M} \subseteq T_p\acute{M}$ for any $p \in \acute{M}$.

Lemma 4.1. *Let $\acute{M}^m$ be an invariant submanifold of a conformal (k,μ)-contact manifold M tangent to ξ. Then*

$$
\sigma(X,\phi Y) = \phi\sigma(X,Y) + \frac{1}{2}\{g(X,\phi Y)(\omega^\sharp)^\perp - g(X,Y)(\phi\omega^\sharp)^\perp\}, \qquad (4.1)
$$

$$
A_{\phi N}X = \phi A_N X + \frac{1}{2}\{\omega(\phi N)X - \omega(N)\phi X\}, \qquad (4.2)
$$

$$
A_N\phi X + \phi A_N X = \omega(N)\phi X, \qquad (4.3)
$$

for all $X, Y \in T\acute{M}$ and $N \in T\acute{M}^\perp$.

Proof. Since $\acute{M}$ is invariant, we get $\phi X = PX$ and $\phi N = fN$ for $X \in T\acute{M}$ and $T\acute{M}^\perp$. Then (4.1) and (4.2) immediately follow from (3.5) and (3.6), respectively. Since σ is self adjoint, from (4.1) we have

$$
\sigma(\phi X,Y) = \phi\sigma(X,Y) + \frac{1}{2}\{g(Y,\phi X)\omega^{\sharp\perp} - g(X,Y)(\phi\omega^\sharp)^\perp\}, \qquad (4.4)
$$

for all $X, Y \in T\acute{M}$. Now taking the inner product of (4.4) with the normal vector field N and using the relation $g(A_N X, Y) = g(\sigma(X,Y), N)$, we get

$$g(A_N\phi X, Y) = -g(A_{\phi N}X, Y) + \frac{1}{2}\{g(X,Y)\omega(\phi N) + g(\phi X, Y)\omega(N)\}.$$

The above equation yields

$$A_N\phi X = -A_{\phi N}X + \frac{1}{2}\{\omega(\phi N)X + \omega(N)\phi X\}, \tag{4.5}$$

for all $X \in T\acute{M}$ and $N \in T\acute{M}^{\perp}$. Now (4.3) follows by substituting (4.2) in (4.5). □

Theorem 4.2. *Let $\acute{M}^m$ be an invariant submanifold of a conformal (k,μ)-contact manifold M tangent to ξ. Then $\acute{M}$ is minimal if and only if the Lee vector field $\omega^\sharp$ of M is tangent to $\acute{M}$.*

Proof. From Gauss formula and (3.1), we have

$$\sigma(X,Y) = \tilde{\nabla}_X Y - \acute{\nabla}_X Y - \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\omega^\sharp\}, \tag{4.6}$$

for all X, Y on $\acute{M}$. Now replacing Y by ϕY in (4.6), we get

$$\begin{aligned}\sigma(X,\phi Y) &= \phi\sigma(X,Y) - (\acute{\nabla}_X\phi)Y + \tilde{g}(X+hX,Y)\tilde{\xi} - \tilde{\eta}(Y)(X+hX)\\ &\quad+\frac{1}{2}\{\omega(Y)\phi X - \omega(\phi Y)X - g(X,Y)\phi\omega^\sharp + g(X,\phi Y)\omega^\sharp\}.\end{aligned}$$

Comparing the tangential and normal parts, we get

$$\sigma(X,\phi Y) = \phi\sigma(X,Y) - \frac{1}{2}\{g(X,Y)(\phi\omega^\sharp)^\perp - g(X,\phi Y)(\omega^\sharp)^\perp\}. \tag{4.7}$$

Since $\xi \in T\acute{M}$, taking $X = \phi X$ in (4.7), we obtain

$$\sigma(\phi X,\phi Y) + \sigma(X,Y) = \{g(X,Y) - \frac{1}{2}\eta(X)\eta(Y)\}(\omega^\sharp)^\perp - g(\phi X, Y)(\phi\omega^\sharp)^\perp, \tag{4.8}$$

for all X, Y on $\acute{M}$. Again, since $\xi \in T\acute{M}$, we put $X = Y = \xi$ in (4.8), and we get

$$\sigma(\xi,\xi) = \frac{1}{2}\omega^{\sharp\perp}. \tag{4.9}$$

Now, let $\{e_i, \phi e_i, \xi | i = 1,2,...,n = \frac{m-1}{2}\}$ be an orthonormal frame on $\acute{M}$ and suppose H is the mean curvature vector field of $\acute{M}$. Then, using (4.8) and (4.9), we have

$$H = \frac{1}{m}\sum_{i=1}^{n}\{\sigma(\xi,\xi) + \sigma(e_i,e_i) + \sigma(\phi e_i,\phi e_i)\} = \frac{1}{2}(\omega^\sharp)^\perp.$$

This completes the proof of the theorem. □

5 Anti-invariant submanifolds

A submanifold $\acute{M}^m$ of a conformal (k,μ)-contact manifold $(M^{2n+1},\phi,\xi,\eta,g)$ is called an anti-invariant submanifold if $\phi T_p\acute{M} \subseteq T_p\acute{M}^\perp$ for any $p \in \acute{M}$. Then we have $PX = 0$ and $fN = 0$ for any $X \in T\acute{M}$ and $N \in T\acute{M}^\perp$.

Lemma 5.1. *Let $\acute{M}^m$ be an anti-invariant submanifold of a conformal (k,μ)-contact manifold M tangent to ξ. Then*

$$\begin{aligned} A_{\phi Y}X = {} & -\phi\sigma(X,Y) - (exp(f))^{\frac{1}{2}}\{g(X+hX,Y)\xi - \eta(Y)(X+hX)\} \\ & +\tfrac{1}{2}\{\omega(\phi Y)X + g(X,Y)\phi\omega^{\sharp\perp}\}, \end{aligned} \tag{5.1}$$

$$\begin{aligned} g([A_{\phi Z},A_{\phi W}]X,Y) = {} & g(\sigma(X,W),\sigma(Y,Z)) - g(\sigma(Y,W),\sigma(X,Z)) \\ & -\tfrac{1}{2}\{\acute{g}(Y,Z)\omega(\sigma(X,W)) - \acute{g}(Y,W)\omega(\sigma(X,Z)) + \acute{g}(X,W)\omega(\sigma(Y,Z)) \\ & -\acute{g}(X,Z)\omega(\sigma(Y,W)) + \omega(\phi Z)\Phi(Y,\sigma(X,W)) - \omega(\phi W)\Phi(Y,\sigma(X,Z)) \\ & +\omega(\phi W)\Phi(X,\sigma(Y,Z)) - \omega(\phi Z)\Phi(X,\sigma(Y,W))\} \\ & -\tfrac{1}{4}\{\omega(\phi W)\omega(\phi X)\acute{g}(Y,Z) + \omega(\phi Z)\omega(\phi Y)\acute{g}(X,W) - \omega(\phi Z)\omega(\phi X)\acute{g}(Y,W) \\ & -\omega(\phi W)\omega(\phi Y)\acute{g}(X,Z) - \|\omega^\sharp\|^2[\acute{g}(X,W)\acute{g}(Y,Z) - \acute{g}(X,Z)\acute{g}(Y,W)]\} \\ & -\tfrac{1}{2}(exp(f))^{\frac{1}{2}}\{2\eta(Z)\Phi(Y+hY,\sigma(X,W)) + 2\eta(W)\Phi(X+hX,\sigma(Y,Z)) \\ & -2\eta(W)\Phi(Y+hY,\sigma(X,Z)) + 2\eta(Z)\Phi(X+hX,\sigma(Y,W)) \\ & +\omega(\phi Z)\eta(Y)\acute{g}(X+hX,W) - \omega(\phi W)\eta(Y)\acute{g}(X+hX,Z) \\ & +\omega(\phi X+\phi hX)\eta(W)\acute{g}(Y,Z) - \omega(\phi X+\phi hX)\eta(Z)\acute{g}(Y,W) \\ & +\omega(\phi W)\eta(X)\acute{g}(Y+hY,Z) - \omega(\phi Z)\eta(X)\acute{g}(Y+hY,W) \\ & +\omega(\phi Y+\phi hY)\eta(Z)\acute{g}(X,W) - \omega(\phi Y+\phi hY)\eta(W)\acute{g}(X,Z) \\ & -\omega(\phi Z)\eta(W)\acute{g}(X+hX,Y) + \omega(\phi W)\eta(Z)\acute{g}(X+hX,Y) \\ & +\omega(\phi Z)\eta(W)\acute{g}(X,Y+hY) - \omega(\phi W)\eta(Z)\acute{g}(X,Y+hY)\} \\ & +exp(f)\{\acute{g}(X+hX,W)\acute{g}(Y+hY,Z) - \acute{g}(X+hX,Z)\acute{g}(Y+hY,W) \\ & +\acute{g}(X+hX,Z)\eta(Y)\eta(W) - \acute{g}(X+hX,W)\eta(Y)\eta(Z) \\ & +\acute{g}(Y+hY,W)\eta(X)\eta(Z) - \acute{g}(Y+hY,Z)\eta(X)\eta(W)\}, \end{aligned} \tag{5.2}$$

for all $X,Y,Z,W \in T\acute{M}$, where $\Phi(X,Y) = g(X,\phi Y)$.

Proof. Since $P \equiv 0$ then (5.1) easily follows from (3.4). Next we obtain (5.2) by substituting (5.1) in $\acute{g}([A_{\phi Z},A_{\phi W}]X,Y) = \acute{g}(A_{\phi W}X,A_{\phi Z}Y) - \acute{g}(A_{\phi Z}X,A_{\phi W}Y)$. □

Proposition 5.2. *Let $\acute{M}^m$ be an anti-invariant submanifold of a conformal (k,μ)-contact manifold M^{2n+1} tangent to ξ. Then $\acute{M}$ has a flat normal connection if and only if*

$$\begin{aligned} \acute{R}(X,Y)Z = {} & \eta(R(X,Y)Z) \\ & +\frac{1}{2}\{\acute{B}(Y,Z)X - \acute{B}(X,Z)Y + \acute{g}(Y,Z)\acute{B}(X,.)^\sharp - \acute{g}(X,Z)\acute{B}(Y,.)^\sharp \\ & +B(X,Z)\eta(Y)\xi - B(Y,Z)\eta(X)\xi + B(Y,\xi)\acute{g}(X,Z)\xi \\ & -B(X,\xi)g(Y,Z)\xi\} + (\frac{1}{4}\|\acute{\omega}^\sharp\|^2 + 1)\{\acute{g}(Y,Z)X - \acute{g}(X,Z)Y\} \\ & +(\frac{1}{4}\|\omega^\sharp\|^2 + 1)\{\acute{g}(X,Z)\eta(Y)\xi - \acute{g}(Y,Z)\eta(X)\xi\} + \{\acute{g}(Y,Z)hX \end{aligned} \tag{5.3}$$

$$+\acute{g}(hY,Z)(X+hX)-\acute{g}(hY,Z)\eta(X)\xi-\acute{g}(X,Z)hY-\acute{g}(hX,Z)(Y+hY)$$
$$+\acute{g}(hX,Z)\eta(Y)\xi-\acute{g}(\phi Y,Z)\phi hX-\acute{g}(\phi hY,Z)\phi(X+hX)+\acute{g}(\phi X,Z)\phi hY$$
$$+\acute{g}(\phi hX,Z)\phi(Y+hY)\}+\frac{1}{2}(exp(f))^{\frac{1}{2}}\{2\eta(Z)\phi\sigma(X+hX,Y)$$
$$-2\eta(Z)\phi\sigma(X,Y+hY)+2\Phi(Z,\sigma(X,Y+hY))\xi$$
$$-2\Phi(Z,\sigma(X+hX,Y))\xi+(exp(f))^{\frac{1}{2}}\{\eta(Z)\acute{g}i(X+hX,Y)\phi\omega^\sharp$$
$$-\eta(Z)\acute{g}(X,Y+hY)\phi\omega^\sharp+\acute{g}(X+hX,Y)\omega(\phi Z)\xi-\acute{g}(X,Y+hY)\omega(\phi Z)\xi\}\}$$
$$-exp(f)\{\acute{g}(Y+hY,Z)(X+hX)-\acute{g}(X+hX,Z)(Y+hY)\},$$

for all $X,Y,Z\in T\acute{M}$*, where* $\acute{\omega}^\sharp$ *and* $\acute{B}=B+\omega\circ\sigma$.

Proof. Since $(\tilde{\nabla}_X\phi)Y=\tilde{g}(X+hX,Y)\xi-\eta(Y)(X+hX)$, we have

$$\begin{aligned}\tilde{R}(X,Y)\phi Z &= \phi\tilde{R}(X,Y)Z-\tilde{g}(Y+hY,Z)(\phi X+\phi hX)+\tilde{g}(X+hX,Z)(\phi Y+\phi hY)\\ &\quad -\tilde{g}(\phi Y+\phi hY,Z)(X+hX)+\tilde{g}(\phi X+\phi hX,Z)(Y+hY),\end{aligned}\tag{5.4}$$

for all vector fields X,Y,Z on a (k,μ)-contact manifold $(M^{2n+1},\tilde{\phi},\tilde{\xi},\tilde{\eta},\tilde{g})$ [5]. By substituting (3.2) in (5.4), we obtain

$$\begin{aligned}&R(X,Y)\phi Z=\phi R(X,Y)Z-\tfrac{1}{2}\{B(X,\phi Z)Y-B(Y,\phi Z)X+B(Y,.)^\sharp g(X,\phi Z)\\ &-B(X,.)^\sharp\acute{g}(Y,\phi Z)-B(X,Z)\phi Y+B(Y,Z)\phi X-\phi B(Y,.)^\sharp\acute{g}(X,Z)\\ &+\phi B(X,.)^\sharp\acute{g}(Y,Z)\}+\tfrac{1}{4}\|\omega^\sharp\|^2\{\acute{g}(X,Z)\phi Y-\acute{g}(Y,Z)\phi X-\acute{g}(X,\phi Z)Y\\ &+\acute{g}(Y,\phi Z)X\}+\{g(X+hX,Z)(\phi Y+\phi hY)-g(Y+hY,Z)(\phi X+\phi hX)\\ &+g(\phi X+\phi hX,Z)(Y+hY)-g(\phi Y+\phi hY,Z)(X+hX)\},\end{aligned}$$

for all $X,Y,Z\in T\acute{M}$, where $B(X,Y)=g(B(X,.)^\sharp,Y)$. Taking the inner product with ϕW and using the Ricci and Gauss equations, we obtain

$$\begin{aligned}&g(R^\perp(X,Y)\phi Z,\phi W)-\acute{g}([A_{\phi Z},A_{\phi W}]X,Y)=\acute{g}(\acute{R}(X,Y)Z,W)\\ &-g(\sigma(X,W),\sigma(Y,Z))+g(\sigma(X,Z),\sigma(Y,W))-\eta(R(X,Y)Z)\eta(W)\\ &-\tfrac{1}{2}\{B(Y,Z)\acute{g}(X,W)-B(X,Z)\acute{g}(Y,W)+B(X,W)\acute{g}(Y,Z)\\ &-B(Y,W)\acute{g}(X,Z)+B(X,Z)\eta(Y)\eta(W)-B(Y,Z)\eta(X)\eta(W)\\ &+B(Y,\xi)\acute{g}(X,Z)\eta(W)-B(X,\xi)\eta(W)\acute{g}(Y,Z)\}\\ &-(\tfrac{1}{4}\|\omega^\sharp\|^2+1)\{\acute{g}(Y,Z)\acute{g}(X,W)-\acute{g}(X,Z)\acute{g}(Y,W)+\acute{g}(X,Z)\eta(Y)\eta(W)\\ &-\acute{g}(Y,Z)\eta(X)\eta(W)\}-\{\acute{g}(Y,Z)\acute{g}(hX,W)+\acute{g}(hY,Z)\acute{g}(X+hX,W)\\ &-\acute{g}(hY,Z)\eta(X)\eta(W)-\acute{g}(X,Z)\acute{g}(hY,W)-\acute{g}(hX,Z)\acute{g}(Y+hY,W)\\ &+\acute{g}(hX,Z)\eta(Y)\eta(W)+\acute{g}(\phi Y,Z)\acute{g}(hX,\phi W)+\acute{g}(\phi hY,Z)\acute{g}(X+hX,\phi W)\\ &-\acute{g}(\phi X,Z)\acute{g}(hY,\phi W)-g(\phi hX,Z)g(Y+hY,\phi W)\},\end{aligned}$$

for all $X,Y,Z,W\in T\acute{M}$. From (5.1) we get

$$\begin{aligned}\Phi(Y,\sigma(X,Z)) &= \Phi(Z,\sigma(X,Y))-(exp(f))^{\frac{1}{2}}\{\acute{g}(X+hX,Z)\eta(Y)-\acute{g}(X+hX,Y)\eta(Z)\}\\ &\quad+\frac{1}{2}\{\omega(\phi Z)\acute{g}(X,Y)-\omega(\phi Y)\acute{g}(X,Z)\},\end{aligned}\tag{5.5}$$

for all $X, Y, Z, W \in T\acute{M}$. By substituting (5.2) in (5.5) and using (5.5), we get

$$
\begin{aligned}
&-\phi R^{\perp}(X,Y)\phi Z = \acute{R}(X,Y)Z - \eta(R(X,Y)Z)\xi\\
&\quad -\tfrac{1}{2}\{\acute{B}(Y,Z)X - \acute{B}(X,Z)Y + \acute{g}(Y,Z)\acute{B}(X,.)^{\sharp} - \acute{g}(X,Z)\acute{B}(Y,.)^{\sharp}\\
&\quad +B(X,Z)\eta(Y)\xi - B(Y,Z)\eta(X)\xi + B(Y,\xi)\acute{g}(X,Z)\xi\\
&\quad -B(X,\xi)g(Y,Z)\xi\} - (\tfrac{1}{4}\|\acute{\omega}^{\sharp}\|^2+1)\{\acute{g}(Y,Z)X - \acute{g}(X,Z)Y\}\\
&\quad -(\tfrac{1}{4}\|\omega^{\sharp}\|^2+1)\{\acute{g}(X,Z)\eta(Y)\xi - \acute{g}(Y,Z)\eta(X)\xi\} - \{\acute{g}(Y,Z)hX\\
&\quad +\acute{g}(hY,Z)(X+hX) - \acute{g}(hY,Z)\eta(X)\xi - \acute{g}(X,Z)hY - \acute{g}(hX,Z)(Y+hY)\\
&\quad +\acute{g}(hX,Z)\eta(Y)\xi - \acute{g}(\phi Y,Z)\phi hX - \acute{g}(\phi hY,Z)\phi(X+hX) + \acute{g}(\phi X,Z)\phi hY\\
&\quad +\acute{g}(\phi hX,Z)\phi(Y+hY)\} - \tfrac{1}{2}(exp(f))^{\frac{1}{2}}\{2\eta(Z)\phi\sigma(X+hX,Y)\\
&\quad -2\eta(Z)\phi\sigma(X,Y+hY) + 2\Phi(Z,\sigma(X,Y+hY))\xi\\
&\quad -2\Phi(Z,\sigma(X+hX,Y))\xi + (exp(f))^{\frac{1}{2}}\{\eta(Z)\acute{g}(X+hX,Y)\phi\omega^{\sharp}\\
&\quad -\eta(Z)\acute{g}(X,Y+hY)\phi\omega^{\sharp} + \acute{g}(X+hX,Y)\omega(\phi Z)\xi - \acute{g}(X,Y+hY)\omega(\phi Z)\xi\}\}\\
&\quad +exp(f)\{\acute{g}(Y+hY,Z)(X+hX) - \acute{g}(X+hX,Z)(Y+hY)\},
\end{aligned}
\tag{5.6}
$$

for all $X, Y, Z \in T\acute{M}$. Thus $R^{\perp} = 0$ if and only if (5.3) holds. □

Let $\acute{M}^m$ be an anti-invariant submanifold of a conformal (k,μ)-contact manifold M^{2n+1}. The normal curvature tensor $R^{\perp}$ of $\acute{M}$ is called recurrent if

$$R^{\perp}(X,Y)N = \theta(X,Y)N, \tag{5.7}$$

for all $X, Y \in T\acute{M}$ and $T\acute{M}^{\perp}$ holds on $\acute{M}$, where θ is a 2-form on $\acute{M}$.

Theorem 5.3. *Let $\acute{M}^m$ be an anti-invariant submanifold of a conformal (k,μ)-contact manifold M^{2n+1} normal to ξ with recurrent normal curvature tensor. Then $\acute{M}$ has a flat normal connection.*

Proof. Since $R^{\perp}$ is recurrent, by (5.7) and using (5.6) we obtain

$$
\begin{aligned}
&\acute{R}(X,Y)Z = \theta(X,Y)Z - \theta(X,Y)\eta(Z)\xi + \eta(R(X,Y)Z)\\
&\quad +\tfrac{1}{2}\{\acute{B}(Y,Z)X - \acute{B}(X,Z)Y + \acute{g}(Y,Z)\acute{B}(X,.)^{\sharp} - \acute{g}(X,Z)\acute{B}(Y,.)^{\sharp}\\
&\quad +B(X,Z)\eta(Y)\xi - B(Y,Z)\eta(X)\xi + B(Y,\xi)\acute{g}(X,Z)\xi\\
&\quad -B(X,\xi)g(Y,Z)\xi\} + (\tfrac{1}{4}\|\acute{\omega}^{\sharp}\|^2+1)\{\acute{g}(Y,Z)X - \acute{g}(X,Z)Y\}\\
&\quad +(\tfrac{1}{4}\|\omega^{\sharp}\|^2+1)\{\acute{g}(X,Z)\eta(Y)\xi - \acute{g}(Y,Z)\eta(X)\xi\} + \{\acute{g}(Y,Z)hX\\
&\quad +\acute{g}(hY,Z)(X+hX) - \acute{g}(hY,Z)\eta(X)\xi - \acute{g}(X,Z)hY - \acute{g}(hX,Z)(Y+hY)\\
&\quad +\acute{g}(hX,Z)\eta(Y)\xi - \acute{g}(\phi Y,Z)\phi hX - \acute{g}(\phi hY,Z)\phi(X+hX) + \acute{g}(\phi X,Z)\phi hY\\
&\quad +\acute{g}(\phi hX,Z)\phi(Y+hY)\} + \tfrac{1}{2}(exp(f))^{\frac{1}{2}}\{2\eta(Z)\phi\sigma(X+hX,Y)\\
&\quad -2\eta(Z)\phi\sigma(X,Y+hY) + 2\Phi(Z,\sigma(X,Y+hY))\xi\\
&\quad -2\Phi(Z,\sigma(X+hX,Y))\xi + (exp(f))^{\frac{1}{2}}\{\eta(Z)\acute{g}(X+hX,Y)\phi\omega^{\sharp}\\
&\quad -\eta(Z)\acute{g}(X,Y+hY)\phi\omega^{\sharp} + \acute{g}(X+hX,Y)\omega(\phi Z)\xi - \acute{g}(X,Y+hY)\omega(\phi Z)\xi\}\}\\
&\quad -exp(f)\{\acute{g}(Y+hY,Z)(X+hX) - \acute{g}(X+hX,Z)(Y+hY)\},
\end{aligned}
$$

for all $X, Y, Z \in T\acute{M}$. Since $\xi \in T^{\perp}\acute{M}$, by taking the inner product from the above equation with a tangent vector field W and contracting it over Z and W, we get $(m-1)\theta(X,Y) = 0$, for all X, Y on $\acute{M}$. Since $(m-1) \neq 0$, we get $\theta(X,Y) = 0$. Then from (5.7), we have $R^{\perp} = 0$. Thus, $\acute{M}$ has a flat normal connection. □

6 Totally umbilical and totally geodesic submanifolds

In this last section, we study totally geodesic and totally umbilical submanifolds of a conformal (k,μ)-contact manifold.

Comparing the tangential and normal components of (2.1), (2.2) and Gauss formula, we obtain the following result:

Lemma 6.1. *Let $\acute{M}^m$ be a submanifold of a conformal (k,μ)-contact manifold M^{2n+1}. Then,*

$$\begin{aligned} \acute{\nabla}_X \xi &= -TX - ThX, \\ \sigma(X,\xi) &= NX, \end{aligned} \tag{6.1}$$

for any $X \in T\acute{M}$.

Definition 6.1. A submanifold $\acute{M}^m$ of a conformal (k,μ)-contact manifold M is said to a CR-submanifold, if there exist two orthogonal complementary distributions D and $D^{\perp}$ of $T\acute{M}$ such that $\xi \in T\acute{M}$ and

1. D is invariant by φ, i.e. $\varphi(D_p) \subset D_p$, $\forall p \in \acute{M}$,
2. $D^{\perp}$ is anti-invariant by φ, i.e. $\varphi(D_p^{\perp}) \subset D_p^{\perp}$, $\forall p \in \acute{M}$.

A CR-submanifold is known to be invariant, anti-invariant and proper if $D^{\perp} = 0$, $D = 0$, and $D \neq 0 \neq D^{\perp}$ respectively.

Lemma 6.2. *Let $\acute{M}^m$ be a submanifold of a conformal (k,μ)-contact manifold M^{2n+1}, and let $\mathcal{K}$ be a distribution on $\acute{M}^m$ with $\xi \in \mathcal{K}$. Then, if $\acute{M}^m$ is $\mathcal{K}$-umbilical, it is also $\mathcal{K}$-totally geodesic.*

Proof. Since $\acute{M}^m$ is a $\mathcal{K}$-umbilical submanifold with $\xi \in \mathcal{K}$, then $\sigma(X,Y) = g(X,Y)K$ for a certain normal K to $\acute{M}^m$, $X, Y \in \mathcal{K}$. In particular, we have

$$\sigma(\xi,\xi) = g(\xi,\xi)K = K,$$

but, from (6.1), we get $\sigma(\xi,\xi) = N\xi = 0$, so $K = 0$. Then, $\sigma(X,Y) = 0$, $X, Y \in \mathcal{K}$, and hence $\acute{M}^m$ is $\mathcal{K}$-totally geodesic. □

From the above Lemma, we have the following:

Theorem 6.3. *A totally umbilical submanifold of a conformal (k,μ)-contact manifold, tangent to ξ, is totally geodesic.*

For a totally geodesic submanifold we have:

Theorem 6.4. *A totally geodesic submanifold of a conformal (k,μ)-contact manifold, tangent to ξ, is invariant.*

Proof. Let $\acute{M}^m$ be a totally geodesic submanifold. Then $\sigma(X,\xi) = 0$ for all X tangent to $\acute{M}^m$. But by (6.1), $NX = \sigma(X,\xi) = 0$ for all X, and therefore $\acute{M}^m$ is invariant. □

So we have the following:

Remark. *Proper CR-submanifolds which are totally geodesic or totally umbilical do not exist.*

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References

[1] R. Abdi and E. Abedi, *Invariant and anti-invariant submanifolds of a conformal Kenmotsu manifold*, Azerbaijan Journal of Mathematics, 5, 1 (2015), 54-63.

[2] E. Abedi, *Submanifolds of a conformal Sasakian manifold*, arXiv:1410.7018v2[math.DG], (2015).

[3] M. Banaru, *A new characterization of the Gray Hervella classes of almost Hermitian manifold*, The 8th International Conference on Differential Geometry and Its Applications, August (2001), Opava Czech Republic, 27-31.

[4] D.E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. 509, Springer-Verlag, Berlin 1976.

[5] D.E. Blair, T. Koufogiorgos and B.J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. 91 (1995), 189-214.

[6] S. Dragomir and L. Ornea, *Locally Conformal Kaehler Geometry*, Prog. Math. 155 (1998).

[7] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J. 24 (1972), 93-103.

[8] P. Libermann, *Sur les structure infinitesimals regulieres*, Bull. Soc. Math. France, 83 (1955), 195-224.

[9] F. Tricerri, *Some examples of Locally conformal Kaehler manifolds*, Rend. Sem. Mat. Torino, 40 (1982), 81-92.

[10] I. Vaisman, *A geometry condition for locally conformal Kaehler manifolds to be Kaehler*, Geom. Dedic. 10 (1-4) (1981), 129-134.

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